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Estimates for the Differences of Certain Positive Linear Operators

Ana Maria Acu [1,*,†], Sever Hodiş [2,†] and Ioan Raşa [2,†]

1 Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, Str. Dr. I. Ratiu, No. 5-7, RO-550012 Sibiu, Romania
2 Department of Mathematics, Faculty of Automation and Computer Science, Technical University of Cluj-Napoca, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania; hodissever@gmail.com (S.H.); ioan.rasa@math.utcluj.ro (I.R.)
* Correspondence: anamaria.acu@ulbsibiu.ro
† These authors contributed equally to this work.

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Abstract: The present paper deals with estimates for differences of certain positive linear operators defined on bounded or unbounded intervals. Our approach involves Baskakov type operators, the kth order Kantorovich modification of the Baskakov operators, the discrete operators associated with Baskakov operators, Meyer–König and Zeller operators and Bleimann–Butzer–Hahn operators. Furthermore, the estimates in quantitative form of the differences of Baskakov operators and their derivatives in terms of first modulus of continuity are obtained.

Keywords: positive linear operators; estimates of differences of operators; Baskakov operators; MKZ-operators; BBH-operators; Kantorovich modifications

1. Introduction

The studies of the differences of positive linear operators has as starting point the Lupaş problem proposed in [1] and became an interesting area of research in Approximation Theory. Gonska et al. [2] gave a solution to Lupaş' problem for a more general case in terms of moduli of continuity. New results on this topic were given by Gonska et al. ([3,4]). In [5], new estimates for the differences of positive linear operators, based on some inequalities involving positive linear functionals, are established. Aral et al. [6] obtained some estimates of the differences of positive linear operators defined on unbounded intervals in terms of weighted modulus of continuity. Estimates in terms of Paltanea modulus of continuity for differences of certain well-known operators were obtained by Gupta et al. [7]. Very recently, estimates of the differences of certain positive linear operators defined on bounded intervals and their derivatives were obtained in [8]. For more details about this topic, the reader is referred to [9–11].

The present paper deals with the estimates of the differences of certain positive linear operators (defined on bounded or unbounded intervals) and their derivatives, in terms of the modulus of continuity. Our study concerns the Baskakov type operators, the kth order Kantorovich modification of the Baskakov operators and the discrete operators associated with Baskakov operators. The main reason to associate a discrete operator to an integral one is its simpler form. Using as measuring tool a K-functional an estimate of the difference between the kth order Kantorovich modification of the Baskakov operators and their associated discrete operators will be established.

Let $I \subseteq \mathbb{R}$ be an interval and H a subset of $C(I)$ containing the monomials $e_i(x) = x^i, i = 0,1,2$. Let $L : H \to C(I)$ be a positive linear operator such that $Le_0 = e_0$. Let L be of the form

$$Lf := \sum_{j=0}^{\infty} A_j(f)p_j,\ f \in H,$$

where $A_j : H \to \mathbb{R}$ are positive linear functionals, $A_j(e_0) = 1$ and $p_j \in C(I)$, $p_j \geq 0$, $\sum_{j=0}^{\infty} p_j = e_0$.

Set $b^{A_j} := A_j(e_1)$ and $\mu_i^{A_j} := A_j(e_1 - b^{A_j}e_0)^i$, $i = 0,1,2;\ j \geq 0$. The discrete operator associated with L is defined by

$$\mathcal{D} : H \to C(I),\ \mathcal{D}f := \sum_{j=0}^{\infty} f(b^{A_j})p_j,\ \ f \in H. \tag{1}$$

For more details about this topic, the reader is referred to [12–14].

The k-th order Kantorovich modification of the operators L is defined by

$$L^{(k)} := D^k \circ L \circ \mathcal{I}_k,$$

where D^k denotes the k-th order ordinary differential operator and

$$\mathcal{I}_k f = f,\ \text{if } k = 0 \text{ and } \left(\mathcal{I}_k f\right)(x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt,\ \text{if } k \in \mathbb{N}.$$

The k-th order Kantorovich modification of certain positive linear operators was introduced and studied in the papers [15–18]. In what follows $\|\cdot\|$ will stand for the supremum norm.

2. Baskakov Type Operators

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$ and $-n/c \in \mathbb{N}$ for $c < 0$. Furthermore let $I_c = [0,\infty)$ for $c \geq 0$ and $I_c = [0, -1/c]$ for $c < 0$. Consider $f : I_c \longrightarrow \mathbb{R}$ given in such a way that the corresponding integrals and series are convergent.

