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Δ-Convergence of Products of Operators in p-Uniformly Convex Metric Spaces

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Abstract: In this paper, we first introduce the new notion of p-strongly quasi-nonexpansive maps on p-uniformly convex metric spaces, and then we study the Δ(weak)-convergence of products of p-strongly quasi-nonexpansive maps on p-uniformly convex metric spaces. Furthermore, using the result, we prove the Δ-convergence of the weighted averaged method for projection operators.

Keywords: convex feasibility problem; p-strongly quasi-nonexpansive maps; p-uniformly convex metric spaces; weighted average projection method; Δ-convergence

MSC: Primary 41A65; 47H09; Secondary 47J25; 47N10

1. Introduction

The problem of finding common points of two subsets has been studied by many mathematicians, e.g., [1–12]. It is called the convex feasibility problem which has several applications (see [4]).

A simple and famous algorithmic method to study the convex feasibility problem is to use iterative methods for projection operators. Indeed, iterative methods in metric spaces have been studied several authors, e.g., [13–16], etc. Specially, we introduce the (midpoint) averaged method for two projection operators as follows: for two projections P_A and P_B, where A and B are closed convex subset of a Hilbert space H, a iterative sequence

$$x_n := \frac{P_A x_{n-1} + P_B x_{n-1}}{2} \tag{1}$$

is called (midpoint) averaged projection method, where x_0 is a point in H.

In [2], the author studied the weak convergence of $\{x_n\}$ given as in Equation (1). In [5], the authors provided some example that is a sequence which is weakly convergent, but not convergent in norm sense.

The averaged projection method in Hilbert spaces (linear space) can be extended to more general spaces (non linear space), e.g., geodesic metric spaces. In [7], Choi defined the weighted averaged projection method in $\mathrm{CAT}(\kappa)$ spaces with $\kappa \geq 0$ by using the notion of geodesic and the author proved that Δ(weak)-convergence for the weighted averaged projection sequence (see also [17] for the case of $\mathrm{CAT}(0)$ spaces). Indeed, in $\mathrm{CAT}(\kappa)$ spaces with $\kappa \geq 0$, we can define the weighted averaged projection method by

$$x_{n+1} = P_A x_n \#_t P_B x_n, \quad n \in \mathbb{N}, \quad t \in (0,1), \tag{2}$$

where $x \#_t y$ is a geodesic connecting two point x and y. In particular, if in (2), we take $t_n = 1/2$ for $n = 0, 1, \cdots$, we have the averaged projection method. In fact, in [9], the authors studied the Δ-convergence of the weighted averaged sequence for general operators on p-uniformly convex metric spaces. Note that every $\mathrm{CAT}(\kappa)$ space with $\kappa \geq 0$ having some diameter condition, is a 2-uniformly convex metric space, (see Example 1).

The main purpose of this paper is to study the Δ-convergence (or weak convergence) of products of p-strongly quasi-nonexpansive maps (see Section 3) on p-uniformly convex metric spaces. Indeed, (2) can be rewritten as

$$x_n = T^n x_0,$$

where $T = P_A \#_t P_B$, $t \in (0,1)$, where $(P_A \#_t P_B)x := P_A x \#_t P_B x$. Thus, the convergence of (2) can be proven by the convergence result of iterates of the products of T.

This paper is organized as follows. In Section 2, we firstly recall the notions of p-uniformly convex metric spaces, and the notion of Δ-convergence of sequence in p-uniformly convex metric spaces. In Section 3, we firstly introduce a new notion of p-strongly quasi-nonexpansive maps on p-uniformly convex metric spaces, and then we study the Δ-convergence result of products of p-strongly quasi-nonexpansive maps. Furthermore, using the result, we study the Δ-convergence of the weighted averaged sequence for two projections defined by (2).

2. Geodesic Metric Spaces

2.1. p-Uniformly Convex Metric Spaces

Let (X, d) be a metric space and x and y be two element in X. A continuous map $\gamma : [0,1] \to X$ is called a *geodesic* joining x and y if it satisfies the following property: $d(\gamma(s), \gamma(t)) = |s - t| d(x, y)$ for any $s, t \in [0,1]$ with $\gamma(0) = x$, $\gamma(1) = y$.

A metric space X is said to be a *geodesic metric space* if for any two points x and y in X, there exists a geodesic γ joining them.

