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Pricing European-Style Options in General Lévy Process with Stochastic Interest Rate

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Abstract: This paper extends the traditional jump-diffusion model to a comprehensive general Lévy process model with the stochastic interest rate for European-style options pricing. By using the Girsanov theorem and Itô formula, we derive the uniform formalized pricing formulas under the equivalent martingale measure. This model contains not only the traditional jump-diffusion model, such as the compound Poisson model, the renewal model, the pure-birth jump-diffusion model, but also the infinite activities Lévy model.

Keywords: Lévy process; stochastic interest rate; Girsanov theorem; option pricing

1. Introduction

European-style options such as vanilla call and put options are some of the most widely traded contracts among financial instruments and were first successfully priced in 1973 by Black and Scholes [1]. However, this framework demonstrates inconsistency with the empirical characteristics of the real market where the presence of stock prices "jumps" and continuous-time trading is physically impossible. Moreover, it is unreasonable to assume that the short-term interest rate is periodically-constant, either. Merton (1975) [2] introduced the European-style option pricing formula with stock prices following the jump-diffusion model. Kou (2002) [3] proposed the double exponential jump-diffusion model and derived the analytical solutions to European call and put options. Carr and Wu (2003) [4] confirmed that there is a diffusion part and a jump part in the S&P 500 index. Zhang, Zhao and Chang (2012) [5] considered the option pricing problem under the equilibrium pricing model where the stock price satisfies the jump-diffusion process. Fu (2012) [6] investigated the equilibrium approach of asset pricing for Lévy process. Liang and Li (2015) [7] introduced Normal Tempered Stable (NTS) process and derived explicit formulae for option pricing and hedging by means of the characteristic function based methods. Deelstra and Simon (2017) [8] considered the pricing of some multivariate European options, when the risky assets involved are modeled by Markov-Modulated Lévy Processes (MMLPs). More recently, Bao and Zhao (2019) [9] studied the pricing of European options under Markovian regime switching exponential Lévy models with stochastic interest rates model. Nowak and Pawłowski (2019) [10] used a Lévy process of jump–diffusion type for description of an underlying asset and derived analytical option pricing formulas using the minimal L^q equivalent martingale measure. Feng, Tan, Jiang and Chen (2020) [11] proposed a European option-pricing model with stochastic volatility and stochastic interest rates and pure-jump Lévy processes.

The purpose of this paper is threefold. Firstly, the paper extends the traditional jump-diffusion model to a comprehensive general Lévy process model which contains not only the traditional jump-diffusion model, such as the compound Poisson model, the renewal model, the pure-birth jump-diffusion model, but also the infinite activities Lévy model. Secondly, to be more realistic, the model puts forward "stochastic interest rate" assumptions as well as "constant interest rate".

Thirdly, by using the Girsanov theorem and Itô formula, we have derived the uniform formalized European-style options pricing formulas under the equivalent martingale measure.

The rest of this paper is organized as follows. Section 2 gives the basic knowledge required for theorem-proof and formula derivation. Section 3 presents the general market model, i.e., the stock price meets the general Lévy process, of which the interest rate is subject to the Vasieck stochastic one. Section 4 derives the uniform formalized European option pricing formula under the market assumption. Section 5 focuses on some special cases. Given these examples, the innovation of the work done in this article can be justified. The Section 6 conducts numerical experiments by using some examples under the framework of our proposed model so as to illustrate its applicable value. The paper is concluded with Section 7.

2. Preliminaries

In this section, we mainly give serval notations and definitions of Lévy process and Lévy measure. The reader may refer to [12–15] for more detail.

Definition 1. *A cádág stochastic process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lévy process satisfying following properties:*

(1) $X(0) = 0$ (a.s.);
(2) X has independent and stationary increments;
(3) X is stochastically continuous, i.e., for all $\varepsilon > 0$ and for all $t \geq 0$

$$\lim_{h\downarrow 0} \mathbb{P}(|X_{t+h} - X_t| > \varepsilon) = 0.$$

Definition 2. *$(X_t)_{t\geq 0}$ is a Lévy process in $\mathbb{R}$, we say that $\nu(A)$ is X's Lévy measure if:*

$$\nu(A) = E\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right]\right., A \in \mathcal{B}(\mathbb{R})$$

where $\mathcal{B}(\mathbb{R})$ is a Borel σ-algebra defined in $\mathbb{R}$.

