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On Sequential Fractional q-Hahn Integrodifference Equations

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Received: 10 April 2020; Accepted: 6 May 2020; Published: 9 May 2020

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Abstract: In this paper, we prove existence and uniqueness results for a fractional sequential fractional q-Hahn integrodifference equation with nonlocal mixed fractional q and fractional Hahn integral boundary condition, which is a new idea that studies q and Hahn calculus simultaneously.

Keywords: fractional q-calculus; fractional Hahn calculus; fractional integral boundary value problems; existence

MSC: 39A10; 39A13; 39A70

1. Introduction

A q-difference operator D_q is an important tool in areas of mathematics and applications [1–4] such as orthogonal polynomials problems and mathematical control theories. Basic definitions and properties for q-difference calculus were presented by Kac and Cheung [5], Al-Salam [6], Agarwal [7], and Annaby and Mansour [8]. There are many research works widely studying the q-difference operators (see [9–23]).

A Hahn difference operator $D_{q,\omega}$ arose from the forward difference operator and the Jackson q-difference operator was introduced by Hahn [24] in 1949. Then, the right inverse of $D_{q,\omega}$ presented in terms of Jackson q-integral and Nörlund sum was proposed by Aldwoah [25,26] in 2009. The Hahn difference operator can be used in studied of families of orthogonal polynomials and approximation problems (see [27–29]). More research works about Hahn difference calculus can be found in [30–39].

The fractional Hahn difference operators was introduced by Brikshavana and Sitthiwirattham [40] in 2017, and Wang et al. [41] in 2018. The extension of this operator has been used in the study of existence results of solution of boundary value problems [42–45], a generalization of Minkowski's inequality [46], and impulsive fractional quantum Hahn operator [47,48].

From the literature, we have found that the study of fractional q-difference and fractional Hahn difference operators simultaneously have not been studied. Therefore, in this article, we devote ourselves to study the boundary value problem for equations that contain both fractional q-difference and Hahn difference operators. Our problem is a nonlocal mixed fractional q and Hahn integral boundary value problem for sequential fractional q-Hahn integrodifference equation of the form

$$
\begin{aligned}
&D_q^{\alpha} D_{q,\omega}^{\beta} u(t) = F\left[t, u(t), \Psi_q^{\theta} u(t), \Upsilon_{q,\omega}^{\phi} u(t)\right], \quad t \in I_q^{I_{q,\omega}^{[\omega_0,T]}}, \\
&u(\eta) = \lambda \mathcal{I}_{q,\omega}^{\gamma} u(\eta), \quad \eta \in (\omega_0, T), \\
&u(T) = \mu \mathcal{I}_q^{\gamma} u(T),
\end{aligned}
\tag{1}
$$

where $I_q^{I_{q,\omega}^{[\omega_0,T]}} := \bigcup_{k=0}^{\infty} I_q^{q^k s + \omega[k]_q}, s \in [\omega_0, T]$; $I_q^x := \{q^n x : n \in \mathbb{N}_0\} \cup \{0\}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $0 < q < 1$; $\omega > 0$; $T > \omega_0$; $\alpha, \beta, \gamma, \theta, \phi \in (0,1]$; $\alpha + \beta \in (1,2]$; $\lambda, \mu \in \mathbb{R}^+$; $F \in C\big([0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\big)$ is given function; and for $\psi \in C\big([0,T] \times [\omega_0, T], [0,\infty)\big)$, $\varphi \in C\big([\omega_0, T] \times [\omega_0, T], [0,\infty)\big)$, we define

$$
\begin{aligned}
\Psi_q^{\theta} u(t) &:= \left(\mathcal{I}_q^{\theta} \psi\, u\right)(t) = \frac{1}{\Gamma_{q,\omega}(\theta)} \int_0^t \left(t - \sigma_q(s)\right)_q^{\underline{\theta-1}} \psi(t,s)\, u(s)\, d_q s, \\
\Psi_{q,\omega}^{\phi} u(t) &:= \left(\mathcal{I}_{q,\omega}^{\phi} \varphi\, u\right)(t) = \frac{1}{\Gamma_{q,\omega}(\phi)} \int_{\omega_0}^t \left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\phi-1}} \varphi(t,s)\, u(s)\, d_{q,\omega} s.
\end{aligned}
$$

This paper is organized as follows. In Section 2, we provide some definitions and lemmas for q-difference and Hahn difference operators. In Section 3, we prove the existence and uniqueness of a solution to problem (1) by using the Banach fixed point theorem. In the last section, we give an example to illustrate our results.

2. Preliminaries

In this section, we recall the notations, definitions, and lemmas for q and Hahn difference calculus. For $q \in (0,1)$, $\omega > 0$, we define

$$
[n]_q := \frac{1-q^n}{1-q} = q^{n-1} + \ldots + q + 1 \qquad \text{and} \qquad [n]_q! := \prod_{k=1}^{n} \frac{1-q^k}{1-q}, \qquad n \in \mathbb{N}.
$$

The q-analogue of the power function $(a-b)_q^{\underline{n}}$ with $n \in \mathbb{N}_0$ is given by

$$
(a-b)_q^{\underline{0}} := 1, \qquad (a-b)_q^{\underline{n}} := \prod_{k=0}^{n-1} (a - bq^k), \qquad a, b \in \mathbb{R}.
$$

The q,ω-analogue of the power function $(a-b)_{q,\omega}^{\underline{n}}$ with $n \in \mathbb{N}_0$ is given by

$$
(a-b)_{q,\omega}^{\underline{0}} := 1, \qquad (a-b)_{q,\omega}^{\underline{n}} := \prod_{k=0}^{n-1} \left[a - (bq^k + \omega[k]_q)\right], \qquad a, b \in \mathbb{R}.
$$