For $2 \leq p < \infty$, a geodesic metric space (X, d) is called *p-uniformly convex* with parameter $c_X > 0$ if there exists a constant $c_X \in (0,1]$ such that for any $z \in X$ and any geodesic $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$

$$d(z, \gamma(t))^p \leq (1-t) d(z, x)^p + t d(z, y)^p - c_X t(1-t) d(x, y)^p, \quad t \in [0,1]. \tag{3}$$

(see [18–20]). For the case of $1 < p < 2$, since p-uniformly convex metric spaces can be considered as 2-uniformly convex metric spaces (see [18]), we only consider the case of $2 \leq p < \infty$ in this paper.

Now, we give some important examples for p-uniformly convex metric spaces.

Example 1. *(1) Let (X, d) be a complete* CAT(0)*-space (or Hadamard space). Then (X, d) is a 2-uniformly convex metric space with parameter $c_X = 1$. (see [18]).*
(2) Let (X, d) be a CAT(κ) *space with* $\mathrm{diam}(X) (= \sup\{d(x,y) \,;\, x, y \in X\}) < \frac{\pi}{2\sqrt{\kappa}}$. *Then (X, d) is a 2-uniformly convex metric space with parameter $c_X \in (0,1)$ (see [18,20]).*

2.2. Δ-Convergence in Geodesic Metric Spaces

We now recall the notion of a weak type convergence in general metric spaces. In [21], the author was firstly introduced the notion of Δ-convergence that is weak type convergence in general metric spaces. Indeed the weak convergence and the Δ-convergence are equivalent in Hilbert spaces. Many authors have been studied the Δ-convergence results in several geodesic metric spaces, see [3,8,9,17,22–25] etc.

Let (X, d) be a geodesic metric space and let $\{x_n\}$ be a bounded sequence in X. Set

$$r(x, \{x_n\}) := \limsup_{n \to +\infty} d(x, x_n), \quad \text{for } x \in X.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) := \left\{ x \in X \,|\, r(x, \{x_n\}) = \inf_{y \in X} r(y, \{x_n\}) \right\}.$$

A sequence $\{x_n\} \subseteq X$ is said to Δ-*converge* (or weakly converge) to $x \in X$ if for any $\{x_{n_k}\} \subseteq \{x_n\}$, x is a unique asymptotic center of $\{x_{n_k}\}$. In this case, x is called the Δ-*limit* of $\{x_n\}$. A point $x \in X$ is called a Δ-*cluster point* of $\{x_n\}$ if there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying that $\{x_{n_k}\}$ Δ-converges to x.

The following result is important to study the Δ-convergence in a complete p-uniformly convex metric space.

Proposition 1 ([1]). *Let X be a complete p-uniformly convex metric space with parameter $c_X > 0$ and $\{x_n\}$ be a bounded sequence in X. Then the following results hold*

(i) *The asymptotic center of $\{x_n\}$ has only one point.*
(ii) *$\{x_n\}$ has a Δ-cluster point.*

For our study, we recall the notion of Fejér monotone sequence in metric spaces. Let $\{x_n\}$ be a sequence in a metric space (X, d) and K be a non-empty subset of X. A sequence $\{x_n\}$ is called *Fejér monotone with respect to (w.r.t) K* if for any $k \in K$

$$d(x_{n+1}, k) \leq d(x_n, k), \quad \text{for any} \quad n \in \mathbb{N}.$$

It is clear that $\{x_n\}$ is a bounded sequence whenever $\{x_n\}$ is Fejér monotone sequence w.r.t some K.

Lemma 1 ([9]). *Let (X, d) be a complete p-uniformly convex metric space with parameter $c_X > 0$ and let K be a nonempty subset of X. Let $\{x_n\} \subseteq X$ be a Fejér monotone sequence w.r.t K. If any Δ-cluster point z of $\{x_n\}$ belongs to K, then $\{x_n\}$ Δ-converges to a point in K.*

3. Δ-Convergence Results

Let (X, d) be a p-uniformly convex metric space. An operator $T : X \to X$ with $\mathrm{Fix}(T) \neq \emptyset$ is said to be *firmly quasi-nonexpansive* if for all $z \in \mathrm{Fix}(T)$

$$d(Tx, z) \leq d(x \#_t Tx, z),$$

where $x\#_t y$ is a geodesic connecting two point x and y, and *p-strongly quasi-nonexpansive* if T is quasi-nonexpansive and if whenever $\{x_n\} \subseteq X$ is bounded, $z \in \mathrm{Fix}(T)$ and $\lim_{n\to+\infty}[d(x_n, z)^p - d(Tx_n, z)^p] = 0$, it follows that $\lim_{n\to+\infty} d(x_n, Tx_n) = 0$. Note that 1-strongly quasi-nonexpansive is called strongly quasi-nonexpansive (see [26]). Furthemore, $2n$-strongly quasi-nonexpansive for $n \in \mathbb{N}$ is n-strongly quasi-nonexpansive since

$$d(x_n, z)^{2n} - d(Tx_n, z)^{2n} = (d(x_n, z)^n - d(Tx_n, z)^n)(d(x_n, z)^n + d(Tx_n, z)^n).$$