Lemma 1. *If $(X_t)_{t\geq 0}$ is a Lévy process, ν_x is a Lévy measure which satisfying following properties:*

$$\int_{\mathbb{R}} (|x|^2 \wedge 1)\, \nu_x(\mathrm{d}x) < \infty.$$

Then, we have X_t can be expressed as

$$X_t = \mu t + \sigma W_t + X_t^l + \lim_{\varepsilon\to 0} \widetilde{X}_t^\varepsilon \tag{1}$$

where

$$X_t^l = \int_0^t \int_{|x|\geq 1} xJ_X(dx, ds)$$

$$\widetilde{X}_t^\varepsilon = \int_0^t \int_{\varepsilon<|x|<1} x\left\{J_X(dx, ds) - \nu_x(dx)ds\right\}$$

and J_x is jump component, defined as

$$J_x(B) = \#\left\{t, X_t - X_{t-}\right\} \in B\}, B \subset R \times [0, \infty]$$

Remark 1. *If $\int_{|x|<1} |x|\nu(\mathrm{d}x) < \infty$, the process X_t in Equation (1) can be represented as*

$$X_t = \mu_0 t + \sigma W_t + \int_0^t \int_{\mathbb{R}} xJ(\mathrm{d}x, \mathrm{d}s) \tag{2}$$

where $\mu_0 = \mu - \int_{|x|<1} x\nu(\mathrm{d}x)$. *If* ν *is a finite measure, the process* $\int_0^t \int_{\mathbb{R}} xJ(\mathrm{d}x, \mathrm{d}s)$ *is a compound Poisson process.*

3. The Market Model

In this section, we propose a comprehensive general Lévy process model with the stochastic interest rate. The model extends to the previous market assumptions in the option pricing literature, such as Merton (1975) [2], Kou (2002) [3], Liang and Li (2015) [7] and etc., so that the model could capture more sophisticated and flexible jump structure of the underlying asset as well as the systematic interest rate risk. In the following, the dynamics of the general Lévy process model with the stochastic interest rate will be firstly specified, with which one can easily see how our model is constructed.

We consider intertemporal economy, the market uncertainty are represented by the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is probability measure function in $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}$ is the information filtration.

Suppose there are only two continuously tradable assets in a financial market, one is risky assets, it may be set as stocks S_t. In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, S_t satisfies the general Lévy process.

$$S_t = S_0 \exp(X_t) \tag{3}$$

where X_t is a Lévy process which has the following form like Equation (2) or Lemma 1:

$$X_t = \left[\mu_t - \int_{|x|<1} x\nu_x\,(\mathrm{d}x)\right] t + \sigma_t W_t + \int_0^t \int_{\mathbb{R}} xJ_x\,(\mathrm{d}x, \mathrm{d}s).$$

σ_t is the overall level of volatility, J_x is the jump component and v_x is the Lévy measure of jump parts.

The other one is risk-free assets, it may be set as bond which has short-term interest rate r_t, satisfing the Vasicek stochastic interest rate model.

$$\mathrm{d}r_t = (\beta_t - \alpha_t r_t)\mathrm{d}t + \theta_t \mathrm{d}B_t, \tag{4}$$

where α_t, β_t, θ_t are deterministic functions of time t, B_t is 1-dimensional standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathrm{d}W_t \mathrm{d}B_t = \rho \mathrm{d}t$ $(0 \le \rho < 1)$. Furthermore, J_x is independent of B_t.

Theorem 1. *In the complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *if the stock price satisfies the general Lévy process in Equation* (3) *and interest rate satisfies the Vasicek stochastic interest rate model in Equation* (4), *the market model can be expressed as*

$$S_T = S_t \exp\left\{ \int_t^T \left[r_s - \frac{1}{2}\,|\overrightarrow{\sigma_s}|^2 - \int_{\mathbb{R}} (e^x - 1)\nu_x\,(\mathrm{d}x) \right] \mathrm{d}s + \int_t^T \overrightarrow{\sigma_s}\,\mathrm{d}Z_s + \int_t^T \int_{\mathbb{R}} xJ_x\,(\mathrm{d}x, \mathrm{d}s) \right\}. \tag{5}$$

and

$$r_s = r_t l(t,s) + \int_t^s \beta_u l(u,t)\,\mathrm{d}u + \int_t^s \overrightarrow{\theta_u}\,\mathrm{d}u \tag{6}$$

where

$$l(t,s) = \exp\left(- \int_t^s \alpha_t\,\mathrm{d}t \right).$$

Proof of Theorem 1. Using the Itô formula for S_t in Equation (3), we get the following stochastic differential equation.

