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Endomorphism Spectra of Double Fan Graphs

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Abstract: There are six different classes of endomorphisms for a graph. The sets of these endomorphisms always form a chain under the inclusion of sets. For a more systematic treatment of different endomorphisms, Böttcher and Knauer proposed the concepts of the endomorphism type and the endomorphism spectrum of a graph in 1992. In this paper, we studied endomorphism types and endomorphism spectra of double fan graphs.

Keywords: endomorphism spectrum; endomorphism type; double fan graph

MSC: 05C25; 20M20

1. Introduction

More and more scholars paid attention to monoids of graphs during the last few years and many important results concerning endomorphism monoids of graphs have been attained ([1–6]). In order to study six classes of endomorphisms of a graph more systematically, Böttcher and Knauer proposed the concepts of the endomorphism type and the endomorphism spectrum in [7]. In [8], Fan explored endomorphism spectra of bipartite graphs with diameter three and girth six. In [9] various endomorphisms of generalized polygons were explored and endomorphism types of generalized polygons were provided. In [10] Hou, Luo, and Cheng characterized the endomorphism monoid of $\overline{P_n}$. The endomorphism type and the endomorphism spectrum of $\overline{P_n}$ were obtained. In [11] Hou, Gu, and Song explored six classes of endomorphisms of fan graphs. Endomorphism types and endomorphism spectra of these graphs were given. In this paper, we explore endomorphism types and endomorphisms spectra of double fan graphs.

2. Preliminary Concepts

Throughout this paper, all graphs are assumed to be finite, undirected, and simple. For a graph X, denote by $V(X)$ and $E(X)$ the vertex set and edge set of X, respectively. Let $v \in V(X)$. Denote $N(v) = \{x \in V(X) | \{x, v\} \in E(X)\}$. Let f be a mapping on $V(X)$. If $\{x_1, x_2\} \in E(X)$ implies that $\{f(x_1), f(x_2)\} \in E(X)$, then f is known as an *endomorphism* of X. If $\{f(a), f(b)\} \in E(X)$ implies that there exist $x_1, x_2 \in V(X)$ with $f(a) = f(x_1)$ and $f(b) = f(x_2)$ such that $\{x_1, x_2\} \in E(X)$, then f is said to be *half-strong*. If $\{f(a), f(b)\} \in E(X)$ implies that the subgraph of X induced by $f^{-1}(f(a)) \cup f^{-1}(f(b))$ has no isolated vertex, then f is said to be *locally strong*. If $\{f(a), f(b)\} \in E(X)$ implies that there exists $x_1 \in f^{-1}(f(a))$ which is adjacent to every vertex of $f^{-1}(f(b))$ and analogously for preimage of $f(b)$, then f is said to be *quasi-strong*. If $\{f(a), f(b)\} \in E(X)$ implies that the subgraph of X induced by $f^{-1}(f(a)) \cup f^{-1}(f(b))$ is complete bipartite, then f is said to be *strong*. If f is bijective, then f is known as an *automorphism* of X. Denote the set of automorphisms, strong endomorphisms, quasi-strong endomorphisms, locally strong endomorphisms, half-strong endomorphisms, and endomorphisms for a graph X by $Aut(X)$, $sEnd(X)$, $qEnd(X)$, $lEnd(X)$, $hEnd(X)$, and $End(X)$, respectively. Clearly

$$End(X) \supseteq hEnd(X) \supseteq lEnd(X) \supseteq qEnd(X) \supseteq sEnd(X) \supseteq Aut(X).$$

The 6-tuple

$$(|End(X)|, |hEnd(X)|, |lEnd(X)|, |qEnd(X)|, |sEnd(X)|, |Aut(X)|)$$

is known as the *endomorphism spectrum* of X and we denote it by $EndospecX$. Let $s_i \in \{0,1\}$, $i = 1,2,3,4,5$, where $s_i = 0$ if and only if the ith element equals to the $(i+1)$th element in $EndospecX$ and $s_i = 1$ otherwise. The integer $\sum_{i=1}^{5} s_i 2^{i-1}$ is known as the *endomorphism type* of X and we denote it by $EndotypeX$.