Example 2. *Let (X, d) be a complete $\mathrm{CAT}(\kappa)$ space $(\kappa \geq 0)$ with $\mathrm{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$, and A be a non-empty closed convex subset. Then the metric projection operator P_A is firmly quasi-nonexpansive (see [1]).*

Lemma 2. *Let (X, d) be a p-uniformly convex metric space with parameter $c_X > 0$. Every firmly quasi-nonexpansive map T on X with with $\mathrm{Fix}(T) \neq \emptyset$ is p-strongly quasi-nonexpansive.*

Proof. Suppose that $T : X \to X$ is a firmly quasi-nonexpansive map with $\mathrm{Fix}(T) \neq \emptyset$ and $\{x_n\}$ is a bounded sequence. Put $z \in \mathrm{Fix}(T)$ is a point such that

$$\lim_{n\to+\infty} [d(x_n, z)^p - d(Tx_n, z)^p] = 0. \tag{4}$$

Then we only show that $\lim_{n\to+\infty} d(x_n, Tx_n) = 0$ for our proof. To do this, we assume that there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ such that

$$d(x_{n_k}, Tx_{n_k}) \geq \epsilon \tag{5}$$

for all $k \in \mathbb{N}$. Since the sequence $\{x_n\}$ is bounded, $\{d(x_n, z)\}$ is also bounded in $\mathbb{R}$. So, we can take a subsequence $\{d(x_{n_k}, z)\}$ of $\{d(x_n, z)\}$ such that $\lim_{k\to+\infty} d(x_{n_k}, z) = \alpha$. Therefore, by using (4), we have

$$\alpha = \lim_{k\to+\infty} d(x_{n_k}, z) = \lim_{k\to+\infty} d(Tx_{n_k}, z). \tag{6}$$

Since T is a firmly quasi-nonexpansive map, we have for all $t \in [0, 1]$

$$\begin{aligned} d(Tx_{n_k}, z)^p &\leq d(x_{n_k} \#_t Tx_{n_k}, z)^p \\ &\leq (1-t)d(x_{n_k}, z)^p + td(Tx_{n_k}, z)^p \leq d(x_{n_k}, z)^p, \end{aligned}$$

which implies that for all $t \in [0, 1]$ (using (6))

$$\lim_{k\to+\infty} d(x_{n_k} \#_t Tx_{n_k}, z) = \alpha.$$

However, we have by (3),

$$\begin{aligned} 0 &\leq \frac{c_X}{2} t(1-t)d(x_{n_k}, Tx_{n_k})^p \\ &\leq (1-t)d(x_{n_k}, z)^p + td(Tx_{n_k}, z)^p - d(x_{n_k} \#_t Tx_{n_k}, z)^p, \end{aligned}$$

which implies that

$$\lim_{k\to+\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

This is a contradict to (5). The proof is completed. □

Lemma 3. *Let (X, d) be a geodesic metric space. If $\{T_i : X \to X\}_{i=1}^m$ is a sequence of p-strongly quasi-nonexpansive maps with $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$, then*

$$\mathrm{Fix}(T_m T_{m-1} \cdots T_1) = \bigcap_{i=1}^m \mathrm{Fix}(T_i).$$

Proof. Using the definition of a p-strongly quasi-nonexpansive map and the similar method in the proof in [26] Lemma 3.3, the proof is clear. □

Using above lemma and same method in [26], we can have the following results.

Lemma 4. *Let (X, d) be a geodesic metric space. If $\{T_i : X \to X\}_{i=1}^m$ is a sequence of p-strongly quasi-nonexpansive maps with $\bigcap_{i=1}^m \mathrm{Fix}(T_i) \neq \emptyset$, then $T := T_m T_{m-1} \cdots T_1$ is also p-strongly quasi-nonexpansive.*

Let (X, d) be a geodesic metric space. Now we recall the notion of convex combinations of two operators. Let T_1 and T_2 be two operators on X. The convex combination of T_1 and T_2 is the operator $T_1 \#_t T_2$ defined by

$$(T_1 \#_t T_2)x := T_1 x \#_t T_2 x.$$

With above setting we have the following result.