For $\alpha \in \mathbb{R}$, the power function is given by

$$
(a-b)_q^{\underline{\alpha}} = a^{\alpha} \prod_{n=0}^{\infty} \frac{1 - \left(\frac{b}{a}\right) q^n}{1 - \left(\frac{b}{a}\right) q^{\alpha+n}}, \; a \neq 0,
$$

$$
(a-b)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^{\alpha} \prod_{n=0}^{\infty} \frac{1 - \left(\frac{b-\omega_0}{a-\omega_0}\right) q^n}{1 - \left(\frac{b-\omega_0}{a-\omega_0}\right) q^{\alpha+n}} = \Big((a-\omega_0) - (b-\omega_0)\Big)_q^{\underline{\alpha}}, \; a \neq \omega_0.
$$

We let the notations, $a_q^{\underline{\alpha}} = a^{\alpha}$, $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^{\alpha}$, and $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$.

The q-gamma and q-beta functions are defined by

$$\begin{aligned}\Gamma_q(x) &:= \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \qquad x \in \mathbb{R}\setminus\{0,-1,-2,...\},\\ B_q(x,s) &:= \int_0^1 t^{x-1}(1-qt)_q^{\underline{s-1}} d_q t = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)},\end{aligned}$$

respectively.

For $k\in\mathbb{N}$, the q-analogue and q,ω-analogue of forward jump operator are defined by

$$\sigma_q^k(t) := q^k t \text{ and } \sigma_{q,\omega}^k(t) := q^k t + \omega[k]_q,$$

respectively. The q-analogue and q,ω-analogue of backward jump operator are defined by

$$\rho_q^k(t) := \frac{t}{q^k}, \text{ and } \rho_{q,\omega}^k(t) := \frac{t-\omega[k]_q}{q^k},$$

respectively.

Definition 1. *For $q\in(0,1)$, the q-differnce of a real function f is defined by*

$$D_q f(t) = \frac{f(t)-f(qt)}{(1-q)t}, \ t\neq 0 \quad \textit{and} \quad D_q f(0) = \lim_{t\to 0} D_q f(t).$$

Let f be a function defined on the interval $[0,T]$. q-integral is defined by

$$\mathcal{I}_q f(t) = \int_0^t f(s)\, d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$$

where the infinite series is convergent.

Definition 2. *For $q\in(0,1)$, $\omega>0$ and f defined on an interval $I\subseteq\mathbb{R}$ which contains $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by*

$$D_{q,\omega} f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega} \quad \textit{for} \ \ t\neq\omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$.

For $a,b\in I\subseteq\mathbb{R}$ with $a<\omega_0<b$ and $[k]_q = \frac{1-q^k}{1-q}$, $k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, the q,ω-interval is defined by

$$\begin{aligned}[a,b]_{q,\omega} &:= \left\{q^k a + \omega[k]_q : k\in\mathbb{N}_0\right\}\cup\left\{q^k b+\omega[k]_q : k\in\mathbb{N}_0\right\}\cup\{\omega_0\}\\ &= [a,\omega_0]_{q,\omega}\cup[\omega_0,b]_{q,\omega}\\ &= (a,b)_{q,\omega}\cup\{a,b\} \ = \ [a,b)_{q,\omega}\cup\{b\} \ = \ (a,b]_{q,\omega}\cup\{a\}.\end{aligned}$$

We note that, for each $s\in[a,b]_{q,\omega}$, the sequence $\{\sigma_{q,\omega}^k(s)\}_{k=0}^{\infty} = \{q^k s+\omega[k]_q\}_{k=0}^{\infty}$ is uniformly convergent to ω_0.

Definition 3. *Let I be any closed interval of $\mathbb{R}$ that contains a,b and ω_0. Letting $f: I\to\mathbb{R}$ be a given function, q,ω-integral of f from a to b is defined by*

$$\int_a^b f(t) d_{q,\omega}t := \int_{\omega_0}^b f(t) d_{q,\omega}t - \int_{\omega_0}^a f(t) d_{q,\omega}t$$

where $\displaystyle\int_{\omega_0}^{x} f(t)d_{q,\omega}t := \big[x(1-q)-\omega\big]\sum_{k=0}^{\infty} q^k f\left(xq^k+\omega[k]_q\right)$, $x \in I$, *and the series converges at* $x = a$ *and* $x = b$ *where the sum of the right-hand side is called the Jackson–Nörlund sum.*

Note that the actual domain of function f is defined on $[a,b]_{q,\omega} \subset I$.

The following fractional q integral, fractional Hahn integral, fractional q difference, and fractional Hahn difference of Riemann–Liouville type are defined.

Definition 4. *Let f be defined on $[0,T]$ and $\alpha \geq 0$, the fractional q-integral of the Riemann–Liouville type is defined by*

$$\begin{aligned}
(\mathcal{I}_q^{\alpha} f)(t) &:= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\underline{\alpha-1}} f(t)d_q s \\
&= \frac{t\,(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(t-q^{n+1}t\right)_q^{\underline{\alpha-1}} f(q^n t) \\
&= \frac{t^{\alpha}(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\,(1-q^{n+1})_q^{\underline{\alpha-1}} f(q^n t),
\end{aligned}$$

and $(\mathcal{I}_q^0 f)(x) = f(x)$.

Definition 5. *Let f be defined on $[\omega_0, T]_{q,\omega}$ and $\alpha, \omega > 0$, $q \in (0,1)$, and the fractional Hahn integral, is defined by*

$$\begin{aligned}
\mathcal{I}_{q,\omega}^{\alpha} f(t) &:= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^t \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} f(s)d_{q,\omega}s \\
&= \frac{[t\,(1-q)-\omega]}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(t-\sigma_{q,\omega}^{n+1}(t)\right)_{q,\omega}^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^{n}(t)\right) \\
&= \frac{(1-q)(t-\omega_0)^{\alpha}}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(1-q^{n+1}\right)_q^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^{n}(t)\right),
\end{aligned}$$

and $(\mathcal{I}_{q,\omega}^0 f)(t) = f(t)$.