$$\begin{aligned}\frac{\mathrm{d}S_t}{S_{t-}} &= \left(\mu_t + \frac{1}{2}\sigma_t^2 + \int_{|x|<1}(e^x - 1 - x)\nu(\mathrm{d}x)\right)\mathrm{d}t + \sigma_t \mathrm{d}W_t \\ &\quad + \int_{|x|\le 1}(e^x - 1)\tilde{J}(\mathrm{d}x, \mathrm{d}t) + \int_{|x|>1}(e^x - 1)J(\mathrm{d}x, \mathrm{d}t) \\ &= \left(\mu_t + \frac{1}{2}\sigma_t^2 + \int_{\mathbb{R}}(e^x - 1 - x\mathbb{1}_{|x|<1})\nu_x(\mathrm{d}x)\right)\mathrm{d}t + \sigma_t \mathrm{d}W_t \\ &\quad + \int_{\mathbb{R}}(e^x - 1)\left[J_x(\mathrm{d}x, \mathrm{d}t) - \nu_x(\mathrm{d}x)\mathrm{d}t\right] \end{aligned} \tag{7}$$

Using the theorem of the existence of equivalent martingale measure, we can find a risk neutral measure $\mathbb{Q}$, equivalent to physical measure $\mathbb{P}$. Under this new equivalent measure, if all jump risks are nonsystematic, Equation (7) can be written as follows

$$\frac{\mathrm{d}S_t}{S_{t-}} = r_t \mathrm{d}t + \sigma_t \mathrm{d}W_t^{\mathbb{Q}} + \int_{\mathbb{R}}(e^x - 1)\left[J_x(\mathrm{d}x, \mathrm{d}t) - \nu_x(\mathrm{d}x)\mathrm{d}t\right]. \tag{8}$$

where σ_t determines the overall level of volatility of the sample paths into continuous, and $W_t^{\mathbb{Q}}$ is 1-dimensional standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. J_x is the jump component and v_x is the Lévy measure of jump parts.

Then, considering the stochastic interest rate in the market model, we assume that $Z_t = (Z_{1t}, Z_{2t})$ is a 2-dimensional standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$\widehat{Z}_{1t} = Z_{1t},\ \widehat{Z}_{2t} = \rho Z_{1t} + \sqrt{1-\rho^2}Z_{2t}$$

where $\widehat{Z}_{1t}$,$\widehat{Z}_{2t}$ are 1-dimensional standard Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathrm{d}\widehat{Z}_{1t}\mathrm{d}\widehat{Z}_{2t} = \rho\mathrm{d}t$. Let us rewrite Equations (4) and (8) as follows

$$\frac{\mathrm{d}S_t}{S_{t-}} = r_t \mathrm{d}t + \overrightarrow{\sigma_t}\mathrm{d}Z_t + \int_{\mathbb{R}}(e^x - 1)[J_x(\mathrm{d}x, \mathrm{d}t) - \nu_x(\mathrm{d}x)\mathrm{d}t], \tag{9}$$

and

$$\mathrm{d}r_t = (\beta_t - \alpha_t r_t)\mathrm{d}t + \overrightarrow{\theta_t}\mathrm{d}Z_t, \tag{10}$$

where $\overrightarrow{\sigma_t} = \sigma_t(\rho, \sqrt{1-\rho^2})$, $\overrightarrow{\theta_t} = (\theta_t, 0)$.

Using the Itô's formula, we obtain the solution of Equation (9)

$$S_T = S_t \exp\left\{\int_t^T \left[r_s - \frac{1}{2}|\overrightarrow{\sigma_s}|^2 - \int_{\mathbb{R}}(e^x - 1)\nu_x(\mathrm{d}x)\right]\mathrm{d}s + \int_t^T \overrightarrow{\sigma_s}\mathrm{d}Z_s + \int_t^T\int_{\mathbb{R}} xJ_x(\mathrm{d}x, \mathrm{d}s)\right\}. \tag{11}$$

For $t \le s \le T$, the Itô's formula for Equation (10) is

$$r_s = r_t l(t,s) + \int_t^s \beta_u l(u,t)\,\mathrm{d}u + \int_t^s \overrightarrow{\theta_u}\,\mathrm{d}u \tag{12}$$

where $l(t,s) = \exp\left(-\int_t^s \alpha_t\,\mathrm{d}t\right)$.