Lemma 5. *Let (X, d) be a p-uniformly convex metric space. If T_1 and T_2 are p-strongly quasi-nonexpansive maps with $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \neq \emptyset$ then $T_1 \#_t T_2$ is p-strongly quasi-nonexpansive and $\mathrm{Fix}(T_1 \#_t T_2) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$ for all $t \in (0, 1)$.*

Proof. It is clear that $\mathrm{Fix}(T_1\#_t T_2) \supseteq \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$. Assume that $x \in \mathrm{Fix}(T_1\#_t T_2)$ and fix a point $z \in \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$. Since T_1 and T_2 are quasi-nonexpansive maps, we have

$$d(x,z)^p = d((T_1\#_t T_2)x,z)^p \leq (1-t)d(T_1x,z)^p + td(T_2x,z)^p \leq d(x,z)^p$$

which implies that

$$(1-t)d(T_1x,z)^p + td(T_2x,z)^p = d(x,z)^p = (1-t)d(x,z)^p + td(x,z)^p.$$

Thus we obtain that for all $t \in (0,1)$

$$(1-t)[d(x,z)^p - d(T_1x,z)^p] + t[d(x,z)^p - d(T_2x,z)^p] = 0.$$

Therefore, by p-strongly quasi-nonexpansivity of T_1 and T_2, we have $T_1x = T_2x = x$. So we have $\mathrm{Fix}(T_1\#_t T_2) \subseteq \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$. Now we show that $T_1\#_t T_2$ is p-strongly quasi-nonexpansive. If $\{x_n\}$ is a bounded sequence in X satisfying

$$\lim_{n\to+\infty}[d(x_n,z)^p - d((T_1\#_t T_2)x_n,z)] = 0.$$

for $z \in \mathrm{Fix}(T_1\#_t T_2) = \mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$, then by (3) we have

$$d((T_1\#_t T_2)x_n,z)^p \leq (1-t)d(T_1x_n,z)^p + td(T_2x_n,z)^p \leq d(x_n,z)^p.$$

Thus we have

$$0 \leq (1-t)[d(x_n,z)^p - d(T_1x_n,z)^p] + t[d(x_n,z)^p - d(T_2x_n,z)^p] \leq d(x_n,z)^p - d((T_1\#_t T_2)x_n,z)^p,$$

which implies that

$$\lim_{n\to+\infty}(1-t)[d(x_n,z)^p - d(T_1x_n,z)^p] + t[d(x_n,z)^p - d(T_2x_n,z)^p] = 0.$$

Since T_1 and T_2 are quasi-nonexpansive, we have that

$$\lim_{n\to+\infty} d(x_n,z)^p - d(T_1x_n,z)^p = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(x_n,z)^p - d(T_2x_n,z)^p = 0.$$

Furthemore, by the fact that T_1 and T_2 are p-strongly quasi-nonexpansive, we conclude that

$$\lim_{n\to+\infty} d(x_n,T_1x_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(x_n,T_2x_n) = 0,$$

which implies that

$$\lim_{n\to+\infty} d(x_n,(T_1\#_t T_2)x_n) = 0.$$

since $d((T_1\#_t T_2)x_n,x_n)^p \leq (1-t)d(T_1x_n,x_n)^p + td(T_2x_n,x_n)^p$. Hence $T_1\#_t T_2$ is p-strongly quasi-nonexpansive. □

A map $T : X \to X$ is called Δ-*demiclosed* if for any Δ-convergent sequence $\{x_n\}$ with $\lim_{n\to+\infty} d(Tx_n,x_n) = 0$, its Δ-limit of $\{x_n\}$ belong to $\mathrm{Fix}(T)$.

It is clear that the identity map I on X is Δ-demiclosed.

Example 3. *(i) Every firmly nonexpansive map T on X, (that is,*

$$d(Tx,Ty) \leq d(x\#_t Tx, y\#_t Ty)$$

for all $x, y \in X$ *and* $t \in [0, 1)$*) is* Δ*-demiclosed. (see [9]).*
(ii) Let (X, d) *be a complete* $\mathrm{CAT}(\kappa)$ *space* $(\kappa \geq 0)$ *(with* $\mathrm{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$*, for* $\kappa > 0$*) and* $A \neq \emptyset$ *be a closed convex subset of* X*. Then* P_A *is* Δ*-demiclosed (see [27]).*

Now we prove the convergence of p-strongly quasi-nonexpansive maps on geodesic metric spaces as following:

Theorem 1. *Let* (X, d) *be a complete* p*-uniformly convex metric space and* $T : X \to X$ *be a* Δ*-demiclosed* p*-strongly quasi-nonexpansive map with* $\mathrm{Fix}(T) \neq \emptyset$*. Then* $\{T^n x\}$ Δ*-converges to a point* $z \in \mathrm{Fix}(T)$ *as* $n \longrightarrow +\infty$*.*