Definition 6. *Let f be defined on $[0,T]$ and $\alpha \geq 0$, the fractional q-derivative of the Riemann–Liouville type of order α, is defined by*

$$\begin{aligned}
(D_q^{\alpha} f)(t) &:= (D_q^N \mathcal{I}_q^{N-\alpha} f)(t) \\
&= \frac{1}{\Gamma_q(-\alpha)}\int_0^t \left(t-\sigma_q(s)\right)_q^{\underline{-\alpha-1}} f(s)d_q s,
\end{aligned}$$

and $(D_q^0 f)(x) = f(x)$, *where N is the smallest integer that is greater than or equal to α.*

Definition 7. *Let f be defined on $[\omega_0, T]_{q,\omega}$ and $\alpha, \omega > 0$, $q \in (0,1)$, the fractional Hahn difference of the Riemann–Liouville type of order α is defined by*

$$\begin{aligned}
D_{q,\omega}^{\alpha} f(t) &:= (D_{q,\omega}^N \mathcal{I}_{q,\omega}^{N-\alpha} f)(t) \\
&= \frac{1}{\Gamma_q(-\alpha)}\int_{\omega_0}^t \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{-\alpha-1}} f(s)d_{q,\omega}s,
\end{aligned}$$

and $D_{q,\omega}^0 f(t) = f(t)$, *where N is the smallest integer that is greater than or equal to α.*

Lemma 1 ([10]). *Letting* $\alpha > 0, q \in (0,1)$ *and* $f : I_q^T \to \mathbb{R}$,

$$\mathcal{I}_q^{\alpha} D_q^{\alpha} f(t) = f(t) + C_1 t^{\alpha-1} + ... + C_N t^{\alpha-N},$$

for some $C_i \in \mathbb{R}, i = \{1, 2, ..., N\}$ *and* $N - 1 < \alpha \leq N, N \in \mathbb{N}$.

Lemma 2 ([40]). *Letting* $\alpha > 0, q \in (0,1), \omega > 0$ *and* $f : I_{q,\omega}^T \to \mathbb{R}$,

$$\mathcal{I}_{q,\omega}^{\alpha} D_{q,\omega}^{\alpha} f(t) = f(t) + C_1 (t - \omega_0)^{\alpha-1} + ... + C_N (t - \omega_0)^{\alpha-N},$$

for some $C_i \in \mathbb{R}, i = \{1, 2, ..., N\}$ *and* $N - 1 < \alpha \leq N, N \in \mathbb{N}$.

Some auxiliary lemmas used to investigate the solution of the linear variant of (1) are provided as follows.

Lemma 3 ([16]). *Let* $\alpha, \beta \geq 0$ *and* $p, q \in (0,1)$. *Then, the following formulas hold:*

$$\int_0^{\eta} (\eta - qt)_q^{\underline{\alpha-1}} t^{\beta} d_q t = \eta^{\alpha+\beta} B_q(\beta + 1, \alpha),$$
$$\int_0^{\eta} \int_0^{s} (\eta - ps)_p^{\underline{\alpha-1}} (s - qt)_q^{\underline{\beta-1}} d_q t d_p s = \frac{\eta^{\alpha+\beta}}{[\beta]_q} B_q(\beta + 1, \alpha).$$

Lemma 4 ([40]). *Letting* $\alpha, \beta > 0$, $p, q \in (0,1)$ *and* $\omega > 0$,

$$\begin{aligned} \int_{\omega_0}^{t} \Big(t - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\alpha-1}} (s - \omega_0)^{\beta} \, d_{q,\omega} s &= (t - \omega_0)^{\alpha+\beta} B_q(\beta + 1, \alpha), \\ \int_{\omega_0}^{t} \int_{\omega_0}^{x} \Big(t - \sigma_{p,\omega}(x)\Big)_{p,\omega}^{\underline{\alpha-1}} \Big(x - \sigma_{q,\omega}(s)\Big)_{q,\omega}^{\underline{\beta-1}} d_{q,\omega} s \, d_{p,\omega} x &= \frac{(t - \omega_0)^{\alpha+\beta}}{[\beta]_q} B_q(\beta + 1, \alpha). \end{aligned}$$

Employing Lemmas 3 and 4, we obtain the solution of the linear variant of problem (1) as shown in the following lemma.

Lemma 5. *Let* $\alpha, \beta, \gamma \in (0,1]$, $\alpha + \beta \in (1,2]$; $0 < q < 1$; $\omega > 0$; $T > \omega_0$; $\lambda, \mu \in \mathbb{R}^+$; $h \in C\big([0,T], \mathbb{R}\big)$ *be a given function. Then, the linear variant problem*

$$\begin{aligned} & D_q^{\alpha} D_{q,\omega}^{\beta} u(t) = h(t), \quad t \in I_q^{I_{q,\omega}^{[\omega_0,T]}}, \\ & u(\eta) = \lambda \mathcal{I}_{q,\omega}^{\gamma} u(\eta), \quad \eta \in (\omega_0, T), \\ & u(T) = \mu \mathcal{I}_q^{\gamma} u(T) \end{aligned} \tag{2}$$

