mathematics

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Crossing Limit Cycles of Planar Piecewise Linear Hamiltonian Systems without Equilibrium Points

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Received: 20 April 2020; Accepted: 6 May 2020; Published: 10 May 2020

Abstract: In this paper, we study the existence of limit cycles of planar piecewise linear Hamiltonian systems without equilibrium points. Firstly, we prove that if these systems are separated by a parabola, they can have at most two crossing limit cycles, and if they are separated by a hyperbola or an ellipse, they can have at most three crossing limit cycles. Additionally, we prove that these upper bounds are reached. Secondly, we show that there is an example of two crossing limit cycles when these systems have four zones separated by three straight lines.

Keywords: piecewise smooth vector field; Hamiltonian system; crossing limit cycles; conics

MSC: Primary 34A36; 34C25; 34A26

1. Introduction

The problem of the existence of limit cycles and mainly the problem of controlling their maximum number are two of the most difficult problems in the qualitative theory of differential systems in the plane. We solve these two problems for the class of discontinuous piecewise differential systems here considered.

We recall that a limit cycle is a periodic orbit of a differential system, which is isolated in the set of all periodic orbits of the system.

Limit cycles appear in a natural way in many applications. Thus, recently, the problem of the existence and the number of limit cycles has also been studied for discontinuous piecewise linear differential systems; this study goes back to Andronov et al. [1] and still has been given attention by researchers, mainly due to its simplicity and to its applications to a large number of phenomena, such as switches in electronic circuits, mechanical systems, etc.; see for instance [2–4], the books [5,6], and the hundreds of references quoted therein.

Lum and Chua [7] conjectured that a continuous planar piecewise linear system with two zones separated by a straight line can exhibit at most one limit cycle. Freire et al. [8] proved this conjecture in 1998. For the planar discontinuous piecewise linear systems, Han and Zhang [9] conjectured that these systems can have at most two crossing limit cycles when we separate them by a straight line, but Huan and Yang [10] gave a numerical example with three limit cycles; this result was proven analytically by Llibre and Ponce [11]. In 2015, Llibre et al. [12] proved that if we separate the planar discontinuous piecewise linear differential centers by a straight line, we cannot have any limit cycle. Recently, in the works [13–16], planar discontinuous linear differential centers separated by an algebraic curve, such as a conic, or a reducible and irreducible cubic, were studied, and it was proven that these differential systems can exhibit at most three crossing limit cycles having two intersection points with the conic of separation; the same result was proven if the curve of separation was a cubic.

In the literature, we find many papers studying piecewise smooth vector fields with two zones, and few papers for three and four zones.

In this paper, we consider planar piecewise linear Hamiltonian systems without equilibrium points.

Our first objective is to provide the exact maximum number of crossing limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points (PHS) and separated by a conic Σ. We follow the Filippov rules for defining the flow of the piecewise differential systems on a line of discontinuity; see [17].

We know that any conic takes nine canonical forms, but the four following forms: $x^2 + 1 = 0$, $x^2 + y^2 = 0$, and $x^2 + y^2 + 1 = 0$ do not separate the plane in connected regions, then we omit them. We do not study the crossing limit cycles separated by the conic $x^2 - 1 = 0$, because in [18], it was proven that PHS with three zones separated by two parallel straight lines have at most one crossing limit cycle.

The second objective of this paper is to study the crossing limit cycles of piecewise smooth differential systems such that in each piece, the differential system is linear, Hamiltonian, and without equilibrium points. Then, easy computations show that such differential system in each piece must have a vector field of the form:

$$X_i(x, y) = (-\lambda_i b_i x + b_i y + \gamma_i, -\lambda_i^2 b_i x + \lambda_i b_i y + \delta_i),$$

$\delta_i \neq \lambda_i \gamma_i$ and $b_i \neq 0$, with $i = 1 \ldots 4$. Their corresponding Hamiltonian function is:

$$H_i(x, y) = (-\lambda_i^2 b_i/2)x^2 + \lambda_i b_i xy - (b_i/2)y^2 + \delta_i x - \gamma_i y.$$

For more details, see [18].

1.1. Crossing Limit Cycles for Planar Piecewise Linear Hamiltonian Systems without Equilibrium Points Separated by a Conic

In this subsection, we give the upper bound of crossing limit cycles of PHS separated by a parabola, **P**: $y - x^2 = 0$, by a hyperbola **H**: $x^2 - y^2 - 1 = 0$, or by an ellipse **E**: $x^2 + y^2 - 1 = 0$.

We consider only the crossing limit cycles that intersect the conics in exactly two points, and for this reason, we will not study the crossing limit cycles separated by two intersecting straight lines $xy = 0$.

Our first main result is the following.

Theorem 1. *The following statements hold.*

- *(a)* *The maximum number of crossing limit cycles of PHS intersecting the parabola **P** at two points is at most two, and this maximum is reached; see Figure 1.*
- *(b)* *The maximum number of crossing limit cycles of PHS intersecting the hyperbola **H** at two points is at most three, and this maximum is reached; see Figure 2.*
- *(c)* *The maximum number of crossing limit cycles of PHS intersecting the ellipse **E** at two points is at most three, and this maximum is reached; see Figure 3.*

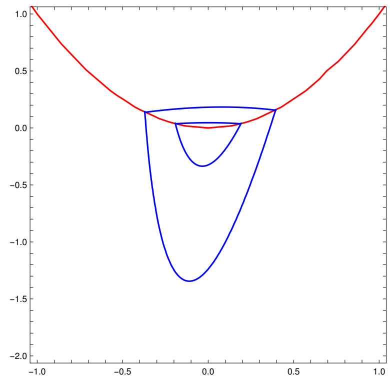

Figure 1. Two crossing limit cycles of planar discontinuous piecewise linear Hamiltonian systems (PHS) intersecting the parabola at two points.