Proof. Let $x \in X$ be given. Define the sequence $\{x_n\}$ by

$$x_1 := x, \quad x_{n+1} := T^n x = T x_n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a Fejér monotone sequence w.r.t $\mathrm{Fix}(T)$ since T is quasi-nonexpansive. Thus the sequence $\{d(x_n, z)\}$ is decreasing and bounded in $\mathbb{R}$ for $z \in \mathrm{Fix}(T)$. Therefore $\{d(x_n, z)\}$ converges to a point in $\mathbb{R}$. Thus, we obtain that

$$\lim_{n \to +\infty} d(x_n, z) = \lim_{n \to +\infty} d(T x_n, z),$$

which implies that

$$\lim_{n \to +\infty} d(x_n, T x_n) = 0. \tag{7}$$

Since $\{x_n\}$ is bounded, by Proposition 1, there exists $\{x_{n_k}\} \subseteq \{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to $z \in X$. Since T is a Δ-demiclosed, by combining (7), we have that $z \in \mathrm{Fix}(T)$. By Lemma 1, we obtain that $\{T^n x\}$ Δ-converges to a point $z \in \mathrm{Fix}(T)$ as $n \longrightarrow +\infty$. □

Lemma 6. *Let* (X, d) *be a* $\mathrm{CAT}(\kappa)$ *space with* $\mathrm{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$*. Let* A_1 *and* A_2 *be two closed convex subsets of* X *with* $A_1 \cap A_2 \neq \emptyset$ *and* P_{A_1} *and* P_{A_2} *be corresponding projection operators, respectively. Then* $P_{A_1} \#_t P_{A_2}$ *is also* Δ*-demiclosed for all* $t \in [0, 1]$*.*

Proof. Put $P = P_{A_1} \#_t P_{A_2}$ for $t \in (0, 1)$. Let $\{x_n\}$ be a (bounded) sequence in X and $z \in X$ such that $d(P x_n, x_n) \to 0$ as $n \to +\infty$ and $\{x_n\}$ Δ-converges to z. Note that since for any $q \in A_1 \cap A_2$,

$$0 \leq d(x_n, q) - d(P x_n, q) \leq d(P x_n, x_n)$$

we have

$$d(x_n, q) - d(P x_n, q) \to 0$$

as $n \to +\infty$ which implies that

$$\lim_{n \to +\infty} d(x_n, q)^2 - d(P x_n, q)^2 = 0.$$

Since for any $q \in A_1 \cap A_2$,

$$\begin{aligned} d(P x_n, q)^2 &\leq (1 - t) d(P_{A_1} x_n, q)^2 + t d(P_{A_2} x_n, q)^2 - c_X t (1 - t) d(P_{A_1} x_n, P_{A_2} x_n)^2 \\ &\leq d(x_n, q)^2 - c_X t (1 - t) d(P_{A_1} x_n, P_{A_2} x_n)^2, \end{aligned}$$

we have that

$$\lim_{n \to +\infty} d(P_{A_1} x_n, P_{A_2} x_n) = 0.$$

Thus we have

$$d(P_{A_1}x_n, Px_n) = td(P_{A_1}x_n, P_{A_2}x_n) \to 0$$

as $n \to +\infty$ which implies that $P_{A_1}z = z$ since P_{A_1} is Δ-demiclosed. By similar method, we have $P_{A_2}z = z$. Since

$$\begin{aligned}\limsup_{n\to+\infty} d(Pz, x_n)^2 &\leq \limsup_{n\to+\infty}\left[(1-t)d(P_{A_1}z, x_n) + td(P_{A_1}z, x_n)^2\right] \\ &= \limsup_{n\to+\infty} d(z, x_n)^2,\end{aligned}$$

by uniqueness of Δ-limit, we conclude that $P(z) = z$. The proof is completed. $\square$

Using Lemmas 5 and 6 and Theorem 1, we have the following result

Theorem 2. *Let (X, d) be a $\mathrm{CAT}(\kappa)$ space with $\mathrm{diam}(X) < \frac{\pi}{2\sqrt{\kappa}}$. Let A_1 and A_2 be two closed convex subsets of X with $A_1 \cap A_2 \neq \emptyset$ and P_{A_1} and P_{A_2} be corresponding (metric) projections, respectively. Then for all $x \in X$ and $t \in (0, 1)$, there exists a point $z \in A_1 \cap A_2$ such that $\left(P_{A_1}\#_t P_{A_2}\right)^n x$ Δ-converges to z as $n \longrightarrow +\infty$.*

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