The Baskakov-type operators are defined as follows (see [19–21])

$$\mathcal{B}_{n,c}(f;x) = \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) f\left(\frac{j}{n}\right),$$

where

$$p_{n,j}^{[c]}(x) = \begin{cases} \dfrac{n^j}{j!} x^j e^{-nx} & , c = 0, \\ \dfrac{n^{c,\bar{j}}}{j!} x^j (1+cx)^{-\left(\frac{n}{c}+j\right)} & , c \neq 0, \end{cases} \tag{2}$$

and $a^{c,\bar{j}} := \prod_{l=0}^{j-1}(a + cl), \quad a^{c,\bar{0}} := 1$.

Denote by $V_n := \mathcal{B}_{n,1}$ the classical Baskakov operators defined as follows:

$$V_n(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) b_{n,k}(x),\ \text{where } b_{n,k}(x) := \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}},\ x \in [0,\infty).$$

The classical Szász–Mirakjan operators are Baskakov type operators with $c = 0$ defined by (see [22–24])

$$S_n(f;x) := e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f\left(\frac{j}{n}\right),\ x \in [0,\infty). \tag{3}$$

Nowdays generalizations of these operators have been studied by several authors. An important type of generalization of these operators has been considered by López-Moreno in [25] as follows

$$L_{n,s}(f;x) = \sum_{k=0}^{\infty} (-1)^s f\left(\frac{k}{n}\right) \frac{\phi_n^{(k+s)}(x)}{n^s} \frac{(-x)^k}{k!},\ x \in [0,\infty), \tag{4}$$

where $f : [0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, and the sequence (ϕ_n) of analytic functions $\phi_n : [0,\infty) \to \mathbb{R}$ verifies the conditions

(i) $\phi_n(0) = 1$, for every $n \in \mathbb{N}$;
(ii) $(-1)^k \phi_n^{(k)}(x) \geq 0$, for every $n \in \mathbb{N}$, $x \in [0,\infty)$, $k \in \mathbb{N}_0$.

The derivative of the operator $L_{n,s}$ has the form (see [25], p. 147)

$$L_{n,s}^{(r)}(f;x) = (-1)^r \sum_{k=0}^{\infty} (-1)^s \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) \frac{\phi_n^{(k+s+r)}(x)}{n^s} \frac{(-x)^k}{k!}. \tag{5}$$

Some examples of operators of the form (4) are the classical Baskakov operators and Szász–Mirakjan operators. These operators are obtained by choosing $s = 0$ and $\phi_n(x) = (1+x)^{-n}$, respectively $\phi_n(x) = e^{-nx}$.

In the following we give the estimates of the differences of Baskakov and Szász–Mirakjan operators and their derivatives.

Lemma 1. *If $t \in [0,1]$ and $r \in \mathbb{N}$, then*

$$(1+t)(1+2t)\dots(1+(r-1)t) - 1 \leq (r!-1)t.$$

Proof. For $t \in [0,1]$ and $r \in \mathbb{N}$, it follows

$$(1+t)(1+2t)\dots(1+(r-1)t) - 1 = c_1 t + c_2 t^2 + \cdots + c_{r-1} t^{r-1} \leq (c_1 + c_2 + \cdots + c_{r-1})t.$$

For $t = 1$ we get $c_1 + c_2 + \cdots + c_{r-1} = r! - 1$. □

Let $\omega(f,\delta)$ be the first order modulus of continuity and $C_b[0,\infty)$ the space of all real-valued, bounded, continuous functions on $[0,\infty)$ endowed with the supremum norm $\|\cdot\|$. Denote

$$V_n^{[r]}(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) b_{n+r,k}(x).$$

Theorem 1. *For $r \geq 0$ the Baskakov operators verify*

$$\left\| (V_n f)^{(r)} - V_n^{[r]}\left(f^{(r)}\right) \right\| \leq \frac{r!-1}{n} \|f^{(r)}\| + \omega\left(f^{(r)}, \frac{r}{n}\right),\ f^{(r)} \in C_b[0,\infty).$$

Proof. Using relation (5) the r^{th} derivative of Baskakov operators can be written as follows:

$$V_n^{(r)}(f;x) = \sum_{k=0}^{\infty} \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) \frac{(n+k+r-1)!}{(n-1)!} (1+x)^{-n-k-r} \frac{x^k}{k!}.$$