Let $f \in End(X)$ and $A \subseteq V(X)$. Denote by $f|_A$ the restriction of f on A. The *endomorphic image* I_f of X under f is a subgraph of X with $V(I_f) = f(V(X))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $s \in f^{-1}(f(a))$ and $t \in f^{-1}(f(b))$ such that $\{s,t\} \in E(X)$. The *endomorphic kernel* ρ_f induced by f is an equivalence relation on $V(X)$ such that for $a, b \in V(X)$, $(a,b) \in \rho_f$ if and only if $f(a) = f(b)$.

Example 1. *Let DF_4 be a double fan graph, as shown in Figure 1. Let*

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & s & t \\ s & 1 & t & 3 & 2 & 2 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & s & t \\ 3 & 2 & 3 & 4 & s & t \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & s & t \\ 1 & 2 & 1 & 2 & s & t \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & s & t \\ 1 & 2 & 3 & 4 & s & s \end{pmatrix},$$

Then it is easy to check that $f_1 \in End(DF_4) \setminus hEnd(DF_4)$, $f_2 \in hEnd(DF_4) \setminus lEnd(DF_4)$, $f_3 \in lEnd(DF_4) \setminus qEnd(DF_4)$ and $f_4 \in sEnd(DF_4) \setminus Aut(DF_4)$,

We refer the reader to [12–15] for all the concepts of semigroup theory and graph theory not defined here.

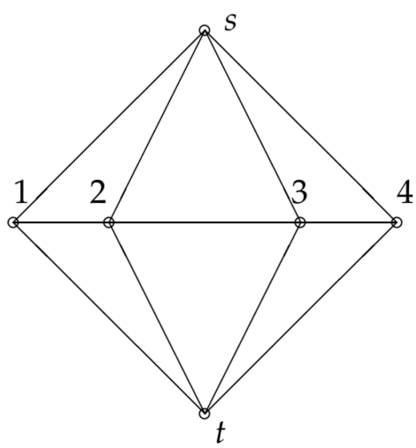

Figure 1. Graph DF_4.

3. Endomorphism Spectra of Double Fan Graphs

Let DF_n be a double fan graph as shown in Figure 2 and $A = \{1, 2, \ldots, n\}$. Denote by $B = \{1, 3, 5, \ldots\}$ the subset of A containing all the odd numbers and by $C = \{2, 4, 6 \ldots\}$ the subset of A containing all the even numbers.

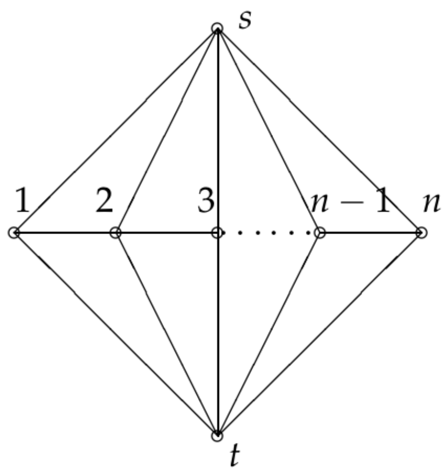

Figure 2. Graph DF_n.

Lemma 1. *([16]) Let X be a graph and* $f \in End(X)$*. Then* $f \in hEnd(X)$ *if and only if* I_f *is an induced subgraph of X.*

Lemma 2. *Let* $f \in End(DF_n)$*. Then* $f(s), f(t) \in \{s,t\}$*, or* $f(s), f(t) \notin \{s,t\}$*.*
(1) If $f(s), f(t) \in \{s,t\}$*, then* $f|_A \in End(P_n)$*.*
(2) If $f(s) = f(t) = i$ *for some* $i \in A$*, then* $I_f \cong K_3$*, or* $I_f \cong F_3$*, or* $I_f \cong DF_3$*.*
(3) If $f(s) = i$ *and* $f(t) = j$ *for some* $i, j \in A$*, then* $|i-j| = 2$*,* $I_f \cong F_3$ *or* $I_f \cong DF_3$*.*

Proof. (1) Let $f \in End(DF_n)$ and $f(s), f(t) \in \{s,t\}$. Then $f(A) \subseteq A$. Since the subgraph of DF_n induced by A is P_n, $f|_A \in End(P_n)$.

(2) If $f(s), f(t) \notin \{s,t\}$, there are three cases:

Case 1. $f(s) = f(t) = i$ for some $i \in A$. Since s and t are adjacent to all vertices of A, $f(s)$ and $f(t)$ are adjacent to all vertices of $f(A)$. Note that i is adjacent to at most 4 vertices. If $\left|V(I_f)\right| = 3$, then $I_f \cong K_3$. If $\left|V(I_f)\right| = 4$, then $I_f \cong F_3$. If $\left|V(I_f)\right| = 5$, then $I_f \cong DF_3$.