Notice that

$$\begin{aligned}\int_t^T r_s\,\mathrm{d}s =& r_t\int_t^T l(t,s)\,\mathrm{d}s+\int_t^T \mathrm{d}s\int_t^s \beta_u l(u,s)\,\mathrm{d}u+\int_t^T \mathrm{d}s\int_t^T \overrightarrow{\theta_u} l(u,s)\,\mathrm{d}Z_u\\ =& H(t)+\int_t^T \mathrm{d}s\int_t^T \overrightarrow{\theta_u} l(u,s)\,\mathrm{d}Z_u\\ =& H(t)+\int_t^T \overrightarrow{\theta_u} h(u,T)\,\mathrm{d}Z_u\end{aligned}$$

where

$$H(t)=r_t\int_t^T l(t,s)\,\mathrm{d}s+\int_t^T \mathrm{d}s\int_t^s \beta_u l(u,s)\,\mathrm{d}u,\ h(u,T)=\int_u^T l(u,s)\,\mathrm{d}s.$$

□

4. Option Pricing under General Lévy Process Model with Stochastic Interest Rate

In this section, we focus on the European-style options pricing based on the market model proposed in Section 3. We derive the uniform formalized options pricing formulas under the equivalent martingale measure, beginning with the comprehensive general Lévy process, in which the Normal Tempered Stable model, the compound Poisson model, the Merton model, the BS model are viewed as special cases.

Theorem 2. *Let the stock price and interest rate satisfy process Equations (5) and (6), and T is the expiration date. Then the price of an European call option $c(t, S_t)$ with the strike price K at time t is given by*

$$\begin{aligned}c(t,S_t)=&\varepsilon_n\Bigg[S_t\exp\left\{-\int_t^T\int_{\mathbb{R}}(e^x-1)\nu(\mathrm{d}x)\,\mathrm{d}s+\int_t^T\int_{\mathbb{R}}xJ_x(\mathrm{d}x),\mathrm{d}s)\right\}\Phi(d_1)\\ &-Ke^{-H(t)+\frac{1}{2}\int_t^T|\overrightarrow{\theta_u}|^2h(u,T)\,\mathrm{d}u}\Phi(d_2)\Bigg]\end{aligned}\tag{13}$$

where

$$d_1=\frac{\ln\frac{S_t}{k}+\int_t^T\frac{1}{2}|\overrightarrow{\sigma_s}|^2\,\mathrm{d}s-\int_{\mathbb{R}}(e^x-1)\nu_x\,(\mathrm{d}x)\mathrm{d}s+\int_t^T\overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u+\int_t^T\int_{\mathbb{R}}xJ_x\,(\mathrm{d}x,\mathrm{d}s)+H(t)}{\sqrt{\int_t^T\left|\overrightarrow{\theta_u}\right|^2h^2(u,T)\,\mathrm{d}u+\int_t^T\left|\overrightarrow{\sigma_u}\right|^2\,\mathrm{d}u+2\int_t^T\overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u}},$$

$$d_2=d_1-\sqrt{\int_t^T\left|\overrightarrow{\theta_u}\right|^2h^2(u,T)\,\mathrm{d}u+\int_t^T\left|\overrightarrow{\sigma_u}\right|^2\,\mathrm{d}u+2\int_t^T\overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u}$$

ε_n is the expectation operator of $\int_t^T\int_{\mathbb{R}}xJ_x\,(\mathrm{d}x,\mathrm{d}s)$, and $\Phi(x)=\int_{-\infty}^x\frac{1}{\sqrt{2\pi}}e^{\frac{t^2}{2}}\,\mathrm{d}t$ is the cumulative distribution function of the standard normal distribution.

Proof of Theorem 2. The European call option that pays $(S_T-K)^+$ at time T. By using the no-arbitrage hypothesis and martingale representation theorem, we split the last integral into two parts

$$c(t,S_t)=\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,\mathrm{d}s}(S_T-K)^+\mid F_t\right]=E_1-E_2.$$