For the differences of Baskakov operators and their derivatives we obtain

$$\begin{aligned}
V_n^{(r)}(f;x) - V_n^{[r]}(f^{(r)};x) &= \sum_{k=0}^{\infty} \left\{ \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) \frac{(n+k+r-1)!}{(n-1)!} \frac{1}{k!} \frac{x^k}{(1+x)^{n+k+r}} - f^{(r)}\left(\frac{k}{n}\right) b_{n+r,k}(x) \right\} \\
&= \sum_{k=0}^{\infty} \left(\frac{(n+r-1)!}{(n-1)!} \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) - f^{(r)}\left(\frac{k}{n}\right) \right) b_{n+r,k}(x) \\
&= \sum_{k=0}^{\infty} \left(\frac{(n+r-1)!}{(n-1)!} \frac{r!}{n^r} \left[\frac{k}{n}, \dots, \frac{k+r}{n}; f\right] - f^{(r)}\left(\frac{k}{n}\right) \right) b_{n+r,k}(x) \\
&= \sum_{k=0}^{\infty} \left[\frac{n(n+1)\dots(n+r-1)}{n^r} f^{(r)}(\xi_{n,k,r}) - f^{(r)}\left(\frac{k}{n}\right) \right] b_{n+r,k}(x),
\end{aligned}$$

with $\frac{k}{n} \le \xi_{n,k,r} \le \frac{k+r}{n}$.

Therefore,

$$\begin{aligned}
\left| V_n^{(r)}(f;x) - V_n^{[r]}(f^{(r)};x) \right| \le \sum_{k=0}^{\infty} &\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{r-1}{n}\right) - 1 \right] |f^{(r)}(\xi_{n,k,r})| \\
&+ \left| f^{(r)}(\xi_{n,k,r}) - f^{(r)}\left(\frac{k}{n}\right) \right| \Bigg] .
\end{aligned}$$

Now Lemma 1 shows that

$$\|V_n^{(r)} f - V_n^{[r]} f^{(r)}\| \le \frac{r!-1}{n} \|f^{(r)}\| + \omega\left(f^{(r)}, \frac{r}{n}\right).$$

□

Theorem 2. *For $r \ge 0$ the Szász–Mirakjan operators verify*

$$\left\| (S_n f)^{(r)} - S_n\left(f^{(r)}\right) \right\| \le \omega\left(f^{(r)}, \frac{r}{n}\right), \ f^{(r)} \in C_b[0,\infty).$$

Proof. From relation (5) the derivative of Szász–Mirakjan operators can be written as

$$S_n^{(r)}(f;x) = \sum_{k=0}^{\infty} n^r \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) \frac{(xn)^k}{k!} e^{-nx}.$$

Therefore,

$$\begin{aligned}
\left| S_n^{(r)}(f;x) - S_n(f^{(r)};x) \right| &\le \sum_{k=0}^{\infty} \left| n^r \Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) - f^{(r)}\left(\frac{k}{n}\right) \right| \frac{(nx)^k}{k!} e^{-nx} \\
&= \sum_{k=0}^{\infty} \left| f^{(r)}(\xi_{n,k,r}) - f^{(r)}\left(\frac{k}{n}\right) \right| \frac{(nx)^k}{k!} e^{-nx},
\end{aligned}$$

where $\frac{k}{n} \le \xi_{n,k,r} \le \frac{k+r}{n}$. Using the above relation the theorem is proved. □

A similar result can be obtained for the operators $L_{n,s}$ introduced by López-Moreno in [25].

Theorem 3. *For $r \ge 0$ the positive linear operators $L_{n,s}$ verify*

$$\left\| (L_{n,s} f)^{(r)} - L_{n,s+r}\left(f^{(r)}\right) \right\| \le \left(1 + \mathcal{O}(n^{-1})\right) \omega\left(f^{(r)}, \frac{r}{n}\right), \ f^{(r)} \in C_b[0,\infty).$$

Proof. We have

$$\begin{aligned}|L_{n,s}^{(r)}(f;x)-L_{n,s+r}(f^{(r)};x)| &\le \sum_{k=0}^{\infty}\frac{(-1)^{r+s+k}\phi_n^{(k+s+r)}(x)}{n^{s+r}}\frac{x^k}{k!}\left|n^r\Delta_{\frac{1}{n}}^r f\left(\frac{k}{n}\right)-f^{(r)}\left(\frac{k}{n}\right)\right| \\ &= \sum_{k=0}^{\infty}\frac{(-1)^{r+s+k}\phi_n^{(k+s+r)}(x)}{n^{s+r}}\frac{x^k}{k!}\left|f^{(r)}(\xi_{n,k,r})-f^{(r)}\left(\frac{k}{n}\right)\right| \\ &\le \omega\left(f^{(r)};\frac{r}{n}\right)\sum_{k=0}^{\infty}\frac{(-1)^{r+s+k}\phi_n^{(k+s+r)}(x)}{n^{s+r}}\frac{x^k}{k!},\end{aligned}$$

where $\dfrac{k}{n}\le\xi_{n,k,r}\le\dfrac{k+r}{n}$.