Case 2. $f(s) = i$ and $f(t) = j$ for some $i, j \in A$. Since s and t are adjacent to all vertices of A, i and j are adjacent to all vertices of $f(A)$. Thus $|i-j| = 2$. If $f(A)$ has two vertices, then $I_f \cong F_3$. If $f(A)$ has three vertices, then $I_f \cong DF_3$. Thus (3) is proved.

Case 3. $f(s) \notin \{s,t\}$ and $f(t) \in \{s,t\}$. Without loss of generalization, suppose that $f(s) = i$ for some $i \in A$ and $f(t) = t$. Since t is adjacent to all vertices of A and $N(t) = A$, $f(A) \subseteq A$. Now $f(A)$ has two adjacent vertices $f(x)$ and $f(y)$. As s is adjacent to all vertices of A, i is adjacent to all vertices of $f(A)$. In particular, i is adjacent to $f(x)$ and $f(y)$. Thus the subgraph of DF_n induced by $\{f(x), f(y), i\}$ is isomorphic to K_3. This is impossible since the subgraph of DF_n induced by A is a path. □

Lemma 3. *Let* DF_n *be a double fan graph. Then* $End(DF_n) = hEnd(DF_n)$ *if and only if* $n \leq 3$*.*

Proof. Necessity. We only need to prove that $End(DF_n) \neq hEnd(DF_n)$ for any $n \geq 4$. Define a mapping f on $V(DF_n)$ by

$$f(x) = \begin{cases} s, & x = 1, \\ 1, & x = 2, \\ t, & x \in \{3,5,7,9,\ldots\}, \\ 3, & x \in \{4,6,8,10,\ldots\}, \\ 2, & x \in \{s,t\}. \end{cases}$$

Then $f \in End(DF_n)$ and $\{s, 3\} \in E(DF_n)$. Note that $f^{-1}(s) = 1$, $f^{-1}(3) \in \{4, 6, 8, 10 \ldots\}$ and $\{u, v\} \notin E(DF_n)$ for any $u \in f^{-1}(s)$ and $v \in f^{-1}(3)$. Then $f \notin hEnd(DF_n)$.

Sufficiency. If $n = 2$, there are only two vertices s and t which are not adjacent in $V(DF_n)$. Let $f \in End(DF_n) \setminus Aut(DF_n)$. Then $f(s) = f(t)$. Note that $N(s) = N(t)$. Then $f \in sEnd(DF_n)$. If $n = 3$, then there are two pairs of vertices in $V(DF_n)$, which are not adjacent. They are 1 and 3, s and t. Let $f \in End(DF_n)$. If $f(i) = f(j)$, then $i = 1$ and $j = 3$, or $i = s$ and $j = t$. Note that $N(1) = N(3)$ and $N(s) = N(t)$. Thus $End(DF_3) = sEnd(DF_3)$. Therefore $End(DF_n) = hEnd(DF_n)$ for $n = 2, 3$. □

Lemma 4. *Let DF_n be a double fan graph with $n \geq 4$, as shown in Figure 3. If n is odd, then* $|End(DF_n)| = 4(n+1)2^{n-1} - 4(2n-1)\binom{n-1}{\frac{n-1}{2}} + 16n - 24 + (10n-12)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4) + 8(n-2)\left(2^{\frac{n+1}{2}} - 2\right)\left(2^{\frac{n-1}{2}} - 2\right)$*; If n is even, then* $|End(DF_n)| = 4(n+1)2^{n-1} - 4n\binom{n}{\frac{n}{2}} + 16n - 24 + (16n-24)(2^{\frac{n}{2}} - 2) + 8(n-2)(2^{\frac{n}{2}} - 2)^2$.

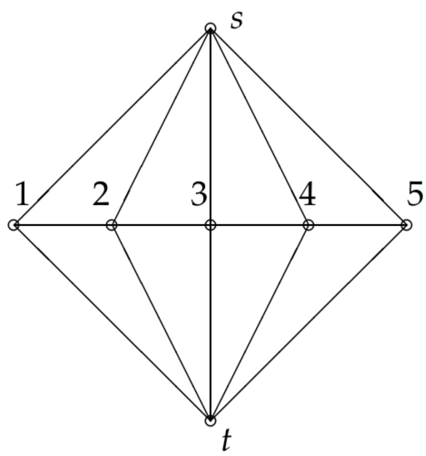

Figure 3. Graph DF_5.