has the unique solution which is in a form

$$\begin{aligned} u(t) \; &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{t} \int_0^{x} \big(t - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\beta-1}} \big(x - \sigma_q(s)\big)_q^{\underline{\alpha-1}} \, h(s) \, d_q s \, d_{q,\omega} x \\ & + \Big\{ \boldsymbol{A}_T \mathcal{O}_{\eta}[h] - \boldsymbol{A}_{\eta} \mathcal{O}_T[h] \Big\} \frac{1}{\Omega \Gamma_q(\beta)} \int_{\omega_0}^{t} \big(t - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1} d_{q,\omega} s \\ & - \Big\{ \boldsymbol{B}_T \mathcal{O}_{\eta}[h] - \boldsymbol{B}_{\eta} \mathcal{O}_T[h] \Big\} \frac{(t - \omega_0)^{\beta-1}}{\Omega} \end{aligned} \tag{3}$$

for $t \in [\omega_0, T]$, *where the functionals* $\mathcal{O}_{\eta}[h]$ *and* $\mathcal{O}_T[h]$ *are defined by*

$$\begin{aligned}
\mathcal{O}_\eta[h] := & -\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{\eta}\int_0^x \left(\eta-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s d_{q,\omega} x \\
& + \frac{\lambda}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{r}\int_0^x \left(\eta-\sigma_{q,\omega}(r)\right)_{q,\omega}^{\underline{\gamma-1}}\left(r-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}} \times \\
& \left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s\, d_{q,\omega} x\, d_{q,\omega} r,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathcal{O}_T[h] := & -\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{T}\int_0^x \left(T-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s d_{q,\omega} x \\
& + \frac{\mu}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{0}^{T}\int_{\omega_0}^{r}\int_0^x \left(T-\sigma_{q}(r)\right)_{q}^{\underline{\gamma-1}}\left(r-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}} \times \\
& \left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s\, d_{q,\omega} x\, d_{q} r,
\end{aligned} \tag{5}$$

and the constants $\boldsymbol{A}_\eta, \boldsymbol{A}_T, \boldsymbol{B}_\eta, \boldsymbol{B}_T, \Omega$ *are defined by*

$$\boldsymbol{A}_\eta := (\eta-\omega_0)^{\beta-1} - \frac{\lambda}{\Gamma_q(\gamma)}\int_{\omega_0}^{T}\left(\eta-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\gamma-1}}(s-\omega_0)^{\beta-1}\, d_{q,\omega} s, \tag{6}$$

$$\boldsymbol{A}_T := (T-\omega_0)^{\beta-1} - \frac{\mu}{\Gamma_q(\gamma)}\int_{0}^{T}\left(T-\sigma_{q}(s)\right)_{q}^{\underline{\gamma-1}}(s-\omega_0)^{\beta-1}\, d_{q} s, \tag{7}$$

$$\begin{aligned}
\boldsymbol{B}_\eta := & \frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{\eta}\left(\eta-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega} s \\
& - \frac{\lambda}{\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{x}\left(\eta-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\gamma-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega} s d_{q,\omega} x,
\end{aligned} \tag{8}$$

$$\begin{aligned}
\boldsymbol{B}_T := & \frac{1}{\Gamma_q(\beta)}\int_{\omega_0}^{T}\left(T-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega} s \\
& - \frac{\mu}{\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{0}^{T}\int_{\omega_0}^{x}\left(T-\sigma_{q}(x)\right)_{q}^{\underline{\gamma-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega} s d_{q} x,
\end{aligned} \tag{9}$$

$$\Omega := \boldsymbol{A}_T\boldsymbol{B}_\eta - \boldsymbol{A}_\eta\boldsymbol{B}_T \neq 0. \tag{10}$$

Proof. Firstly, we take fractional q-integral of order α for (2). Then, we have

$$\begin{aligned}
D_{q,\omega}^{\beta} u(t) &= C_0 t^{\alpha-1} + \frac{(1-q)t^\alpha}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty} q^k\left(1-q^{k+1}\right)_q^{\underline{\alpha-1}} h\left(\sigma_q^k(t)\right) \\
&= C_0 t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)}\int_0^t \left(t-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(x)\, d_q s,
\end{aligned} \tag{11}$$

for $t \in I_{q,\omega}^{[\omega_0,T]} := \{q^n s + \omega[n]_q : s \in [\omega_0, T], n \in \mathbb{N}_0\} \cup \{\omega_0\}$.

Taking fractional Hahn integral of order β for (11), we obtain

$$
\begin{aligned}
u(t) = \; & C_1(t-\omega_0)^{\beta-1} + \frac{C_0}{\Gamma_q(\beta)}(1-q)(t-\omega_0)^{\beta}\sum_{k=0}^{\infty} q^k \left(1-q^{k+1}\right)_q^{\underline{\beta-1}} \left(\sigma_{q,\omega}^k(t)\right)^{\alpha-1} \\
& + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}(1-q)^2(t-\omega_0)^{\beta}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} q^{h+k}\left(1-q^{h+1}\right)_q^{\underline{\beta-1}} \times \\
& \left(1-q^{k+1}\right)_q^{\underline{\alpha-1}} \left(\sigma_{q,\omega}^h(t)\right)^{\alpha} h\left(\sigma_q^k\left(\sigma_{q,\omega}^h(t)\right)\right) \\
= \; & C_1(t-\omega_0)^{\beta-1} + \frac{C_0}{\Gamma_q(\beta)}\int_{\omega_0}^{t}\left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega}s \\
& + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{t}\int_{0}^{x}\left(t-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}}\left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s d_{q,\omega}x,
\end{aligned} \tag{12}
$$

for $t \in [\omega_0, T]$.