Since $L_{n,s}(1;x)=1+\mathcal{O}(n^{-1})$ (see [25], Lemma 2) we get

$$\left\|(L_{n,s}f)^{(r)}-L_{n,s+r}\left(f^{(r)}\right)\right\|\le\left(1+\mathcal{O}(n^{-1})\right)\omega\left(f^{(r)},\frac{r}{n}\right),\ f^{(r)}\in C_b[0,\infty),\ r\ge 0.$$

□

3. The kth Order Kantorovich Modification of the Baskakov Operators

The k-th order Kantorovich modifications of the operators $\mathcal{B}_{n,c}$ are defined by

$$\mathcal{B}_{n,c}^{(k)}:=D^k\circ\mathcal{B}_{n,c}\circ\mathcal{I}_k.$$

For $n>(k+1)c$ denote

$$\mathcal{K}_{n,c}^{(k)}:=\frac{(n-ck)^k}{(n-ck)^{c,\overline{k}}}D^k\mathcal{B}_{n-ck,c}\mathcal{I}_k.$$

Let $n,c,k\ge 0$, $n>(k+1)c$, be fixed. Using the well known representation of $B_{n,c}^{(k)}$ (see [20]) we can write

$$\begin{aligned}\mathcal{K}_{n,c}^{(k)}(f;x) &= \sum_{j=0}^{\infty}k!p_{n,j}^{[c]}(x)\left[\frac{j}{n-ck},\frac{j+1}{n-ck},\dots,\frac{j+k}{n-ck};\mathcal{I}_kf\right] \\ &= \sum_{j=0}^{\infty}p_{n,j}^{[c]}(x)f(\xi_j),\ \ \frac{j}{n-ck}<\xi_j<\frac{j+k}{n-ck}.\end{aligned}\tag{6}$$

The domain of $\mathcal{K}_{n,c}^{(k)}$ is a linear subspace $H_{n,c}^{(k)}$ of $C[0,\infty)$ if $c\ge 0$, or $C\,[0,-1/c]$ if $c<0$, containing the polynomial functions. For $j\ge 0$ and $f\in H_{n,c}^{(k)}$ let

$$\begin{aligned}F_j(f) &= k!\left[\frac{j}{n-ck},\frac{j+1}{n-ck},\dots,\frac{j+k}{n-ck};\mathcal{I}_kf\right], \\ G_j(f) &= f\left(\frac{2j+k}{2(n-ck)}\right).\end{aligned}$$

The discrete operators (1) associated with $\mathcal{K}_{n,c}^{(k)}$ are given by

$$\mathcal{D}_{n,c}^{(k)}(f;x)=\sum_{j=0}^{\infty}p_{n,j}^{[c]}(x)G_j(f).$$

In order to estimate the difference between $\mathcal{K}_{n,c}^{(k)}$ and $\mathcal{D}_{n,c}^{(k)}$ we use as measuring tool the K-functional (see [26,27])

$$K_2(f;\lambda)=\inf\left\{\|f-g\|+\lambda\|g''\|;g\in X\right\},\ \lambda>0,\ f\in C_b[0,\infty),$$

where $X := \{g \in C_b[0,\infty) |$ there exists $g'' \in C_b[0,\infty)\}$.

Theorem 4. *Let* $f \in H^{(k)}_{n,c} \cap C_b[0,\infty)$. *Then*

$$\|\mathcal{K}^{(k)}_{n,c} f - \mathcal{D}^{(k)}_{n,c} f\| \le 2K_2\left(f; \frac{k}{48(n-ck)^2}\right). \tag{7}$$

Proof. We have

$$\begin{aligned}
b^{F_j} &= F_j(e_1) = k!\left[\frac{j}{n-ck}, \frac{j+1}{n-ck}, \dots, \frac{j+k}{n-ck}; \frac{e_{k+1}}{(k+1)!}\right] = \frac{1}{k+1}\sum_{i=0}^{k}\frac{j+i}{n-ck} = \frac{2j+k}{2(n-ck)},\\
b^{G_j} &= \frac{2j+k}{2(n-ck)} = b^{F_j},\\
F_j(e_2) &= k!\left[\frac{j}{n-ck}, \frac{j+1}{n-ck}, \dots, \frac{j+k}{n-ck}; \frac{2e_{k+2}}{(k+2)!}\right]\\
&= \frac{2}{(k+1)(k+2)(n-ck)^2}\left[\sum_{i=0}^{k}(j+i)^2 + \sum_{i=0}^{k-1}\sum_{l=i+1}^{k}(j+i)(j+l)\right]\\
&= \frac{1}{12(n-ck)^2}(12j^2 + 12jk + 3k^2 + k).
\end{aligned}$$