Proof. Let $f \in End(DF_n)$. By Lemma 2, $f(s), f(t) \in \{s, t\}$ or $f(s), f(t) \notin \{s, t\}$.

(1) Suppose that $f(s), f(t) \in \{s, t\}$. By Lemma 2 (1), $f|_A \in End(P_n)$. Thus the number of endomorphisms of DF_n such that $f(s) = s$ and $f(t) = t$ is equal to $|End(P_n)|$. A similar argument will show that the number of endomorphisms is equal to $|End(P_n)|$ for the case of $f(s) = t$ and $f(t) = s$, $f(s) = f(t) = s$ and $f(s) = f(t) = t$. Hence there are $4|End(P_n)|$ endomorphisms in this case. It is known from [17] that

$$|End(P_n)| = \begin{cases} (n+1)2^{n-1} - (2n-1)\binom{n-1}{\frac{n-1}{2}}, & if\ n\ is\ odd, \\ (n+1)2^{n-1} - n\binom{n}{\frac{n}{2}}, & if\ n\ is\ even. \end{cases}$$

(2) Suppose that $f(s), f(t) \notin \{s, t\}$. By Lemma 2, there are three cases:

Case 1. Assume that $I_f \cong K_3$. Then $|f(A)| = 2$ and $\rho_f = \{B, C, [s, t]\}$. There are $2(n-1)$ subgraphs in DF_n isomorphic to K_3. Select a subgraph X of DF_n such that $X \cong K_3$. Without loss of generality, let $V(X) = \{i, j, k\}$, where $i, j \in A$ and $k \in \{s, t\}$. Since $f(s), f(t) \notin \{s, t\}$, $f(s) = i$ or $f(s) = j$. At the same time, f can map $\{B, C\}$ to $V(X) \setminus \{f(s)\}$ in two ways. Therefore there are $8(n-1)$ endomorphisms in this case.

Case 2. Assume that $I_f \cong F_3$. Then $V(I_f) = \{i, j, s, t\}$ for some $i, j \in A$ with $\{i, j\} \in E(DF_n)$, or $V(I_f) = \{i, j, k, m\}$ for some $\{i, j, k\} \in A$ and $m \in \{s, t\}$ with $\{i, j\} \in E(DF_n)$ and $\{j, k\} \in E(DF_n)$. There are three cases:

(i) Assume that $f(s) = f(t)$ and $V(I_f) = \{i, j, s, t\}$. If n is odd, then $|B| = \frac{n+1}{2}$ and $|C| = \frac{n-1}{2}$. Now there are $n-1$ subgraphs in DF_n which are isomorphic to I_f and there are $2^{\frac{n+1}{2}} - 2$ ways to divide B into two non-empty subsets B_1 and B_2. Clearly, there are two ways to map $\{B_1, B_2\}$ to $\{s, t\}$ and there are two ways to map $\{C, [s, t]\}$ to $\{i, j\}$. Similarly, there are $2^{\frac{n-1}{2}} - 2$ ways to divide C into two non-empty subsets C_1 and C_2. Now there are two ways to map $\{C_1, C_2\}$ to $\{s, t\}$ and there are two ways to map $\{B, [s, t]\}$ to $\{i, j\}$. Therefore, there are $4(n-1)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4)$ endomorphisms in this case. If n is even, a similar argument will show that there are $8(n-1)(2^{\frac{n}{2}} - 2)$ endomorphisms.

(ii) Assume that $f(s) = f(t) = j$ and $V(I_f) = \{i, j, k, m\}$. Then there are $2(n-2)$ endomorphism images. If n is odd, there are $2^{\frac{n+1}{2}} - 2$ ways to divide B into two non-empty subsets B_1 and B_2 and there are $2^{\frac{n-1}{2}} - 2$ ways to divide C into two non-empty subsets C_1 and C_2. If $f(C) = m$, then there are two ways to map $\{B_1, B_2\}$ to $\{i, k\}$. If $f(B) = m$, then there are two ways to map $\{C_1, C_2\}$ to $\{i, k\}$. Therefore, there are $2(n-2)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4)$ endomorphisms in this case. If n is even, a similar argument will show that there are $4(n-2)(2^{\frac{n}{2}} - 2)$ endomorphisms.