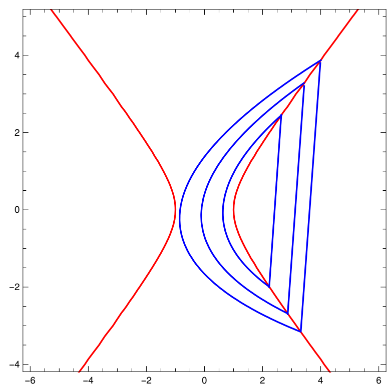

Figure 2. Three crossing limit cycles of PHS intersecting the hyperbola at two points.

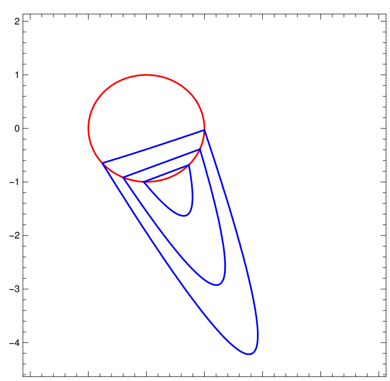

Figure 3. Three crossing limit cycles of PHS intersecting the ellipse at two points.

The proof of Theorem 1 is given in Section 2.

1.2. Crossing Limit Cycles for Planar Piecewise Linear Hamiltonian Systems without Equilibrium Points with Four Zones

In this subsection, we study the existence of crossing limit cycles of the planar piecewise linear Hamiltonian systems without equilibrium points with four zones:

$$X(x,y) = \begin{cases} X_1(x,y), & x \le -1, \\ X_2(x,y), & -1 \le x \le 0, \\ X_3(x,y), & 0 \le x \le 1, \\ X_4(x,y), & x \ge 1. \end{cases} \tag{1}$$

satisfying the condition:

C. The vector fields X_1, X_2, X_3, and X_4 are linear and Hamiltonian without equilibrium points.

Our second results are the following.

Theorem 2. *Continuous planar piecewise Hamiltonian systems without equilibrium points with four zones satisfying* ***C*** *have no crossing limit cycles.*

Theorem 3. *There are discontinuous planar piecewise Hamiltonian systems without equilibrium points with four zones satisfying* ***C****, exhibiting exactly two crossing limit cycles; see Figure 4.*

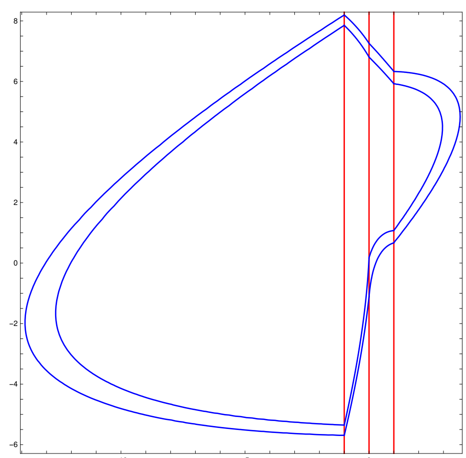

Figure 4. Two crossing limit cycles of PHS with four zones.

The proofs of Theorems 2 and 3 are given in Section 3.

2. Proof of Theorem 1

Proof of Statement (a) of Theorem 1. In the region $R_1 = \{(x,y) : y - x^2 \ge 0\}$, we consider the planar discontinuous piecewise linear Hamiltonian systems without equilibrium points:

$$\dot{x} = -\lambda_1 b_1 x + b_1 y + \gamma_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \delta_1, \tag{2}$$

with $b_1 \ne 0$ and $\delta_1 \ne \lambda_1 \gamma_1$. The corresponding Hamiltonian function is:

$$H_1(x,y) = -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \delta_1 x - \gamma_1 y. \tag{3}$$

In the region $R_2 = \{(x,y) : y - x^2 \le 0\}$, we consider:

$$\dot{x} = -\lambda_2 b_2 x + b_2 y + \gamma_2, \quad \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \delta_2, \tag{4}$$

with $b_2 \neq 0$ and $\delta_2 \neq \lambda_2\gamma_2$. The corresponding Hamiltonian function is:

$$H_2(x,y) = -(\lambda_2^2 b_2/2)x^2 + \lambda_2 b_2 xy - (b_2/2)y^2 + \delta_2 x - \gamma_2 y. \tag{5}$$

In order to have a crossing limit cycle that intersects the parabola $y - x^2 = 0$ at the points (x_i, y_i) and (x_k, y_k), these points must satisfy the following system:

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k) = 0, \\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0, \\ &y_i - x_i^2 = 0, \\ &y_k - x_k^2 = 0. \end{aligned} \tag{6}$$

We suppose that the two systems (2) and (4) have three crossing limit cycles. Then, System (6) must have three pairs of points as solutions, namely $p_i = (r_i, r_i^2)$ and $q_i = (s_i, s_i^2)$, with $i = 1, 2, 3$. Due to the fact that these points satisfy System (6) and if we consider the points $p_1 = (r_1, r_1^2)$ and $q_1 = (s_1, s_1^2)$, solving the first two equations of (6) with respect to the parameters γ_1 and γ_2, we get:

$$\begin{aligned} \gamma_1 = \; & \frac{1}{2(r_1+s_1)}(-r_1 r_1^3 - b_1 r_1^2 s_1 - b_1 r_1 s_1^2 - b_1 s_1^3 + 2\delta_1 + 2b_1 r_1^2 \lambda_1 + 2b_1 r_1 s_1 \lambda_1 \\ & +2b_1 s_1^2 \lambda_1 - b_1 r_1 \lambda_1^2 - b_1 s_1 \lambda_1^2), \end{aligned}$$

and γ_2 has the same expression that γ_1 changes $(b_1, \lambda_1, \delta_1)$ by $(b_2, \lambda_2, \delta_2)$.