From the above relations we get

$$\mu_2^{F_j} = \frac{k}{12(n-ck)^2}, \; j \ge 0. \tag{8}$$

Then

$$\left|\mathcal{K}^{(k)}_{n,c}(f;x) - \mathcal{D}^{(k)}_{n,c}(f;x)\right| \le \sum_{j=0}^{\infty}\left|F_j(f) - G_j(f)\right| p^{[c]}_{n,j}(x) = \sum_{j=0}^{\infty}\left|F_j(f) - f(b^{F_j})\right| p^{[c]}_{n,j}(x). \tag{9}$$

For $g \in X$ and $j \ge 0$ we have by Taylor expansion

$$\left|g(t) - g(b^{F_j}) - g'(b^{F_j})(t - b^{F_j})\right| \le \frac{1}{2}\|g''\|(t - b^{F_j})^2, \; t \ge 0.$$

Applying the functional F_j we get

$$|F_j(g) - g(b^{F_j})| \le \frac{1}{2}\|g''\|\mu_2^{F_j}.$$

Combined with (8) and (9), this leads to

$$\left\|\mathcal{K}^{(k)}_{n,c} g - \mathcal{D}^{(k)}_{n,c} g\right\| \le \|g''\|\frac{k}{24(n-ck)^2}, \; g \in X.$$

Furthermore,

$$\begin{aligned}
\|\mathcal{K}^{(k)}_{n,c} f - \mathcal{D}^{(k)}_{n,c} f\| &\le \|\mathcal{K}^{(k)}_{n,c} f - \mathcal{K}^{(k)}_{n,c} g\| + \|\mathcal{K}^{(k)}_{n,c} g - \mathcal{D}^{(k)}_{n,c} g\| + \|\mathcal{D}^{(k)}_{n,c} g - \mathcal{D}^{(k)}_{n,c} f\|\\
&\le 2\|f - g\| + \|g''\|\frac{k}{24(n-ck)^2}.
\end{aligned}$$

Taking the infimum over $g \in X$ we get (7).

□

The estimates of the differences between the Baskakov operators $\mathcal{B}_{n,c}$ and the k-th Kantorovich modification of Baskakov operators $\mathcal{K}_{n,c}^{(k)}$, respectively the discrete operators associated with $\mathcal{K}_{n,c}^{(k)}$, in terms of the first order modulus of continuity will be enumerate in the next results.

Proposition 1. *Let $c = 0$ and $f \in H_{n,0}^{(k)}$. Then*

(i) $\|S_n f - \mathcal{K}_{n,0}^{(k)} f\| \leq \omega\left(f; \frac{k}{n}\right)$,

(ii) $\|S_n f - \mathcal{D}_{n,0}^{(k)} f\| \leq \omega\left(f; \frac{k}{2n}\right)$.

Proof. For fixed n, k and $c = 0$ we have according to (6)

$$p_{n,j}(x) = \frac{n^j}{j!} x^j e^{-nx},$$

$$\mathcal{K}_{n,0}^{(k)}(f; x) = \sum_{j=0}^{\infty} p_{n,j}^{[0]}(x) f(\xi_j), \ \frac{j}{n} < \xi_j < \frac{j+k}{n},$$

$$\mathcal{D}_{n,0}^{(k)}(f; x) = \sum_{j=0}^{\infty} p_{n,j}^{[0]}(x) f\left(\frac{2j+k}{2n}\right).$$

Combined with (3), these relations prove Proposition 1. □

Proposition 2. *Let $f \in C\,[0, -1/c]$ and $c < 0$. Then*

(i) $\|\mathcal{B}_{n,c} f - \mathcal{K}_{n,c}^{(k)} f\| \leq \omega\left(f; \frac{k}{n-ck}\right)$,

(ii) $\|\mathcal{B}_{n,c} f - \mathcal{D}_{n,c}^{(k)} f\| \leq \omega\left(f; \frac{k}{2(n-ck)}\right)$.

Proof. For $c < 0$ we have $c = -\frac{n}{l}$, $l \in \mathbb{N}$, and

$$p_{n,j}(x) = \left(1 - \frac{n}{l}x\right)^{l-j} x^j \frac{n^j}{l^j} \binom{l}{j},$$

$$\mathcal{B}_{n,c}(f; x) = \sum_{j=0}^{l} p_{n,j}^{[c]}(x) f\left(\frac{j}{n}\right),$$

$$\mathcal{K}_{n,c}^{(k)}(f; x) = \sum_{j=0}^{l} p_{n,j}^{[c]}(x) f(\xi_j), \quad \frac{j}{n-ck} < \xi_j < \frac{j+k}{n-ck},$$

$$\mathcal{D}_{n,c}^{(k)}(f; x) = \sum_{j=0}^{l} p_{n,j}^{[c]}(x) f\left(\frac{2j+k}{2(n-ck)}\right).$$

Using the above relations the proposition is proved. □

Proposition 3. *Let $c > 0$, $f \in H_{n,c}^{(k)}$ and $f' \in C_b[0, \infty)$. Then*

(i) $\|\mathcal{B}_{n,c} f - \mathcal{K}_{n,c}^{(k)} f\| \leq \frac{k(1+cx)}{n-ck} \|f'\|$,

(ii) $\|\mathcal{B}_{n,c} f - \mathcal{D}_{n,c}^{(k)} f\| \leq \frac{k(1+2cx)}{2(n-ck)} \|f'\|$.