We begin by considering E_1 so,

$$\begin{aligned}E_1=&\ \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,\mathrm{d}s}S_T\cdot\mathbb{1}_{\{S_T\geq K\}}\right]\\ =&\ \mathbb{E}_t^{\mathbb{Q}}\left[S_t\exp\left\{\int_t^T\left[-\frac{1}{2}|\overrightarrow{\sigma_s}|^2-\int_{\mathbb{R}}(e^x-1)\nu_x(\mathrm{d}x)\right]\mathrm{d}s+\int_t^T\overrightarrow{\sigma_s}\,\mathrm{d}Z_s+\int_t^T\int_{\mathbb{R}}xJ_x\,(\mathrm{d}x,\mathrm{d}s)\right\}\mathbb{1}_{\{\ln S_T\geq\ln K\}}\right]\\ =&\ \exp\left\{-\int_t^T\int_{\mathbb{R}}(e^x-1)\nu_x(\mathrm{d}x)\mathrm{d}s\right\}\varepsilon_n\left[S_T\exp\left\{\int_t^T-\frac{1}{2}|\overrightarrow{\sigma_s}|^2\mathrm{d}s+\int_t^T\overrightarrow{\sigma_s}\mathrm{d}Z_s+\int_t^T\int_{\mathbb{R}}xJ_x\,(\mathrm{d}x,\mathrm{d}s)\right\}\right]\mathbb{1}_{\{A+B\geq F(t)\}}\end{aligned}$$

where

$$A = \int_t^T \overrightarrow{\theta_u} h(u,T)\,\mathrm{d}Z_u,\; B = \int_t^T \overrightarrow{\sigma_u}\,\mathrm{d}Z_u,$$

$$F(t) = \ln\frac{K}{S_t} + \int_t^T \left[\frac{1}{2}|\overrightarrow{\sigma_s}|^2 + \int_{\mathbb{R}} (e^x-1)\nu_x(\mathrm{d}x)\right]\mathrm{d}s - \int_t^T \int_{\mathbb{R}} xJ_x(\mathrm{d}x,\mathrm{d}s) - H(t).$$

Let $\mathbb{Q}^*$ be a new measure. Then the Girsanov's differential can be written as

$$\xi = \frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}} \mid F_t = \exp\left\{\int_t^T -\frac{1}{2}|\overrightarrow{\sigma_s}|^2\,\mathrm{d}s + \int_t^T \overrightarrow{\sigma_s}\,\mathrm{d}Z_s\right\}.$$

By using Girsanov's theorem, and under the new probability measure we obtain a 2-dimensional standard Brownian motion $Z_t^* = Z_t - \int_t^T \overrightarrow{\sigma_s}\,\mathrm{d}s$ where $\mathbb{E}^{\mathbb{Q}}(\xi I_X) = \mathbb{E}^{\mathbb{Q}^*}(I_X)$.

Now let

$$A^* = \int_t^T \overrightarrow{\theta_u} h(u,T)\,\mathrm{d}Z_u^*,\; B^* = \int_t^T \overrightarrow{\sigma_u}\,\mathrm{d}Z_u^*,$$

then

$$A^*+B^* \sim N\left(0, \int_t^T \left|\overrightarrow{\theta_u}\right|^2 h^2(u,T)\,\mathrm{d}u + \int_t^T |\overrightarrow{\sigma_u}|^2\,\mathrm{d}u + 2\int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u\right)$$

Hence,

$$\begin{aligned} E_1 &= \exp\left\{-\int_t^T\int_{\mathbb{R}}(e^x-1)\nu_x(\mathrm{d}x)\mathrm{d}s\right\}\varepsilon_n\left[S_t\exp\left\{\int_t^T\int_{\mathbb{R}} xJ_x(\mathrm{d}x,\mathrm{d}s)\right\}\mathbb{1}_{\left\{A^*+B^*\geq F(t)-\int_t^T|\overrightarrow{\sigma_u}|^2\,\mathrm{d}u-\int_t^T\overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u\right\}}\right] \\ &= \exp\left\{-\int_t^T\int_{\mathbb{R}}(e^x-1)\nu_x(\mathrm{d}x)\mathrm{d}s\right\}\varepsilon_n\left[S_t\exp\left\{\int_t^T\int_{\mathbb{R}} xJ_x(\mathrm{d}x,\mathrm{d}s)\right\}\Phi(d_1)\right] \end{aligned}$$

where

$$d_1 = \frac{\ln\frac{S_t}{k} + \int_t^T \frac{1}{2}|\overrightarrow{\sigma_s}|^2\,\mathrm{d}s - \int_{\mathbb{R}}(e^x-1)\nu_x(\mathrm{d}x)\mathrm{d}s + \int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u + \int_t^T\int_{\mathbb{R}} xJ_x(\mathrm{d}x,\mathrm{d}s) + H(t)}{\sqrt{\int_t^T \left|\overrightarrow{\theta_u}\right|^2 h^2(u,T)\,\mathrm{d}u + \int_t^T \left|\overrightarrow{\sigma_u}\right|^2\,\mathrm{d}u + 2\int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\,\mathrm{d}u}}.$$