If the second points $p_2 = (r_2, r_2^2)$ and $q_2 = (s_2, s_2^2)$ satisfy System (6), then solving the first two equations of (6) with respect to the parameters δ_1 and δ_2, we get:

$$\begin{aligned} \delta_1 = \; & \frac{b_1}{2(r_1 - r_2 + s_1 - s_2)}(-r_1^3 r_2 - r_1 r_2^3 + r_1^2 r_2 s_1 - r_2^3 s_1 + r_1 r_2 s_1^2 + r_2 s_1^3 + r_1^3 s_2 \\ & -r_1 r_2^2 s_2 + r_1^2 s_1 s_2 - r_2^2 s_1 s_2 + r_1 s_1^2 s_2 + s_1^3 s_2 - r_1 r_2 s_2^2 - r_2 s_1 s_2^2 - r_1 s_2^3 - s_1 s_2^3 \\ & -2r_1^2 r_2 \lambda_1 + 2r_1 r_2^2 \lambda_1 - 2r_1 r_2 s_1 \lambda_1 + 2r_2^2 s_1 \lambda_1 - 2r_2 s_1^2 \lambda_1 - 2r_1^2 s_2 \lambda_1 + 2r_1 r_2 s_2 \lambda_1 \\ & -2r_1 s_1 s_2 \lambda_1 + 2r_2 s_1 s_2 \lambda_1 - 2s_1^2 s_2 \lambda_1 + 2r_1 s_2^2 \lambda_1 + 2s_1 s_2^2 \lambda_1), \end{aligned}$$

and δ_2 has the same expression that δ_1 changes (b_1, λ_1) by (b_2, λ_2).

Finally, we suppose that the points $p_3 = (r_3, r_3^2)$ and $q_3 = (s_3, s_3^2)$ satisfy System (6), then the parameters λ_1 and λ_2 must be $\lambda_1 = A/B$ where:

$$\begin{aligned} A = \; & r_1^3(r_2 - r_3 + s_2 - s_3) + r_1^2 s_1(r_2 - r_3 + s_2 - s_3) + r_2^3(r_3 - s_1 + s_3) + r_2^2 s_2(r_3 - s_1 \\ & +s_3) + r_1(-r_2^3 + r_3^3 - r_3 s_1^2 - r_2^2 s_2 + s_1^2 s_2 - s_2^3 + r_2(s_1^2 - s_2^2) + r_3^2 s_3 - s_1^2 s_3 + r_3 s_3^2 \\ & +s_3^3) + (s_1 - s_2)(r_3^3 + r_3^2 s_3 + (s_1 - s_3)(s_2 - s_3)(s_1 + s_2 + s_3) - r_3(s_1^2 + s_1 s_2 + s_2^2 \\ & -s_3^2)) - r_2(r_3^3 - s_1^3 + s_1 s_2^2 + r_3^2 s_3 - s_2^2 s_3 + s_3^3 + r_3(-s_2^2 + s_3^2)), \\ B = \; & 2((s_1 - s_2)(r_3^2 + (s_1 - s_3)(s_2 - s_3) - r_3(s_1 + s_2 - s_3)) + r_1^2(r_2 - r_3 + s_2 - s_3) \\ & +r_2^2(r_3 - s_1 + s_3) + r_1(-r_2^2 + r_3^2 - r_3 s_1 + r_2(s_1 - s_2) + s_1 s_2 - s_2^2 + r_3 s_3 - s_1 s_3 \\ & +s_3^2) - r_2(r_3^2 + r_3(-s_2 + s_3) - (s_1 - s_3)(s_1 - s_2 + s_3))). \end{aligned}$$

λ_2 has the same expression that λ_1 changes b_1 by b_2.

We replace γ_1, λ_1, and δ_1 in the expression of $H_1(x,y)$ and γ_2, λ_2, and δ_2 in the expression of $H_2(x,y)$, and we obtain $H_1(x,y) = H_2(x,y)$. Therefore, the piecewise linear differential system becomes a linear differential system, which does not have limit cycles. Therefore, the maximum number of crossing limit cycles in this case is two.

Example with two limit cycles. Consider the planar discontinuous piecewise linear Hamiltonian system without equilibrium points separated by the parabola **P**:

$$\dot{x} = 5.5x - 0.5y + 3, \quad \dot{y} = 60.5x - 5.5y + 0.2;$$

in the region R_1, its corresponding Hamiltonian function is:

$$H_1(x,y) = 30.25x^2 - 5.5xy + 0.2x + 0.25y^2 - 3y.$$

The second system is:

$$\dot{x} = 0.2x - 0.1y - 0.778814, \quad \dot{y} = 0.4x - 0.2y + 0.00727332;$$

in the region R_2, its corresponding Hamiltonian function is:

$$H_2(x,y) = 0.2x^2 - 0.2xy + 0.00727332x + 0.05y^2 + 0.778814y.$$

Now system (6) has the two solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1) = (0.191567.., 0.0366978.., -0.191502.., 0.036673..)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (0.395114.., 0.156115.., -0.372692.., 0.138899..)$. Which provide the two limit cycles shown in Figure 1. This completes the proof of Statement (a) of Theorem 1. □

Proof of Statement (b) of Theorem 1. In the region $R_1 = \{(x,y) : x^2 - y^2 - 1 \geq 0\}$, we consider the PHS given in (2). Its corresponding Hamiltonian function is given by Equation (3).