Proof. We have $\mathcal{K}_{n,c}^{(k)}(f;x) = \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) f(\xi_j)$, $\frac{j}{n-ck} < \xi_j < \frac{j+k}{n-ck}$. Therefore,

$$\begin{aligned}\left|\mathcal{K}_{n,c}^{(k)}(f;x) - \mathcal{B}_{n,c}(f;x)\right| &\leq \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left|f(\xi_j) - f\left(\frac{j}{n}\right)\right| \leq \|f'\| \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left|\xi_j - \frac{j}{n}\right| \\ &\leq \|f'\| \| \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left(\frac{j+k}{n-ck} - \frac{j}{n}\right) = \|f'\| \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left(\frac{k}{n-ck} + \frac{ck}{n-ck}\frac{j}{n}\right) \\ &= \|f'\| \left(\frac{k}{n-ck} + \frac{ck}{n-ck}x\right) = \|f'\| \frac{k(1+cx)}{n-ck},\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{B}_{n,c}(f;x) - \mathcal{D}_{n,c}^{(k)}(f;x)\right| &\leq \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left|f\left(\frac{j}{n}\right) - f\left(\frac{2j+k}{2(n-ck)}\right)\right| \leq \|f'\| \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left|\frac{j}{n} - \frac{2j+k}{2(n-ck)}\right| \\ &= \|f'\| \sum_{j=0}^{\infty} p_{n,j}^{[c]}(x) \left(\frac{k}{2(n-ck)} + \frac{ck}{n-ck}\frac{j}{n}\right) \\ &= \|f'\| \left(\frac{k}{2(n-ck)} + \frac{ckx}{n-ck}\right) = \|f'\| \frac{k(1+2cx)}{2(n-ck)}.\end{aligned}$$

□

4. The Meyer–König and Zeller Operators

Meyer–König and Zeller [28] introduced the operators defined for $f \in C[0,1]$ as follows

$$\mathcal{M}_n(f;x) = \begin{cases} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k (1-x)^{n+1} f\left(\frac{k}{n+k}\right), \; x \in [0,1), \\ f(1), x = 1. \end{cases}$$

Let $\hat{\mathcal{M}}_n := D \circ \mathcal{M}_n \circ \mathcal{I}_1$ be the Kantorovich modification of the MKZ-operators ([29]). Denote $I_{n,k} := \left[\frac{k}{k+n}, \frac{k+1}{k+n+1}\right]$. For the operator $\hat{\mathcal{M}}_n$ the following explicit form can be obtained:

$$\hat{\mathcal{M}}_n(f;x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} (1-x)^n x^k \frac{(n+k)(n+k+1)}{n} \int_{I_{n,k}} f(t)dt.$$

Indeed,

$$\begin{aligned}\hat{\mathcal{M}}_n(f;x) &= \sum_{k=0}^{\infty} k(1-x)^{n+1} x^{k-1} \binom{n+k}{k} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right) - \sum_{k=0}^{\infty} (n+1)(1-x)^n x^k \binom{n+k}{k} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right) \\ &= \sum_{k=1}^{\infty} (n+1)(1-x)^{n+1} x^{k-1} \binom{n+k}{k-1} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right) - \sum_{k=0}^{\infty} (n+1)(1-x)^n x^k \binom{n+k}{k} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right) \\ &= \sum_{k=0}^{\infty} (n+1)(1-x)^n x^k \binom{n+k+1}{k} (\mathcal{I}_1 f)\left(\frac{k+1}{n+k+1}\right) \\ &- \sum_{k=0}^{\infty} (n+1)(1-x)^n x^{k+1} \binom{n+k+1}{k} (\mathcal{I}_1 f)\left(\frac{k+1}{n+k+1}\right) - \sum_{k=0}^{\infty} (n+1)(1-x)^n x^k \binom{n+k}{k} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right) \\ &= \sum_{k=0}^{\infty} (n+1)(1-x)^n x^k \binom{n+k+1}{k} (\mathcal{I}_1 f)\left(\frac{k+1}{n+k+1}\right)\end{aligned}$$