(iii) Assume that $f(s) = i$ and $f(t) = k$ for some $i, j \in A$ and $V(I_f) = \{i, j, k, m\}$. Then $|i - j| = 2$ and there are $2(n-2)$ endomorphisms images. Note that $f(s) = i$ and $f(t) = k$. Now there are two ways map $\{B, C\}$ to $\{m, j\}$. Analogously for the case of $f(s) = k$ and $f(t) = i$. Therefore, there are $8(n-2)$ endomorphisms in this case.

Therefore, if $I_f \cong F_3$ and n is odd, then there are $(6n-8)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4) + 8(n-2)$ endomorphisms. If $I_f \cong F_3$ and n is even, then there are $(12n-16)(2^{\frac{n}{2}} - 2) + 8(n-2)$ endomorphisms.

Case 3. Assume that $I_f \cong DF_3$. Then there are $n-2$ subgraphs in DF_n isomorphic to DF_3. Select a subgraph Z of DF_n such that $Z \cong DF_3$. Then Z has three vertices in A. Suppose $V(Z) = \{i, j, k, s, t\}$ for some $i, j, k \in A$ such that $\{i, j\} \in E(DF_n)$ and $\{j, k\} \in E(DF_n)$. Then $f(s) = i$ and $f(t) = k$, or $f(s) = k$ and $f(t) = i$, or $f(s) = f(t) = j$.

(i) If n is odd, then $|B| = \frac{n+1}{2}$ and $|C| = \frac{n-1}{2}$. Note that there are $2^{\frac{n+1}{2}} - 2$ ways to divide B into two non-empty subsets B_1 and B_2 and there are $2^{\frac{n-1}{2}} - 2$ ways to divide C into two non-empty subsets C_1 and C_2.

If $f(s) = f(t) = j$, then there are $2^{\frac{n+1}{2}} - 2$ ways to map the vertices in B to $\{i, k\}$ and there are $2^{\frac{n-1}{2}} - 2$ ways to map vertices in C to $\{s, t\}$, or there are $2^{\frac{n+1}{2}} - 2$ ways to map the vertices in B to $\{s, t\}$ and there are $2^{\frac{n-1}{2}} - 2$ ways to map the vertices in C to $\{i, j\}$. Therefore, there are $8(n-2)(2^{\frac{n+1}{2}} - 2)(2^{\frac{n-1}{2}} - 2)$ endomorphisms in this case.

If $f(s) = i$, $f(t) = k$ and $f(C) = j$, then there are $2^{\frac{n+1}{2}} - 2$ ways to map the vertices in B to $\{s, t\}$. If $f(s) = i$, $f(t) = k$ and $f(B) = j$, then there are $2^{\frac{n-1}{2}} - 2$ ways to map vertices in C to $\{s, t\}$. It is also analogous for the case of $f(s) = k$ and $f(t) = i$. Therefore, there are $2(n-2)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4)$ endomorphisms in this case.

(ii) If n is even, then $|B| = |C| = \frac{n}{2}$. Thus, there are $2^{\frac{n}{2}} - 2$ ways to divide B into two non-empty subsets B_1 and B_2 and there are $2^{\frac{n}{2}} - 2$ ways to divide C into two non-empty subsets C_1 and C_2.

If $f(s) = f(t) = j$, then there are $8(n-2)(2^{\frac{n}{2}} - 2)^2$ endomorphisms. If $f(s) = i$ and $f(t) = k$, then there are $2(n-2)(2^{\frac{n}{2}} - 2)$ endomorphisms. Analogously for the case of $f(s) = k$ and $f(t) = i$. Then there are $4(n-2)(2^{\frac{n}{2}} - 2)$ endomorphisms.

Therefore, if $I_f \cong DF_3$ and n is odd, then there are $8(n-2)(2^{\frac{n+1}{2}} - 2)(2^{\frac{n-1}{2}} - 2) + 2(n-2)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4)$ endomorphisms. If $I_f \cong DF_3$ and n is even, then there are $8(n-2)(2^{\frac{n}{2}} - 2)^2 + 4(n-2)(2^{\frac{n}{2}} - 2)$ endomorphisms.