In the region $R_2 = \{(x,y) : x^2 - y^2 - 1 \leq 0\}$, we consider the PHS given in (2). Its corresponding Hamiltonian function is given by Equation (5).

In order to have a crossing limit cycle that intersects the hyperbola $x^2 - y^2 - 1 = 0$ at the points (x_i, y_i) and (x_k, y_k), these points must satisfy the system:

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k) = 0, \\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0, \\ &x_i^2 - y_i^2 = 1, \\ &x_k^2 - y_k^2 = 1. \end{aligned} \tag{7}$$

We assume that the two systems (2) and (4) have four crossing limit cycles. Therefore, System (7) must have four pairs of points $p_i = (\cosh r_i, \sinh r_i)$ and $q_i = (\cosh s_i, \sinh s_i)$ for $i = 1, 2, 3, 4$ as solutions. Since these points satisfy System (7), we consider the points $p_1 = (\cosh r_1, \sinh r_1)$ and $q_1 = (\cosh s_1, \sinh s_1)$, and from (7), we obtain that the parameters γ_1 and γ_2 must be:

$$\begin{aligned} \gamma_1 = \; & \frac{1}{2(\sinh r_1 - \sinh s_1)}(2\delta_1 \cosh r_1 - b_1\lambda_1^2 \cosh^2 r_1 + b_1\lambda_1^2 \cosh^2 s_1 - 2\cosh s_1(\delta_1 \\ & + b_1\lambda_1 \sinh s_1) + b_1(-\sinh^2 r_1 + \lambda_1 \sinh(2r_1) + \sinh^2 s_1). \end{aligned}$$

If we change $(b_1, \lambda_1, \delta_1)$ by $(b_2, \lambda_2, \delta_2)$ in the expression of γ_1, we get the expression of γ_2.

We suppose that the second points $p_2 = (\cosh r_2, \sinh r_2)$ and $q_2 = (\cosh s_2, \sinh s_2)$ satisfy System (7), then the parameters δ_1 and δ_2 must be:

$$\begin{aligned}
\delta_1 = \quad & \frac{1}{4\Big(\cosh\left(\frac{r_1-2r_2+s_1}{2}\right)-\cosh\left(\frac{r_1+s_1-2s_2}{2}\right)\Big)}(b_1\operatorname{csch}(\frac{r_1-s_1}{2})(-\lambda_1^2\cosh^2 r_1 \\
& \sinh r_2+\lambda_1^2\cosh^2 s_1\sinh r_2-\sinh^2 r_1\sinh r_2+\lambda_1\sinh(2r_1)\sinh r_2+\sinh r_1\sinh^2 r_2 \\
& -\lambda_1\sinh r_1\sinh(2r_2)+\lambda_1^2\cosh^2 r_2(\sinh r_1-\sinh s_1)-\sinh^2 r_2\sinh s_1+\lambda_1\sinh(2r_2) \\
& \sinh s_1+\sinh r_2\sinh^2 s_1+\lambda_1^2\cosh^2 s_2(-\sinh r_1+\sinh s_1)-\lambda_1\sinh r_2\sinh(2s_1) \\
& +\lambda_1^2\cosh^2 r_1\sinh s_2-\lambda_1^2\cosh^2 s_1\sinh s_2+\sinh^2 r_1\sinh s_2-\lambda_1\sinh(2r_1)\sinh s_2 \\
& -\sinh^2 s_1\sinh s_2+\lambda_1\sinh(2s_1)\sinh s_2-\sinh r_1\sinh^2 s_2+sinhs_1\sinh^2 s_2 \\
& +\lambda_1(\sinh r_1-\sinh s_1)\sinh(2s_2))).
\end{aligned}$$

If we change (b_1, λ_1) by (b_2, λ_2) in the expression of δ_1, we obtain δ_2.

Now, we suppose that points $p_3 = (\cosh r_3, \sinh r_3)$ and $q_3 = (\cosh s_3, \sinh s_3)$ satisfy System (7), then we obtain two values of λ_1 (we name them $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$) and two values of λ_2 (we name them $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$). The first value of λ_1 is given by $\lambda_1^{(1)} = (A-(1/2)\sqrt{B})/C$ and $\lambda_1^{(2)} = (A+(1/2)\sqrt{B})/C$, where:

$$\begin{aligned}
A = \quad & -\sinh\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)+\sinh\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right) \\
& -\sinh\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)+\sinh\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right) \\
& -\sinh\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)+\sinh\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right) \\
& -\sinh\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)+\sinh\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right) \\
& +\sinh\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)-\sinh\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right) \\
& +\sinh\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)-\sinh\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right),
\end{aligned}$$

$$\begin{aligned}
B = \quad & -4\Big(\cosh\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)-\cosh\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right) \\
& +\cosh\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)-\cosh\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right) \\
& -\cosh\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)+\cosh\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right) \\
& -\cosh\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)+\cosh\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right) \\
& +\cosh\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)-\cosh\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right) \\
& +\cosh\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)-\cosh^2\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right)) \\
& +4(\sinh\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)-\sinh\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right) \\
& +\sinh\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)-\sinh\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right) \\
& +\sinh\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)-\sinh\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right) \\
& +\sinh\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)-\sinh\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right) \\
& -\sinh\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)+\sinh\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right) \\
& -\sinh\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)+\sinh^2\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right)\Big),
\end{aligned}$$