$$-\sum_{k=0}^{\infty}(n+1)(1-x)^n x^k \binom{n+k}{k-1} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right) - \sum_{k=0}^{\infty}(n+1)(1-x)^n x^k \binom{n+k}{k} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right)$$
$$= \sum_{k=0}^{\infty}(n+1)(1-x)^n x^k \binom{n+k+1}{k} (\mathcal{I}_1 f)\left(\frac{k+1}{n+k+1}\right)$$
$$- \sum_{k=0}^{\infty}(n+1)(1-x)^n x^k \binom{n+k+1}{k} (\mathcal{I}_1 f)\left(\frac{k}{n+k}\right)$$
$$= (n+1)\sum_{k=0}^{\infty}\binom{k+n+1}{k}(1-x)^n x^k \int_{I_{n,k}} f(t)dt$$
$$= \sum_{k=0}^{\infty}\binom{n+k-1}{k}(1-x)^n x^k \frac{(n+k)(n+k+1)}{n} \int_{I_{n,k}} f(t)dt.$$

The discrete operators (1) associated with $\hat{\mathcal{M}}_n$ are given by

$$\mathcal{D}_n(f;x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}(1-x)^n x^k f(u_{n,k}), \text{ where } u_{n,k} = \frac{1}{2}\left(\frac{k}{k+n} + \frac{k+1}{k+n+1}\right).$$

Theorem 5. *Let $f \in C[0,1]$. Then*

(i) $\|\hat{\mathcal{M}}_n f - \mathcal{M}_{n-1} f\| \le 2\omega\left(f; \frac{1}{n}\right)$,

(ii) $\|\mathcal{M}_{n-1} f - \mathcal{D}_n f\| \le \omega\left(f; \frac{1}{n}\right)$,

(iii) $\|\hat{\mathcal{M}}_n f - \mathcal{D}_n f\| \le \omega\left(f; \frac{1}{n}\right)$.

Proof. (i) Let $\mathcal{A}_{n-1}(f;x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)(1-x)^n x^k \binom{n+k-1}{k}$. We have

$$\begin{aligned}\left|\hat{\mathcal{M}}_n(f;x) - \mathcal{M}_{n-1}(f;x)\right| &\le \left|\hat{\mathcal{M}}_n(f;x) - \mathcal{A}_{n-1}(f;x)\right| + \left|\mathcal{A}_{n-1}(f;x) - \mathcal{M}_{n-1}(f;x)\right| \\ &\le \sum_{k=0}^{\infty}\binom{n+k-1}{k}(1-x)^n x^k \left| f(\xi_{n,k}) - f\left(\frac{k}{n+k}\right)\right| \\ &+ \sum_{k=0}^{\infty}\binom{n+k-1}{k}(1-x)^n x^k \left| f\left(\frac{k}{n+k}\right) - f\left(\frac{k}{n+k-1}\right)\right|,\end{aligned}$$

where $\frac{k}{k+n} < \xi_{n,k} < \frac{k+1}{n+k+1}$. Since

$$\left| f(\xi_{n,k}) - f\left(\frac{k}{n+k}\right)\right| \le \omega\left(f; \frac{1}{n}\right),$$
$$\left| f\left(\frac{k}{n+k}\right) - f\left(\frac{k}{n+k-1}\right)\right| \le \omega\left(f; \frac{1}{n}\right),$$

we get $\|\hat{\mathcal{M}}_n f - \mathcal{M}_{n-1} f\| \le 2\omega\left(f; \frac{1}{n}\right)$.

(ii) Using

$$\left|u_{n,k}-\frac{k}{n+k-1}\right| = \left|\frac{1}{2}\left(\frac{k}{k+n}-\frac{k}{n+k-1}\right)+\frac{1}{2}\left(\frac{k+1}{k+n+1}-\frac{k}{n+k-1}\right)\right|$$
$$\leq \frac{1}{2}\frac{k}{(n+k)(n+k-1)}+\frac{1}{2}\frac{|n-1-k|}{(n-1+k)}\frac{1}{n+k+1}$$
$$\leq \frac{1}{2}\left(\frac{1}{n+k}+\frac{1}{n+k+1}\right)\leq\frac{1}{n},$$

we obtain $\|\mathcal{M}_{n-1}f-\mathcal{D}_nf\|\leq\omega\left(f;\frac{1}{n}\right)$.