Let us now consider E_2,

$$\begin{aligned} E_2 =& K\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s\,\mathrm{d}s} I_{\{S_T\geq K\}}] \\ =& Ke^{-H(t)+\frac{1}{2}\int_t^T\left|\overrightarrow{\theta_u}\right|^2 h^2(u,T)\,\mathrm{d}u}\mathbb{E}^{\mathbb{Q}}[e^{-A-\frac{1}{2}\int_t^T\left|\overrightarrow{\theta_u}\right|^2 h^2(u,T)\,\mathrm{d}u}\mathbb{1}_{\{A+B\geq F(t)\}}] \end{aligned}$$

Similarly, let $\overline{\mathbb{Q}}$ be a new measure. Then,

$$\frac{\mathrm{d}\overline{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} \mid F_T = \exp\left\{-A - \int_t^T \frac{1}{2}|\overrightarrow{\theta_u}|^2 g^2(u,T)\,\mathrm{d}u\right\}.$$

Hence, $Z_t^{\overline{\mathbb{Q}}} = Z_t + \int_0^t \overrightarrow{\theta_u} g(u,T)\,\mathrm{d}u$ is a standard Brownian motion under measure $\overline{\mathbb{Q}}$. Therefore, we have

$$E_2 = Ke^{-H(t)+\frac{1}{2}\int_t^T\left|\overrightarrow{\theta_u}\right|^2 h^2(u,T)\,\mathrm{d}u}\varepsilon_n[\Phi(d_2)],$$

where

$$d_2 = d_1 - \sqrt{\int_t^T |\overrightarrow{\theta_u}|^2 h^2(u,T)\mathrm{d}u + \int_t^T |\overrightarrow{\sigma_u}|^2\mathrm{d}u + 2\int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\mathrm{d}u}.$$

□

Theorem 3. *Let the stock price and rate satisfy process Equations (5) and (6). The T is the expiration date. Then the value function $c(t, S_t)$ at time t of the European put option with strike K is given by*

$$\begin{aligned} p(t,S_t) =&\varepsilon_n \Bigg[K e^{-H(t)+\frac{1}{2}\int_t^T |\overrightarrow{\theta_u}|^2 h(u,T)\,\mathrm{d}u} \Phi(-d_2) \\ &- S_t \exp\left\{ -\int_t^T \int_{\mathbb{R}} (e^x-1)\nu(\mathrm{d}x)\,\mathrm{d}s + \int_t^T \int_{\mathbb{R}} x J_x(\mathrm{d}x,\mathrm{d}s) \right\} \Phi(-d_1) \Bigg], \end{aligned} \tag{14}$$

where d_1, d_2 are same as in Theorem 2.

5. Some Special Cases

In this section, we give some special Lévy processes of exponential type.

Remark 2. *If $\int_{\mathbb{R}}(e^x-1)\nu_x(\mathrm{d}x)\mathrm{d}t = \lambda\mathbb{E}(e^{\xi}-1)\mathrm{d}t$, $\int_{\mathbb{R}}(e^x-1)J_x(\mathrm{d}x,\mathrm{d}t) = (e^{\xi}-1)\mathrm{d}N_t$, this Lévy process is the compound Poisson process. Then, Equation (13) becomes*

$$\begin{aligned} c(t,S_t) = \sum_{n=0}^{\infty} \mathbb{P}_n(T-t)\overline{\varepsilon_n} \Bigg[& S_t \exp\left\{ -\lambda\mathbb{E}(e^{\xi}-1)\tau + \sum_{i=0}^{n} x_i \right\} \Phi(d_1) \\ & - K e^{-H(t)+\frac{1}{2}\int_t^T |\overrightarrow{\theta_u}|^2 h(u,T)\,\mathrm{d}u} \Phi(d_2) \Bigg] \end{aligned} \tag{15}$$

where

$$d_1 = \frac{\ln\frac{S_t}{K} + \int_t^T \frac{1}{2}|\overrightarrow{\sigma_s}|^2 \mathrm{d}s - \lambda\mathbb{E}\left(e^{\xi}-1\right)\tau + \int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\mathrm{d}u + \sum_{i=1}^{n} x_i + H(t)}{\sqrt{\int_t^T |\overrightarrow{\theta_u}|^2 h^2(u,T)\mathrm{d}u + \int_t^T |\overrightarrow{\sigma_u}|^2 \mathrm{d}u + 2\int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\mathrm{d}u}},$$

$$d_2 = d_1 - \sqrt{\int_t^T |\overrightarrow{\theta_u}|^2 h^2(u,T)\mathrm{d}u + \int_t^T |\overrightarrow{\sigma_u}|^2 \mathrm{d}u + 2\int_t^T \overrightarrow{\theta_u}^T h(u,T)\overrightarrow{\sigma_u}\mathrm{d}u},$$