and the expression of C is:

$$
\begin{aligned}
C = \quad & \cosh\left(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\right) - \cosh\left(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\right) \\
& + \cosh\left(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\right) - \cosh\left(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\right) \\
& - \cosh\left(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\right) + \cosh\left(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\right) \\
& - \cosh\left(\frac{r_1 + r_2 - r_3 + 3s_1 + s_2 - s_3}{2}\right) + \cosh\left(\frac{r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3}{2}\right) \\
& + \cosh\left(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\right) - \cosh\left(\frac{r_1 - r_2 + 3r_3 + s_1 - s_2 + s_3}{2}\right) \\
& + \cosh\left(\frac{r_1 - r_2 + r_3 + 3s_1 - s_2 + s_3}{2}\right) - \cosh\left(\frac{r_1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\right).
\end{aligned}
$$

We get the expression of $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ by changing b_1 by b_2 in the expression of $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, respectively.

We replace γ_1, δ_1, and $\lambda_1^{(i)}$ in the expression of $H_1(x,y)$ and γ_2, δ_2, and $\lambda_2^{(i)}$ in the expression of $H_2(x,y)$, and we obtain $H_1(x,y) = H_2(x,y)$, for $i = 1,2$. Hence, in these cases, the piecewise linear differential system becomes a linear differential system, which does not have limit cycles. Therefore, the maximum number of crossing limit cycles in this case is two.

Now, we consider either $\lambda_1^{(2)}$ and $\lambda_2^{(1)}$ or $\lambda_1^{(1)}$ and $\lambda_2^{(2)}$, by replacing the expressions of γ_1, δ_1, and $\lambda_1^{(2)}$ (resp. $\lambda_1^{(1)}$) in the expression of $H_1(x,y)$ and γ_2, δ_2, and $\lambda_2^{(1)}$ (resp. $\lambda_2^{(2)}$) in the expression of $H_2(x,y)$; we have $H_1(x,y) \neq H_2(x,y)$.

Then, we assume that points $p_4 = (\cosh r_4, \sinh r_4)$ and $q_4 = (\cosh s_4, \sinh s_4)$ satisfy System (7), then we obtain $b_1 = 0$ and $b_2 = 0$. This is a contradiction because by the assumptions, they are not zero. Then, we proved that the maximum number of crossing limit cycles for PHS separated by a hyperbola is at most three.

Example with three limit cycles. We consider a PHS separated by the hyperbola **H**:

$$\dot{x} = -0.14..x + 1.4y + \frac{1}{5}, \quad \dot{y} = -0.014..x + 0.14y + 1.9, \tag{8}$$

in the region $R_1 = \{(x,y) : x^2 - y^2 - 1 \leq 0\}$. It has the Hamiltonian function:

$$H_1(x,y) = -0.007..x^2 + 0.14xy + 1.9x - 0.7y^2 - \frac{y}{5}.$$

Now, we consider the second PHS:

$$\dot{x} = 5x - \frac{y}{2} - 7.14286.., \quad \dot{y} = 50x - 5y - 67.8571.., \tag{9}$$

in the region $R_2 = \{(x,y) : x^2 - y^2 - 1 \geq 0,\ x > 1\}$. This differential system has the Hamiltonian function:

$$H_2(x,y) = 25x^2 - 5xy - 67.8571..x + \frac{y^2}{4} + 7.14286..y.$$

The PHS (8)–(9) has exactly three crossing limit cycles, because the system of equations:

$$
\begin{aligned}
& H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0, \\
& H_2(\alpha,\beta) - H_2(\gamma,\delta) = 0, \\
& \alpha^2 - \beta^2 - 1 = 0, \\
& \gamma^2 - \delta^2 - 1 = 0,
\end{aligned}
\tag{10}
$$

has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1) = (3.99376..., 3.86653..., 3.31341..., -3.1589..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (3.43842..., 3.28979..., 2.86513..., -2.68496..)$, and $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (2.64219..., 2.44565..., 2.2285..., -1.99154...)$; see Figure 2. □

Proof of Statement (c) of Theorem 1. We consider the PHS given in (2) in the region $R_1 = \{(x, y) : x^2 + y^2 - 1 \geq 0\}$, with its corresponding Hamiltonian function (3).

We consider the PHS given in (2) in the region $R_2 = \{(x, y) : x^2 + y^2 - 1 \leq 0\}$, with its corresponding Hamiltonian function (5).

In order that Systems (2) and (4) have crossing limit cycles intersecting the ellipse $x^2 + x^2 - 1 = 0$ at the points (x_i, y_i) and (x_k, y_k), they must satisfy the system:

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k) = 0, \\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0, \\ &x_i^2 + y_i^2 = 1, \\ &x_k^2 + y_k^2 = 1. \end{aligned} \tag{11}$$

We suppose that Systems (2) and (4) have four crossing limit cycles. Therefore, System (11) must have four pairs of points $p_i = (\cos r_i, \sin r_i)$ and $q_i = (\cos s_i, \sin s_i)$ for $i = 1, 2, 3, 4$ as solutions. Therefore, if we consider the points $p_1 = (\cos r_1, \sin r_1)$ and $q_1 = (\cos s_1, \sin s_1)$, from (11), we obtain that the parameters γ_1 and γ_2 must be:

$$\begin{aligned} \gamma_1 = \; & \frac{1}{4(\sin r_1 - \sin s_1)}(4\delta_1 \cos r_1 + b_1 \cos(2r_1) - b_1\lambda_1^2 \cos(2r_1) - 4\delta_1 \cos s_1 - b_1 \cos(2s_1) \\ & + b_1\lambda_1^2 \cos(2s_1) + 2b_1\lambda_1 \sin(2r_1) - 2b_1\lambda_1 \sin(2s_1)). \end{aligned}$$

If we change $(b_1, \lambda_1, \delta_1)$ by $(b_2, \lambda_2, \delta_2)$ in the expression of γ_1, we get the expression of γ_2.