Taking fractional q-integral of order γ for (12), we have

$$
\begin{aligned}
\mathcal{I}_q^{\gamma} u(t) = \; & \frac{C_1}{\Gamma_q(\gamma)}\int_0^t \left(t-\sigma_q(s)\right)_q^{\underline{\gamma-1}}(s-\omega_0)^{\beta-1}\, d_q s \\
& + \frac{C_0}{\Gamma_q(\beta)\Gamma_q(\gamma)}\int_0^t\int_{\omega_0}^{x}\left(t-\sigma_q(x)\right)_q^{\underline{\gamma-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega}s\, d_q x, \\
& + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)}\int_0^t\int_{\omega_0}^{r}\int_0^x \left(t-\sigma_q(r)\right)_q^{\underline{\gamma-1}}\left(r-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}} \times \\
& \left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s\, d_{q,\omega}x\, d_q r,
\end{aligned} \tag{13}
$$

for $t \in [0, T]$.

In addition, we take fractional Hahn integral of order γ for (12) to get

$$
\begin{aligned}
\mathcal{I}_{q,\omega}^{\gamma} u(t) = \; & \frac{C_1}{\Gamma_q(\gamma)}\int_{\omega_0}^t \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\gamma-1}}(s-\omega_0)^{\beta-1}\, d_{q,\omega} s \\
& + \frac{C_0}{\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{\omega_0}^t\int_{\omega_0}^{x}\left(t-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\gamma-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1}\, d_{q,\omega}s\, d_{q,\omega} x, \\
& + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{\omega_0}^t\int_{\omega_0}^{r}\int_0^x \left(t-\sigma_{q,\omega}(r)\right)_{q,\omega}^{\underline{\gamma-1}}\left(r-\sigma_{q,\omega}(x)\right)_{q,\omega}^{\underline{\beta-1}} \times \\
& \left(x-\sigma_q(s)\right)_q^{\underline{\alpha-1}} h(s)\, d_q s\, d_{q,\omega}x\, d_{q,\omega} r,
\end{aligned} \tag{14}
$$

for $t \in [\omega_0, T]$.

Substituting $t = \eta$ into (12) and (14), and employing the first condition of (2), we have

$$
\mathbf{A}_{\eta} C_1 + \mathbf{B}_{\eta} C_0 = \mathcal{O}_{\eta}[h]. \tag{15}
$$

Taking $t = T$ into (12) and (13), and employing the second condition of (2), we have

$$
\mathbf{A}_{T} C_1 + \mathbf{B}_{T} C_0 = \mathcal{O}_{T}[h]. \tag{16}
$$

Solving Equations (15) and (16), we obtain

$$
C_1 = \frac{\mathbf{B}_{\eta}\mathcal{O}_T[h] - \mathbf{B}_T\mathcal{O}_{\eta}[h]}{\Omega} \quad \text{and} \quad C_0 = \frac{\mathbf{A}_T\mathcal{O}_{\eta}[h] - \mathbf{A}_{\eta}\mathcal{O}_T[h]}{\Omega}.
$$

where $\mathcal{O}_{\eta}[h], \mathcal{O}_T[h], \mathbf{A}_{\eta}, \mathbf{A}_T, \mathbf{B}_{\eta}, \mathbf{B}_T$ and Ω are defined by Equations (4)–(10).

Substituting C_0 and C_1 into (12), we obtain the solution (3). □

3. Existence Results

In this section, the existence and uniqueness result for the mixed q-Hahn problem (1) is studied. Let $\mathcal{C} = C([\omega_0, T], \mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \max_{t\in[\omega_0,T]} |u(t)|.$$

The operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{F}u)(t) := & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{t} \int_0^x (t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}} (x-\sigma_q(s))_q^{\underline{\alpha-1}} \times \\ & F\left[s, u(s), \Psi_q^{\theta} u(s), \Upsilon_{q,\omega}^{\phi} u(s)\right] d_q s \, d_{q,\omega} x \\ & + \left\{\mathbf{A}_T \mathcal{O}_\eta[F_u] - \mathbf{A}_\eta \mathcal{O}_T[F_u]\right\} \frac{1}{\Omega\Gamma_q(\beta)} \int_{\omega_0}^{t} (t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}} s^{\alpha-1} d_{q,\omega} s \\ & - \left\{\mathbf{B}_T \mathcal{O}_\eta[F_u] - \mathbf{B}_\eta \mathcal{O}_T[F_u]\right\} \frac{(t-\omega_0)^{\beta-1}}{\Omega} \end{aligned} \tag{17}$$

where the functionals $\mathcal{O}_\eta[F_u]$, $\mathcal{O}_T[F_u]$ are defined by

$$\begin{aligned}\mathcal{O}_\eta[F_u] := & -\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{\eta} \int_0^x (\eta-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x-\sigma_q(s))_q^{\underline{\alpha-1}} \times \\ & F\left[s, u(s), \Psi_q^{\theta} u(s), \Upsilon_{q,\omega}^{\phi} u(s)\right] d_q s d_{q,\omega} x \\ & + \frac{\lambda}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)} \int_{\omega_0}^{\eta} \int_{\omega_0}^{r} \int_0^x (\eta-\sigma_{q,\omega}(r))_{q,\omega}^{\underline{\gamma-1}} (r-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} \times \\ & (x-\sigma_q(s))_q^{\underline{\alpha-1}} F\left[s, u(s), \Psi_q^{\theta} u(s), \Upsilon_{q,\omega}^{\phi} u(s)\right] d_q s \, d_{q,\omega} x \, d_{q,\omega} r, \end{aligned} \tag{18}$$

$$\begin{aligned}\mathcal{O}_T[F_u] := & -\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T} \int_0^x (T-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x-\sigma_q(s))_q^{\underline{\alpha-1}} \times \\ & F\left[s, u(s), \Psi_q^{\theta} u(s), \Upsilon_{q,\omega}^{\phi} u(s)\right] d_q s d_{q,\omega} x \\ & + \frac{\mu}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)} \int_0^{T} \int_{\omega_0}^{r} \int_0^x (T-\sigma_q(r))_q^{\underline{\gamma-1}} (r-\sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} \times \\ & (x-\sigma_q(s))_q^{\underline{\alpha-1}} F\left[s, u(s), \Psi_q^{\theta} u(s), \Upsilon_{q,\omega}^{\phi} u(s)\right] d_q s \, d_{q,\omega} x \, d_q r, \end{aligned} \tag{19}$$

and the constants $\mathbf{A}_\eta, \mathbf{A}_T, \mathbf{B}_\eta, \mathbf{B}_T, \Omega$ are defined by (6)–(10), respectively.