ξ is the jump size, λ is the jump intensity, $\tau = T-t$, $\mathbb{P}_n(T-t) = \mathbb{P}_n(\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}$, and $\overline{\varepsilon_n}$ is the expectation operator of ξ.

As seen from this example, once the distribution function or expression of ν_x and the specific form of J_x, we can get the corresponding option pricing formula according to formula Equations (13) and (14), not just limited to the stock price satisfying the specific jump diffusion process model. This is the most important innovation of this article.

Remark 3. *If the short-rate is a constant, then we can simplify Equations (13) and (14), then obtain the Merton model in paper [2].*

Remark 4. *If the stoke price is a diffusion process, which means that there are no jumps, and the short-rate is a constant, then Equations (13) and (14) yields the well-known Black-Scholes [1] formula.*

6. Numerical Experiment

In this section, we conduct the empirical investigation via some example models under our general model framework. We concentrate on four alternative models to price options: the Black-Scholes model termed BS, the Merton model termed Merton, the Normal Tempered Stable model, termed NTS, the compound Poisson process with stochastic interest rate, termed CPSIR, the NTS process with stochastic interest rate, termed NTSSIR. The model selection aims at covering and comparing the following features: a model exhibiting infinite jump activity and small jumps versus a model exhibiting finite jump activity and large jumps; a model with stochastic interest rate versus a model without one; The BS model is a limiting case of every other model and viewed as the benchmark in our empirical study. Daily option prices are used to estimate parameters, while the models' performance, both in-sample and out-of-sample, are tested respectively.

The option prices used in this study are from the AAPL stock options traded in the Chicago Board Options Exchange. We employ the delayed market quotes on 15 April 2020 as the in-sample data to calibrate the risk-neutral parameters, with the underlying price \$284.43, and those on 16 April 2020, are used for the out-of-sample test, within the underlying price \$286.69. To ensure sufficient liquidity and to alleviate the influences of price discreetness during the valuation, we preclude the option quotes that lower than \$0.3 in the sample data.

Model parameters are estimated by minimizing loss function that measures pricing errors between model prices and market prices. Here we adopt the mean logarithmic square error (MLSE) as loss function to avoid overweight in-the-money options contracts in mean square error (MSE) method and non-convergence problem in the calculation for deep-out-of-the-money and deep-in-the-money options in implied volatility MSE (IVMSE) method. The function is defined by Equation (16).

$$Loss^{MLSE} = \frac{1}{N_i N_j} \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \left(\ln C_{i,j}^{Market} - \ln C_{i,j}^{Model} \right)^2 \tag{16}$$

where N_i and N_j denoting the number of maturities and the number of strikes in daily sample, respectively. $C_{i,j}^{Market}$ is the mid-prices of AAPL stock options and $C_{i,j}^{Model}$ is the model-determined prices for a given parameter set.

Compared with the BS model of which only the volatility parameter σ needs to be estimated, the Merton and NTS model requires both volatility parameter σ and the jump-related parameters (λ, ξ) be estimated, where λ is the jump intensity and ξ is the jump size. The difference between these two models is that the jump structure of Merton model is finite activity and large, while the counterpart of NTS model turns out to be infinite activity and small. In light of the CPSIR and NTSSIR model, the volatility parameter σ, the jump-related parameters (λ, ξ) along with stochastic interest rate $(\alpha, \beta, \theta, \rho)$ are estimated. These two models increase the stochastic interest rate, and the jump structure in CPSIR model is finite activity and large jumps whereas in NTSSIR model is infinite activity and small jumps.

Table 1. Parameter estimate. This table shows the estimated parameter on 15 April 2020. The parameters are back out using loss function MLSE. For each parameter, we report the mean level and the standard deviation (in parentheses).