Now, if the second points $p_2 = (\cos r_2, \sin r_2)$ and $q_2 = (\cos s_2, \sin s_2)$ satisfy System (11), then the parameters δ_1 and δ_2 take the values:

$$\begin{aligned} \delta_1 = \; & \frac{r_1 \cos((r_1 + s_1)/2) \csc((r_2 - s_2)/2) \csc((r_1 - r_2 + s_1 - s_2)/2)}{4\Big(\sin r_1 - \sin s_1\Big)}(\lambda_1^2 \cos^2 r_2 \sin r_1 \\ & -\lambda_1^2 \cos^2 s_2 \sin r_1 - 2\lambda_1 \cos r_2 \sin r_1 \sin r_2 + \sin r_1 \sin^2 r_2 + 2\lambda_1 \cos(r_1 + s_1) \\ & \sin r_2 \sin(r_1 - s_1) - \lambda_1^2 \cos^2 r_2 \sin s_1 + \lambda_1^2 \cos^2 s_2 \sin s_1 + 2\lambda_1 \cos r_2 \sin r_2 \sin s_1 \\ & - \sin^2 r_2 \sin s_1 - \sin r_2 \sin(r_1 - s_1) \sin(r_1 + s_1) + \lambda_1^2 \sin r_2 \sin(r_1 - s_1) \sin(r_1 + s_1) \\ & -2\lambda_1 \cos(r_1 + s_1) \sin(r_1 - s_1) \sin s_2 + \sin(r_1 - s_1) \sin(r_1 + s_1) \sin s_2 - \lambda_1^2 \\ & \sin(r_1 - s_1) \sin(r_1 + s_1) \sin s_2 - \sin r_1 \sin^2 s_2 + \sin s_1 \sin^2 s_2 + \lambda_1 \sin r_1 \sin(2s_2) \\ & -\lambda_1 \sin s_1 \sin(2s_2)). \end{aligned}$$

If we change (b_1, λ_1) by (b_2, λ_2) in the expression of δ_1, we obtain δ_2.

If we assume that the points $p_3 = (\cos r_3, \sin r_3)$ and $q_3 = (\cos s_3, \sin s_3)$ satisfy System (11), then we obtain two values of λ_1, namely $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, and two values of λ_2, namely $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$, such that $\lambda_1^{(1)} = (A + \sqrt{B})/C$ and $\lambda_1^{(2)} = (A - \sqrt{B})/C$, where:

$$
\begin{aligned}
A = \quad & -\sin\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)+\sin\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)\\
& -\sin\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)+\sin\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right)\\
& -\sin\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)+\sin\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right)\\
& -\sin\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)+\sin\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right)\\
& +\sin\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)-\sin\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right)\\
& +\sin\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)-\sin\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
B = \quad & \cos\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)-\cos\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)\\
& +\cos\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)-\cos\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right)\\
& -\cos\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)+\cos\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right)\\
& -\cos\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)+\cos\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right)\\
& +\cos\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)-\cos\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right)\\
& +\cos\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)-\cos^2\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right)\\
& +(\sin\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)-\sin\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)\\
& +\sin\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)-\sin\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right)\\
& +\sin\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)-\sin\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right)\\
& +\sin\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)-\sin\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right)\\
& -\sin\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)+\sin\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right)\\
& -\sin\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)+\sin^2\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right),
\end{aligned}
$$

and the expression of C is:

$$
\begin{aligned}
C = \quad & \cos\left(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}\right)-\cos\left(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}\right)\\
& +\cos\left(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}\right)-\cos\left(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}\right)\\
& -\cos\left(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}\right)+\cos\left(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}\right)\\
& -\cos\left(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}\right)+\cos\left(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}\right)\\
& +\cos\left(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}\right)-\cos\left(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}\right)\\
& +\cos\left(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}\right)-\cos\left(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}\right).
\end{aligned}
$$

The expressions of $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ are the same as the expressions of $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, respectively, if we change b_1 by b_2.

We replace γ_1, δ_1, and $\lambda_1^{(i)}$ in the expression of $H_1(x,y)$ and γ_2, δ_2 and $\lambda_2^{(i)}$ in the expression of $H_2(x,y)$, and we obtain $H_1(x,y) = H_2(x,y)$ for $i = 1,2$. Therefore, the maximum number of crossing limit cycles in these cases is two.

Now, we consider either $\lambda_1^{(2)}$ and $\lambda_2^{(1)}$ or $\lambda_1^{(1)}$ and $\lambda_2^{(2)}$, by replacing the expressions of γ_1, δ_1, and $\lambda_1^{(2)}$ (resp. $\lambda_1^{(1)}$) in the expression of $H_1(x,y)$ and γ_2, δ_2 and $\lambda_2^{(1)}$ (resp. $\lambda_2^{(2)}$) in the expression of $H_2(x,y)$, and we get two different expressions of the Hamiltonian functions $H_1(x,y)$ and $H_2(x,y)$.