The problem (1) has solution if and only if the operator $\mathcal{F}$ has fixed point. We show the proof in the following theorem.

Theorem 1. *Assume that $F : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\psi : [0,T] \times [\omega_0, T] \to [0,\infty)$ and $\varphi : [\omega_0, T] \times [\omega_0, T] \to [0,\infty)$ are continuous with $\psi_0 = \max\{\psi(t,s) : (t,s) \in [0,T] \times [\omega_0,T]\}$ and $\varphi_0 = \max\{\varphi(t,s) : (t,s) \in [\omega_0,T] \times [\omega_0,T]\}$. In addition, suppose that the following conditions hold:*

(H_1) *There exist constants $\ell_1, \ell_2, \ell_3 > 0$ such that for each $t \in [0,T]$ and $u, v \in \mathbb{R}$,*

$$\begin{aligned} & \left| F\left[t, u, \Psi_q^{\theta} u, \Upsilon_{q,\omega}^{\phi} u\right] - F\left[t, v, \Psi_q^{\theta} v, \Upsilon_{q,\omega}^{\phi} v\right] \right| \\ \leq \; & \ell_1 |u-v| + \ell_2 \left|\Psi_q^{\theta} u - \Psi_q^{\theta} v\right| + \ell_3 \left|\Upsilon_{q,\omega}^{\phi} u - \Upsilon_{q,\omega}^{\phi} v\right|. \end{aligned}$$

(H_2) $\mathcal{L}\,\Xi < 1$,

where

$$\mathcal{L} := \ell_1 + \ell_2\psi_0 \frac{T^{\theta}}{\Gamma_q(\theta+1)} + \ell_3\varphi_0 \frac{(T-\omega_0)^{\phi}}{\Gamma_q(\phi+1)}, \tag{20}$$

$$\Xi := \frac{T^{\alpha}(T-\omega_0)^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} + \Phi_1\Theta_T + \Phi_2\Theta_{\eta}, \tag{21}$$

$$\Phi_1 := \frac{\eta^{\alpha}(\eta-\omega_0)^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} \left|1 - \frac{\lambda(\eta-\omega_0)^{\gamma}}{\Gamma_q(\gamma+1)}\right|, \tag{22}$$

$$\Phi_2 := \frac{T^{\alpha}(T-\omega_0)^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} \left|1 - \frac{\mu T^{\gamma}}{\Gamma_q(\gamma+1)}\right|, \tag{23}$$

$$\Theta_T := \frac{1}{|\Omega|}\left\{|\boldsymbol{A}_T|\frac{T^{\alpha-1}(T-\omega_0)^{\beta}}{\Gamma_q(\beta+1)} + |\boldsymbol{B}_T|(T-\omega_0)^{\beta-1}\right\}, \tag{24}$$

$$\Theta_{\eta} := \frac{1}{|\Omega|}\left\{|\boldsymbol{A}_{\eta}|\frac{T^{\alpha-1}(T-\omega_0)^{\beta}}{\Gamma_q(\beta+1)} + |\boldsymbol{B}_{\eta}|(T-\omega_0)^{\beta-1}\right\}. \tag{25}$$

Then, problem (1) *has a unique solution.*

Proof. Firstly, we verify $\mathcal{F}$ map bounded sets into bounded sets in $B_L = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \le L\}$. Let $K = \max_{t \in I^T_{q,\omega}} |F(t,0,0,0)|$, L be a constant satisfied with

$$L \ge \frac{K\,\Xi}{1-\mathcal{L}\,\Xi}, \tag{26}$$

and the notation $|\mathcal{S}(t,u,0)| = \left|F\left[t,u,\Psi^{\theta}_q u, \Upsilon^{\phi}_{q,\omega}u\right] - F[t,0,0,0]\right| + |F[t,0,0,0]|$.

For each $t \in [0,T]$ and $u \in B_L$

$$\begin{aligned} \left|\mathcal{O}_{\eta}[F_u]\right| \le & \left|\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{\eta}\int_0^x (\eta-\sigma_{q,\omega}(x))^{\underline{\beta-1}}_{q,\omega}(x-\sigma_q(s))^{\underline{\alpha-1}}_{q}|\mathcal{S}(s,u,0)|\,d_q s d_{q,\omega}x \right.\\ & - \frac{\lambda}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)}\int_{\omega_0}^{\eta}\int_{\omega_0}^{r}\int_0^x (\eta-\sigma_{q,\omega}(r))^{\underline{\gamma-1}}_{q,\omega}(r-\sigma_{q,\omega}(x))^{\underline{\beta-1}}_{q,\omega}\times \\ & \left.(x-\sigma_q(s))^{\underline{\alpha-1}}_{q}|\mathcal{S}(s,u,0)|\,d_q s\,d_{q,\omega}x\,d_{q,\omega}r\right| \\ \le & \ [\mathcal{L}\|u\|_{\mathcal{C}} + K]\,\Phi_1 \\ \le & \ [\mathcal{L}\,L + K]\,\Phi_1. \end{aligned} \tag{27}$$