Parameters	BS	Merton	NTS	CPSIR	NTSSIR
σ	0.1642 (0.0241)	0.1431 (0.162)	0.1274 (0.0184)	0.1394 (0.0132)	0.1537 (0.0174)
α				1.3163 (0.0428)	1.9341 (0.0479)
β				0.0372 (0.0017)	0.0413 (0.0023)
θ				0.3297 (0.0463)	0.4361 (0.0519)
ρ				0.8265 (0.0631)	0.7639 (0.0495)
λ		3.2652 (0.0371)	4.1295 (0.0831)	2.6952 (0.0415)	2.4981 (0.0317)
ξ		0.6549 (0.0143)	0.7524 (0.0275)	0.7934 (0.0184)	0.6352 (0.0163)

Table 1 presents the estimated parameters. Taking into the small standard deviations the model parameters have shown, the stability is justified.

Then, we assess and investigate the in-sample and the out-of-sample pricing errors so as to evaluate five model performances. The indicators employed to measure the magnitude of the pricing errors are Mean Absolute Errors (MAE) and Mean Absolute Percentage Errors (MAPE). Furthermore, the option data has been divided into three categories, in line with the moneyness S/K, where S and K denote the AAPL stock price and the exercise price individually. OTM, ATM, ITM represent the out-of-the-money options, at-the-money options, in-the-money options, respectively.

Table 2. In-sample pricing errors. For a given model, we compute the price of each option on on 15 April 2020, using the parameters estimated in Table 1. The group under the heading MAPE reports the sample average of the absolute difference between the market price and the model price for each option in a given moneyness category; the group under the heading MAE reports the sample average of the value derived through dividing the MAE by the market price.

Models	**MAPE**			**MAE**		
	OTM	**ATM**	**ITM**	**OTM**	**ATM**	**ITM**
BS	31.62%	17.64%	6.59%	$0.54	$0.97	$11.86
Merton	28.67%	14.69%	5.97%	$0.49	$0.88	$10.76
NTS	16.45%	9.64%	5.71%	$0.27	$0.52	$6.24
CPSIR	15.36%	7.62%	5.18%	$0.23	$0.46	$5.46
NTSSIR	13.71%	6.94%	5.07%	$0.22	$0.44	$5.23

Table 3. Out-of-sample pricing errors. For a given model, we compute the price of each option on on 16 April 2020, using the parameters estimated in Table 1. The group under the heading MAPE reports the sample average of the absolute difference between the market price and the model price for each option in a given moneyness category; the group under the heading MAE reports the sample average of the value derived through dividing the MAE by the market price.

Models	**MAPE**			**MAE**		
	OTM	**ATM**	**ITM**	**OTM**	**ATM**	**ITM**
BS	35.72%	23.49%	9.25%	$0.69	$1.28	$13.74
Merton	29.53%	16.51%	7.63%	$0.57	$1.07	$11.4
NTS	19.36%	12.74%	7.38%	$0.38	$0.71	$7.6
CPSIR	16.27%	9.65%	6.94%	$0.31	$0.6	$6.4
NTSSIR	14.48%	8.91%	6.83%	$0.29	$0.54	$5.9

Tables 2 and 3 are in-sample and out-of-sample pricing errors corresponding to the estimated parameters in Table 1, separately. Several conclusions can be drawn from the empirical results. Firstly, regarding the reported MAE and MAPE value, we have noticed that the consideration of the stochastic interest rate contributes to models' performance. Secondly, the finite activity and large jumps in NTS process are more suitable for pricing AAPL stock options, in comparison to the compound Poisson process and Merton model. On the whole, NTSSIR shows the best both in-sample and out-of-sample performance, capable of fitting market prices as well as producing jump structure.

It should be noted, we only provide one kind of practical utility in determining the prices of AAPL stock options. As our model provides a generalized European-style options price formula, the practitioners can compare and elect more suitable jump process in accordance with the underlying asset distributions.

7. Conclusions

This paper extends the traditional jump-diffusion model to a comprehensive general Lévy process model with a stochastic interest rate for European-style options pricing. The model includes various special models, such as the Poisson process where the stock price follows the jump intensity and the amplitude is constant, or the compound Poisson process where the jump intensity is unchanged and the jump amplitude changes, and pricing model of pure-birth process with varying jump intensity and amplitude. According to the pricing formula Equations (13) and (14) given in this paper, all European option pricing expressions corresponding to their respective models can be derived. In addition, compared with other article assumptions, the article assumes that the stock price is subject to a general index Lévy process so that the relative jump structure has no range limit. It is more convenient to handle in the calculation and practical utility.

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