Then, we assume that points $p_4 = (\cos r_4, \sin r_4)$ and $q_4 = (\cos s_4, \sin s_4)$ satisfy System (11), and by solving this system, we obtain $b_1 = 0$ and $b_2 = 0$, which is a contradiction. Then, we proved that the maximum number of crossing limit cycles for PHS separated by an ellipse is at most three.

Example with three limit cycles. In the region $R_1 = \{(x,y) : x^2 + y^2 - 1 \geq 0\}$, we consider the linear PHS:

$$\dot{x} = 2.53..x + 1.1..y - 0.6.., \quad \dot{y} = -5.819..x - 2.53..y - 0.4..; \tag{12}$$

its Hamiltonian function is:

$$H_1(x,y) = -2.9095..x^2 - 2.53..xy - 0.4..x - 0.55..y^2 + 0...6y.$$

In the region $R_2 = \{(x,y) : x^2 + y^2 - 1 \leq 0\}$, we consider the linear PHS:

$$\dot{x} = -0.308696..x + 0.71..y + 0.0732085.., \quad \dot{y} = -0.134216..x + 0.308696..y + 0.0488056.. \tag{13}$$

Its Hamiltonian function is:

$$H_2(x,y) = -0.0671078..x^2 + 0.308696..xy + 0.0488056..x - 0.355..y^2 - 0.0732085..y.$$

The linear PHS (12)–(13) has exactly three crossing limit cycles, because the system of equations:

$$\begin{aligned} &H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0, \\ &H_2(\alpha,\beta) - H_2(\gamma,\delta) = 0, \\ &\alpha^2 + \beta^2 - 1 = 0, \\ &\gamma^2 + \delta^2 - 1 = 0, \end{aligned} \tag{14}$$

has three real solutions $(\alpha_1, \beta_1, \gamma_1, \delta_1) = (-0.0450412..., -0.998985..., 0.730814..., -0.682576...)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (-0.40163..., -0.915802..., 0.92153..., -0.388307...)$, and $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (-0.760814.., -0.64897.., 0.99956.., -0.0296781..)$ $\square$

We mention that the proof of Theorem 1 can be also analyzed using the results of [19].

3. Proof of Theorems 2 and 3

Proof of Theorem 2. Consider a continuous linear Hamiltonian differential system separated by the straight lines $x = -1$, $x = 0$, and $x = 1$. According to the continuity of the vector field X, we obtain:

$$X_1(-1,y) = X_2(-1,y), \;\; X_2(0,y) = X_3(0,y) \;\; \textit{and} \;\; X_3(1,y) = X_4(0,y), \;\; \forall y \in \mathbb{R},$$

which imply that:

$$\begin{aligned} &b_1 = b_2 = b_3 = b_4 = b, \\ &\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta, \\ &\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma, \\ &\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda. \end{aligned}$$

Therefore, from System (1), the piecewise vector field becomes the vector field:

$$X(x,y) = (-\lambda bx + by + \gamma, -\lambda^2 bx + \lambda by + \delta), \delta \neq \lambda\gamma, b \neq 0.$$

Since this linear differential system has no equilibrium point, it has no periodic orbits, then no limit cycles. This completes the proof of Theorem 2. □

Proof of Theorem 3. If the PHS with four zones have crossing limit cycles, then there are crossing points $(-1, y_0)$, $(-1, y_5)$, $(0, y_1)$, $(0, y_4)$, and $(1, y_2)$, $(1, y_3)$ satisfying:

$$\begin{aligned} &H_1(-1, y_0) = H_1(-1, y_5),\\ &H_2(-1, y_0) = H_2(0, y_1),\\ &H_2(-1, y_5) = H_2(0, y_4),\\ &H_3((0, y_1) = H_3(1, y_2),\\ &H_3((0, y_4) = H_3(1, y_3),\\ &H_4((1, y_2) = H_4(1, y_3), \end{aligned} \tag{15}$$

or equivalently:

$$(y_0 - y_5)(2b_1\lambda_1 + b_1 y_0 + b_1 y_5 + 2\gamma_1) \;=\; 0, \tag{16}$$

$$-b_2\lambda_2^2 - b_2 y_0^2 - 2b_2\lambda_2 y_0 + b_2 y_1^2 - 2\delta_2 - 2\gamma_2 y_0 + 2\gamma_2 y_1 \;=\; 0, \tag{17}$$

$$-b_2\lambda_2^2 + b_2 y_4^2 - b_2 y_5^2 - 2b_2\lambda_2 y_5 - 2\delta_2 + 2\gamma_2 y_4 - 2\gamma_2 y_5 \;=\; 0, \tag{18}$$

$$b_3\lambda_3^2 - b_3 y_1^2 + b_3 y_2^2 - 2b_3\lambda_3 y_2 - 2\delta_3 - 2\gamma_3 y_1 + 2\gamma_3 y_2 \;=\; 0, \tag{19}$$

$$b_3\lambda_3^2 + b_3 y_3^2 - 2b_3\lambda_3 y_3 - b_3 y_4^2 - 2\delta_3 + 2\gamma_3 y_3 - 2\gamma_3 y_4 \;=\; 0, \tag{20}$$

$$(y_2 - y_3)(-2b_4\lambda_4 + b_4 y_2 + b_4 y_3 + 2\gamma_4) \;=\; 0. \tag{21}$$