Similary,

$$\left|\mathcal{O}_T[F_u]\right| \le [\mathcal{L}\,L + K]\,\Phi_2. \tag{28}$$

From (27) and (28), we find that

$$\begin{aligned} |(\mathcal{F}u)(t)| \le & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_{\omega_0}^{T}\int_0^x (T-\sigma_{q,\omega}(s))^{\underline{\beta-1}}_{q,\omega}(x-\sigma_q(s))^{\underline{\alpha-1}}_{q}|\mathcal{S}(s,u,0)|\,d_q s\,d_{q,\omega}x \\ & + \left\{|\mathbf{A}_T|\,|\mathcal{O}_{\eta}[F_u]| + |\mathbf{A}_{\eta}|\,|\mathcal{O}_T[F_u]|\right\}\frac{1}{\Omega\Gamma_q(\beta)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))^{\underline{\beta-1}}_{q,\omega}s^{\alpha-1}d_{q,\omega}s \\ & + \left\{|\mathbf{B}_T|\,|\mathcal{O}_{\eta}[F_u]| + |\mathbf{B}_{\eta}|\,|\mathcal{O}_T[F_u]|\right\}\frac{(T-\omega_0)^{\beta-1}}{\Omega} \\ \le & \ \Xi\,[\mathcal{L}\,L + K] \\ \le & \ L. \end{aligned} \tag{29}$$

Therefore, we obtain $\|\mathcal{F}u\|_{\mathcal{C}} \leq L$,which implies that $\mathcal{F}B_L \subset B_L$.

Next, we aim to prove that $\mathcal{F}$ is contraction. Let the notation

$$\mathcal{H}|u-v|(t) = \left| F\left[t, u(t), \Psi_q^{\theta}u(t), \Upsilon_{q,\omega}^{\phi}u(t)\right] - F\left[t, v(t), \Psi_q^{\theta}v(t), \Upsilon_{q,\omega}^{\phi}v(t)\right] \right|,$$

for each $t \in [0, T]$ and $u, v \in \mathcal{C}$. From (18), we find that

$$\begin{aligned}
&\left|\mathcal{O}_\eta[F_u] - \mathcal{O}_\eta[F_v]\right| \\
\leq\ & \left| \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{\eta} \int_0^x (\eta - \sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} (x - \sigma_q(s))_q^{\underline{\alpha-1}} \mathcal{H}|u-v|(s)\, d_q s d_{q,\omega} x \right. \\
& - \frac{\lambda}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)} \int_{\omega_0}^{\eta} \int_{\omega_0}^{r} \int_0^x (\eta - \sigma_{q,\omega}(r))_{q,\omega}^{\underline{\gamma-1}} (r - \sigma_{q,\omega}(x))_{q,\omega}^{\underline{\beta-1}} \times \\
& \left. (x - \sigma_q(s))_q^{\underline{\alpha-1}}\, \mathcal{H}|u-v|(s)\, d_q s\, d_{q,\omega} x\, d_{q,\omega} r \right| \\
\leq\ & \left(\ell_1|u-v| + \ell_2\left|\Psi_q^{\theta}u - \Psi_q^{\theta}v\right| + \ell_3\left|\Upsilon_{q,\omega}^{\phi}u - \Upsilon_{q,\omega}^{\phi}v\right|\right) \times \\
& \left| \frac{\eta^{\alpha}(\eta-\omega_0)^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} - \frac{\lambda\eta^{\alpha}(\eta-\omega_0)^{\beta+\gamma}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)\Gamma_q(\gamma+1)} \right| \\
\leq\ & \left(\ell_1 + \ell_2\psi_0 \frac{T^{\theta}}{\Gamma_q(\theta+1)} + \ell_3\varphi_0 \frac{(T-\omega_0)^{\phi}}{\Gamma_q(\phi+1)} \right) |u-v|\, \Phi_1 \\
\leq\ & \mathcal{L}\,\Phi_1 \|u-v\|_{\mathcal{C}}.
\end{aligned}$$

Similary, from (19), we have

$$\left|\mathcal{O}_T[F_u] - \mathcal{O}_T[F_v]\right| \leq \mathcal{L}\,\Phi_2\|u-v\|_{\mathcal{C}}.$$

Next, we find that

$$\begin{aligned}
&\left|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\right| \\
\leq\ & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T} \int_0^x (T - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}} (x - \sigma_q(s))_q^{\underline{\alpha-1}} \mathcal{H}|u-v|(s)\, d_q s\, d_{q,\omega} x \\
& + \left\{ |\mathbf{A}_T|\left|\mathcal{O}_\eta[F_u] - \mathcal{O}_\eta[F_v]\right| + |\mathbf{A}_\eta|\left|\mathcal{O}_T[F_u] - \mathcal{O}_T[F_v]\right| \right\} \frac{1}{\Omega\Gamma_q(\beta)} \int_{\omega_0}^{T} (T - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\beta-1}} \times \\
& s^{\alpha-1} d_{q,\omega} s + \left\{ |\mathbf{B}_T|\left|\mathcal{O}_\eta[F_u] - \mathcal{O}_\eta[F_v]\right| + |\mathbf{B}_\eta|\left|\mathcal{O}_T[F_u] - \mathcal{O}_T[F_v]\right| \right\} \frac{(T-\omega_0)^{\beta-1}}{\Omega} \\
\leq\ & \|u-v\|_{\mathcal{C}}\, \mathcal{L} \left[\frac{T^{\alpha}(T-\omega_0)^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} + \frac{\Phi_1}{|\Omega|} \left\{ |\mathbf{A}_T| \frac{T^{\alpha-1}(T-\omega_0)^{\beta}}{\Gamma_q(\beta+1)} + |\mathbf{B}_T|(T-\omega_0)^{\beta-1} \right\} \right. \\
& \left. + \frac{\Phi_2}{|\Omega|} \left\{ |\mathbf{A}_\eta| \frac{T^{\alpha-1}(T-\omega_0)^{\beta}}{\Gamma_q(\beta+1)} + |\mathbf{B}_\eta|(T-\omega_0)^{\beta-1} \right\} \right] \\
\leq\ & \mathcal{L}\,\Xi\,\|u-v\|_{\mathcal{C}}.
\end{aligned} \tag{30}$$