In a similar way one can prove (iii). □

5. The BBH Operators

Bleimann, Butzer and Hahn [30] introduced the positive linear operator defined as follows:

$$\mathcal{L}_n(f;x)=\sum_{k=0}^{n}f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k(1+x)^{-n},\ x\in[0,\infty),\ f\in C[0,n].$$

Let $\hat{\mathcal{L}}_{n-1}:=\mathcal{D}\circ\mathcal{L}_n\circ\mathcal{I}_1$ be the Kantorovich modification of the BBH-operators. For the operator $\hat{\mathcal{L}}_{n-1}$ the following explicit form can be obtained:

$$\hat{\mathcal{L}}_{n-1}(f;x)=\sum_{k=0}^{n-1}n\binom{n-1}{k}x^k(1+x)^{-n-1}\int_{\frac{k}{n-k+1}}^{\frac{k+1}{n-k}}f(t)dt,\ f\in C[0,n].$$

Indeed,

$$\hat{\mathcal{L}}_{n-1}(f;x)=\sum_{k=0}^{n}\binom{n}{k}\left[kx^{k-1}(1+x)^{-n}-nx^k(1+x)^{-n-1}\right](\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)$$
$$=\sum_{k=1}^{n}n\binom{n-1}{k-1}x^{k-1}(1+x)^{-n}(\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)-\sum_{k=0}^{n}n\binom{n}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)$$
$$=\sum_{k=0}^{n-1}n\binom{n-1}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k+1}{n-k}\right)+\sum_{k=0}^{n-1}n\binom{n-1}{k}x^{k+1}(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k+1}{n-k}\right)$$
$$-\sum_{k=1}^{n}n\binom{n}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)$$
$$=\sum_{k=0}^{n-1}n\binom{n-1}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k+1}{n-k}\right)+\sum_{k=1}^{n-1}n\binom{n-1}{k-1}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)$$
$$-\sum_{k=1}^{n-1}n\binom{n}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)$$
$$=\sum_{k=0}^{n-1}n\binom{n-1}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k+1}{n-k}\right)-\sum_{k=0}^{n-1}n\binom{n-1}{k}x^k(1+x)^{-n-1}(\mathcal{I}_1f)\left(\frac{k}{n-k+1}\right)$$
$$=\sum_{k=0}^{n-1}n\binom{n-1}{k}x^k(1+x)^{-n-1}\int_{\frac{k}{n-k+1}}^{\frac{k+1}{n-k}}f(t)dt.$$

Denote $I_{n,k}=\left[\frac{k}{n+2-k},\frac{k+1}{n+1-k}\right]$, $|I_{n,k}|=\dfrac{n+2}{(n+1-k)(n+2-k)}$. Then,

$$\hat{\mathcal{L}}_n(f;x)=\frac{1}{(1+x)^2}\sum_{k=0}^{n}\binom{n+2}{k}x^k(1+x)^{-n}\left|I_{n,k}\right|^{-1}\int_{I_{n,k}}f(t)dt,\ f\in C[0,n+1] \tag{10}$$

Let $\int_{I_{n,k}} f = T_{n,k}(f) + R_{n,k}(f)$ be the trapezoidal quadrature formula on $I_{n,k}$, based on $m = \lfloor \sqrt{n}|I_{n,k}| \rfloor + 1$ knots, where

$$T_{n,k}(f) = \frac{|I_{n,k}|}{2m}\left[f\left(\frac{k}{n-k+2}\right) + 2\sum_{i=1}^{m-1} f\left(\frac{k}{n-k+2} + i\frac{|I_{n,k}|}{m}\right) + f\left(\frac{k+1}{n-k+1}\right)\right],$$

$$|R_{n,k}(f)| \le \frac{|I_{n,k}|^3\|f''\|}{12\left(\lfloor \sqrt{n}|I_{n,k}| \rfloor + 1\right)^2} \le \frac{|I_{n,k}|\|f''\|}{12n},$$

and $\lfloor x \rfloor$ is the integer part of x.

If in (10) the integral is replaced by its approximation $T_{n,k}(f)$ from the trapezoidal quadrature formula, we get

$$\mathcal{A}_n(f;x) := \frac{1}{(1+x)^2}\sum_{k=0}^{n}\binom{n+2}{k}x^k(1+x)^{-n}\left|I_{n,k}\right|^{-1} T_{n,k}(f).$$

Proposition 4. *The BBH operators $\mathcal{L}_n$ verify:*

$$|\mathcal{L}_n(f;x) - \mathcal{A}_n(f;x)| \le \frac{\|f''\|}{12n},\ f \in C^2[0,n+1].$$

Proof. We get

$$\begin{aligned}|\mathcal{L}_n(f;x) - \mathcal{A}_n(f;x)| &\le \sum_{k=0}^{n}\binom{n+2}{k}x^k(1+x)^{-n-2}\left|I_{n,k}\right|^{-1}|R_{n,k}(f)| \\ &\le \sum_{k=0}^{n}\binom{n+2}{k}x^k(1+x)^{-n-2}\frac{\|f''\|}{12n} \le \frac{\|f''\|}{12n}.\end{aligned}$$

□

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