As $y_0 \neq y_5$ and $y_2 \neq y_3$, we can solve Equation (16) for y_5, as well as we can solve Equation (21) for y_3. Substituting the obtained values of y_5 and y_3 into Equations (18) and (20), respectively, we obtain the following two equations:

$$\begin{aligned} &\gamma_2\left(\frac{2\gamma_1}{b_1} + 2\lambda_1 + y_0 + y_4\right) - \delta_2 - \frac{1}{2b_1^2}(b_2(b_1(2\lambda_1 - \lambda_2 + y_0 - y_4) + 2\gamma_1)\\ &(b_1(2\lambda_1 - \lambda_2 + y_0 + y_4) + 2\gamma_1)) = 0, \end{aligned} \tag{22}$$

and:

$$\begin{aligned} &b_3(b_4(\lambda_3 - 2\lambda_4 + y_2 - y_4) + 2\gamma_4)(b_4(\lambda_3 - 2\lambda_4 + y_2 + y_4) + 2\gamma_4)\\ &-2b_4(b_4(\delta_3 + \gamma_3(-2\lambda_4 + y_2 + y_4)) + 2\gamma_3\gamma_4) = 0. \end{aligned} \tag{23}$$

First, we solve Equation (17) for y_0, and we get:

$$y_0 = (1/b_2)(-b_2\lambda_2 - \gamma_2 \pm \sqrt{b_2^2 y_1^2 + 2b_2\gamma_2\lambda_2 - 2b_2\delta_2 + 2b_2\gamma_2 y_1 + \gamma_2^2}); \tag{24}$$

then, we solve Equation (19) for y_2, and we get:

$$y_2 = (1/b_3)(+b_3\lambda_3 - \gamma_3 \pm \sqrt{b_3^2 y_1^2 - 2b_3\gamma_3\lambda_3 + 2b_3\delta_3 + 2b_3\gamma_3 y_1 + \gamma_3^2}). \tag{25}$$

Substituting (24) into (22), we obtain two equations $f_{1,2}(y_1, y_4) = 0$ depending on y_1 and y_4. Then, substituting (25) into (23), we obtain two equations $g_{1,2}(y_1, y_4) = 0$ depending on y_1 and y_4.

Therefore, we compute the product $F(y_1, y_2) = f_1(y_1, y_2) f_2(y_1, y_2) = 0$ and $G(y_1, y_2) = g_1(y_1, y_2) g_2(y_1, y_2) = 0$, and we obtain two quartic polynomial equations with the variables y_1 and y_4.

By using Bézout's theorem, we obtain that the number of solutions of the system:

$$F(y_1, y_4) = 0, \quad G(y_1, y_4) = 0. \tag{26}$$

is bounded by the product of the degrees of the polynomials $F(y_1, y_4)$ and $G(y_1, y_4)$. If (y_1, y_4) is a solution of these equations, (y_4, y_1) is also a solution. Therefore, we obtain that the number of different solutions of System (26) is at most eight, which is an upper bound for the maximum number of limit cycles that can have the PHS (15). Due to the higher degree of this system and the number of its parameters, we only can give an example with two limit cycles.

Example with two limit cycles. Consider the vector fields $X = (X_1, X_2, X_3, X_4)$ such that:

$$\begin{aligned}
X_1(x, y) &= (-\frac{x}{2} + 2y - 3, -\frac{x}{8} + \frac{y}{2} + 3), \\
X_2(x, y) &= (2 + 2x - 2y, 2x - 2y + 30), \\
X_3(x, y) &= (4 + 4x + 2y, 13 - 8x - 4y), \\
X_4(x, y) &= (-\frac{x}{2} + y - 3, -\frac{x}{4} + \frac{y}{2} - 3).
\end{aligned}$$

Their corresponding Hamiltonian functions are given, respectively, by:

$$\begin{aligned}
H_1(x, y) &= -\frac{x^2}{16} + \frac{xy}{2} + 3x - y^2 + 3y, \\
H_2(x, y) &= x^2 - 2xy + 30x + y^2 - 2y, \\
H_3(x, y) &= -4x^2 - 4xy + 13x - y^2 - 4y, \\
H_4(x, y) &= -\frac{x^2}{8} + \frac{xy}{2} - 3x - \frac{y^2}{2} + 3y.
\end{aligned}$$

The first crossing limit cycle intersects the straight lines of discontinuity in the following points: $(-1, -5.69679...)$ and $(-1, 8.19679...)$; $(0, -1.11032...)$ and $(0, 7.25999...)$; and $(1, 0.66814...)$ and $(1, 6.33186...)$. The second crossing limit cycle intersects the straight lines of discontinuity in the points: $(-1, -5.35506...)$ and $(-1, 7.85506...)$; $(0, 0.177417...)$ and $(0, 0.177417...)$; and $(1, 1.07357...)$ and $(1, 5.92643...)$. The crossing limit cycles of X are shown in Figure 4. $\square$

4. Conclusions

We considered four classes of discontinuous piecewise differential systems formed by linear Hamiltonian systems without equilibrium points in the plane separated either by a parabola, a hyperbola, an ellipse, or three parallel lines. For each class, we provided the maximum number of crossing limit cycles that the differential systems of the class can exhibit. Furthermore, we provided examples exhibiting the maximum number of limit cycles for each class.

We characterized the maximum number of crossing limit cycles for classes of discontinuous piecewise differential systems formed by linear Hamiltonian systems without equilibrium points separated by conics, but it remains to study these maximum numbers when the separation is done by cubics, or more general algebraic curves.

Author Contributions: Conceptualization, R.B. and J.L.; methodology, J.L.; software, R.B.; validation, R.B. and J.L.; formal analysis, R.B. and J.L.; investigation, R.B. and J.L.; writing–original draft, R.B. and J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación Grant MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca Grant 2017SGR1617, and the H2020 European Research Council Grant MSCA-RISE-2017-777911.

Conflicts of Interest: The authors declare no conflict of interest.

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