From the above discussion, we finish our proof. □

Lemma 5. *Let DF_n be a double fan graph with $n \geq 4$. Then $|hEnd(DF_4)| = |End(DF_4)| - 16$. If n is odd, then $|hEnd(DF_n)| = |End(DF_n)| - 8(n-2)\left(1 + \sum_{i=1}^{\frac{n-5}{2}} a_i\right)$; If n is even and $n \geq 6$, then $|hEnd(DF_n)| =$*

$|End(DF_n)| - 8\,(n-2)\left(1+\sum_{i=1}^{\frac{n-6}{2}} a_i + b_{\frac{n-4}{2}+1}\right)$, *where* $a_m = b_1 2^{m-1} + b_2 2^{m-2} + \cdots + b_{m-2}2^2 + 5b_{m-1} + 2b_{m-2}$ $(m \geq 3)$ *and* $b_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right]$ *is the Fibonacci sequence.*

Proof. Let $f \in End(DF_n)$. By Lemma 2, $f(s), f(t) \in \{s,t\}$, or $f(s), f(t) \notin \{s,t\}$. If $f(s), f(t) \in \{s,t\}$, then $f|_A \in End(P_n)$. It is easy to check that $f \in hEnd(X)$. If $f(s), f(t) \notin \{s,t\}$, then $I_f \cong K_3$, or $I_f \cong F_3$, or $I_f \cong DF_3$. If $I_f \cong K_3$ or $I_f \cong F_3$, then I_f is an induced subgraph, and so $f \in hEnd(X)$.

In the following, we assume that $I_f \cong DF_3$. Clearly, there are $n-2$ subgraphs in DF_n isomorphic to DF_3. Select a subgraph Y of DF_n such that $Y \cong DF_3$. Without loss of generality, suppose $V(Y) = \{i,j,k,s,t\}$ for some $i,j,k \in A$ such that $\{i,j\} \in E(DF_n)$ and $\{j,k\} \in E(DF_n)$. Let $A_1 \subset B$ such that $1 \in A_1$, $A_2 = \{k | k \in C \textit{ and } |k-j| = 1 \textit{ for some } j \in A_1\}$, $A_3 = B \backslash A_1$ and $A_4 = C \backslash A_2$. Now f is not half-strong if and only if f maps $\{A_1, A_3\}$ to $\{s,t\}$ and maps $\{A_2, A_4\}$ to $\{i,k\}$, or f maps $\{A_2, A_4\}$ to $\{s,t\}$ and maps $\{A_1, A_3\}$ to $\{i,k\}$. There are three cases:

Case 1. Assume that $n = 4$. Then there are only one way to divide A into four non-empty subsets A_1, A_2, A_3, A_4. It is easy to see that there are 2 ways to map $\{A_1, A_3\}$ to $\{s,t\}$ and there are 2 ways to map $\{A_2, A_4\}$ to $\{i,k\}$. Similarly, there are 2 ways to map $\{A_2, A_4\}$ to $\{s,t\}$ and there are 2 ways to map $\{A_1, A_3\}$ to $\{i,k\}$. Note that there are two subgraphs in DF_4 isomorphic to DF_3. Thus there are 16 endomorphisms of DF_4 which are not half-strong.

Case 2. Assume that n is odd. Then $|A_1| = 1 + \sum_{i=1}^{\frac{n-5}{2}} a_i$, where $a_m = b_1 2^{m-1} + b_2 2^{m-2} + \cdots + b_{m-2}2^2 + 5b_{m-1} + 2b_{m-2}$ $(m \geq 3)$ and $b_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right]$ is the Fibonacci sequence. It is easy to see that there are 2 ways to map $\{A_1, A_3\}$ to $\{s,t\}$ and there are 2 ways to map $\{A_2, A_4\}$ to $\{i,k\}$. Similarly, there are 2 ways to map $\{A_2, A_4\}$ to $\{s,t\}$ and there are 2 ways to map $\{A_1, A_3\}$ to $\{i,k\}$. Note that there are $n-2$ subgraphs in DF_n isomorphic to DF_3. Thus, there are $8\,(n-2)\left(1+\sum_{i=1}^{\frac{n-5}{2}} a_i\right)$ endomorphisms of DF_3 which are not half-strong in this case.

Case 3. Assume that n is even and $n \geq 6$. Then $|A_1| = 1 + \sum_{i=1}^{\frac{n-6}{2}} a_i + b_{\frac{n-4}{2}+1}$. Thus, there are $8\,(n-2)\left(1+\sum_{i=1}^{\frac{n-6}{2}} a_i + b_{\frac{n-4}{2}+1}\right)$ endomorphisms of DF_3, which are not half-strong.