By (H_2), we can conclude that $\mathcal{F}$ is a contraction. From Banach fixed point theorem, $\mathcal{F}$ has a fixed point. Therefore, problem (1) has a unique solution. □

4. Example

In this section, we give an example of nonlocal fractional q and Hahn integral boundary value problem for sequential fractional q-Hahn integrodifference equation:

$$
\begin{aligned}
D^{\frac{1}{3}}_{\frac{1}{2}} D^{\frac{3}{4}}_{\frac{1}{2},\frac{2}{3}} u(t) &= \frac{1}{(1000e^2+t^2)\,(1+|u(t)|)} \Big[e^{-(4t+\frac{\pi}{3})} \left(u^2+2|u|\right) + e^{-(\frac{e}{3}+\cos^2\pi t)} \left| \Psi^{\frac{1}{2}}_{\frac{1}{2}} u(t) \right| \\
&\quad + e^{-(1+\sin^2\pi t)} \left| \Psi^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}} u(t) \right| \Big], \quad t \in I_{\frac{1}{2}}^{I^{[\frac{4}{3},10]}_{\frac{1}{2},\frac{2}{3}}} \qquad (31)\\
u(5) &= \frac{1}{10\pi} \mathcal{I}^{\frac{1}{5}}_{\frac{1}{2},\frac{2}{3}} u(5), \\
u(10) &= \frac{1}{20E} \mathcal{I}^{\frac{1}{5}}_{\frac{1}{2}} u(10),
\end{aligned}
$$

where $\psi(t,s) = \frac{e^{-|s-t|}}{(t+20)^3}$ and $\varphi(t,s) = \frac{e^{-2|s-t|}}{(t+30)^2}$.

Here, $\alpha = \frac{1}{3}$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{5}$, $\theta = \frac{1}{2}$, $\phi = \frac{2}{5}$, $q = \frac{1}{2}$, $\omega = \frac{2}{3}$, $\omega_0 = \frac{\omega}{1-q} = \frac{4}{3}$, $T = 10$, $\eta = 5$, $\lambda = \frac{1}{10\pi}$, $\mu = \frac{1}{20e}$, and $F\left[t, u(t), \Psi^{\theta}_{q} u(t), \Upsilon^{\phi}_{q,\omega} u(t)\right] = \frac{1}{(1000e^2+t^2)(1+|u(t)|)} \times \left[e^{-(4t+\frac{\pi}{3})} \left(u^2+2|u|\right) + e^{-(\frac{e}{3}+\cos^2\pi t)} \left| \Psi^{\frac{1}{2}}_{\frac{1}{2}} u(t) \right| + e^{-(1+\sin^2\pi t)} \left| \Psi^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}} u(t) \right| \right]$.

After calculating, we get

$$
|\mathbf{A}_\eta| \approx 0.7567, \quad |\mathbf{A}_T| \approx 0.5984, \quad |\mathbf{B}_\eta| \approx 0.9962, \quad |\mathbf{B}_T| \approx 1.1816, \quad \text{and} \quad |\Omega| \approx 0.2980.
$$

For all $t \in [0,10]$ and $u, v \in \mathbb{R}$, we find that

$$
\begin{aligned}
&\left| F\left[t, u, \Psi^{\theta}_{q} u, \Upsilon^{\phi}_{q,\omega} u\right] - F\left[t, v, \Psi^{\theta}_{q} v, \Upsilon^{\phi}_{q,\omega} v\right] \right| \\
&\leq \frac{1}{1000e^{2+\frac{\pi}{3}}} |u-v| + \frac{1}{1000e^{2+\frac{e}{3}}} \left| \Psi^{\theta}_{q} u - \Psi^{\theta}_{q} v \right| + \frac{1}{1000e^3} \left| \Upsilon^{\phi}_{q,\omega} u - \Upsilon^{\phi}_{q,\omega} v \right|.
\end{aligned}
$$

Thus, (H_1) holds with $\ell_1 = 0.0000475$, $\ell_2 = 0.0000547$, and $\ell_3 = 0.0000498$.

Next, we find that

$$
\psi_0 = 0.00125, \quad \varphi_0 = 0.00111, \quad \mathcal{L} = 0.000461, \quad \Phi_1 = 4.9572, \quad \Phi_2 = 12.1191, \quad \Theta_T = 4.6218, \quad \Theta_\eta = 4.8705 \quad \text{and} \quad \Xi = 92.4997.
$$

Since

$$
\mathcal{L}\,\Xi \approx 0.0426 < 1,
$$

we see that the condition (H_2) holds.

Hence, by Theorem 1, problem (31) has a unique solution.

5. Conclusions

We have proved existence and uniqueness results of the sequential fractional q-Hahn integrodifference equation with nonlocal mixed fractional q and fractional Hahn integral boundary condition (1) by using the Banach fixed point theorem, and the existence of at least a solution by Schauder's fixed point theorem. Our problem contains both fractional q-difference and fractional Hahn difference operators, which is a new idea.

Author Contributions: Conceptualization, T.D., N.P. and T.S.; Formal analysis, T.D., N.P. and T.S.; Funding acquisition, N.P.; Investigation, T.D., N.P. and T.S.; Methodology, T.D., N.P. and T.S.; Writing—original draft, T.D., N.P. and T.S.; Writing—review and editing, T.D., N.P. and T.S. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract No. KMUTNB-61-GOV-D-64.

Acknowledgments: This research was supported by Chiang Mai University.

Conflicts of Interest: The authors declare that they have no competing interests.

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