From the above discussion, we get the results of Lemma 5 to finish our proof. □

Lemma 6. *Let DF_n be a double fan graph. Then $hEnd(DF_n) = lEnd(DF_n)$ if and only if $n \leq 3$.*

Proof. Necessity. We only need to prove that $hEnd(DF_n) \neq lEnd(DF_n)$ for any $n \geq 4$. Define a mapping f on $V(DF_n)$ by

$$f(x) = \begin{cases} 3, & x = 1, \\ x, & otherwise. \end{cases}$$

Then $f \in hEnd(DF_n)$. It is not hard to show that $\{3,4\} = \{f(3), f(4)\} \in E(I_f)$, $f^{-1}(3) = \{1,3\}$ and $f^{-1}(4) = \{4\}$. Note that $\{1,4\} \notin E(DF_n)$. Then $f \notin lEnd(DF_n)$. Therefore, $hEnd(DF_n) \neq lEnd(DF_n)$.

Sufficiency. If $n = 2,3$, then $End(DF_n) = sEnd(DF_n)$ by the proof of Lemma 3. Therefore, $hEnd(DF_n) = lEnd(DF_n)$ for $n = 2,3$. □

Lemma 7. *Let $f \in End(DF_n)$ be such that $f(s), f(t) \in \{s,t\}$. Then $f \in lEnd(DF_n)$ if and only if $f|_A \in lEnd(P_n)$.*

Proof. Necessity. Let $f \in lEnd(DF_n)$ be such that $f(s), f(t) \in \{s,t\}$. Then $f|_A \in End(P_n)$. Let $i,j \in V(I_{f|_A})$ be such that $\{i,j\} \in E(I_{f|_A})$. Since f is locally-strong, for every $i' \in f^{-1}(i)$ there exists $j' \in f^{-1}(j)$ such that $\{i',j'\} \in E(P_n)$ and analogously for preimage of j. Note that $f|_A^{-1}(i) = f^{-1}(i) \subseteq A$ and $f|_A^{-1}(j) = f^{-1}(j) \subseteq A$. Then $f|_A \in lEnd(P_n)$.

Sufficiency. Let $i,j \in V(I_f)$ be such that $\{i,j\} \in E(I_f)$. If $i \in \{s,t\}$, then $f^{-1}(i) \in \{s,t\}$. Clearly, s and t are adjacent to all vertices of $f^{-1}(j)$. If $i,j \notin \{s,t\}$, then $i,j \in A$. Now we have $\{i,j\} \in E(I_f|_A)$. Since $f|_A \in lEnd(P_n)$, for every preimage $i' \in f|_A^{-1}(i)$ there exists a preimage $j' \in f|_A^{-1}(j)$ such that $\{i',j'\} \in E(P_n)$ and analogously for preimage of j. Note that $f|_A{}^{-1}(i) = f^{-1}(i) \subseteq A$ and $f|_A{}^{-1}(j) = f^{-1}(j) \subseteq A$. Then $f \in lEnd(DF_n)$. □

Lemma 8.

$$|lEnd(DF_n)| = \begin{cases} 8\sum_{l|(n-1)}(n-l) + 24n - 36\,, & \textit{if } n \textit{ is odd and } n \geq 4, \\ 8\sum_{l|(n-1)}(n-l) + 16n - 24\,, & \textit{if } n \textit{ is even and } n \geq 4. \end{cases}$$

Proof. Let $f \in End(DF_n)$. Firstly, suppose that $f(s), f(t) \in \{s,t\}$. By Lemma 7, $f \in lEnd(DF_n)$ if and only if $f|_A \in lEnd(P_n)$. It is known from [18] that $|lEnd(P_n)| = 2\sum_{l|(n-1)}(n-l)$. Now we have $f(s) = s$ and $f(t) = t$, or $f(s) = t$ and $f(t) = s$, or $f(s) = f(t) = s$, or $f(s) = f(t) = t$. Thus, we have $|lEnd(DF_n)| = 8\sum_{l|(n-1)}(n-l)$ endomorphisms in this case.

Next, suppose that $f(s) = f(t) = i$ for some $i \in A$. By Lemma 2, $I_f \cong K_3$, or $I_f \cong F_3$, or $I_f \cong DF_3$. There are three cases.

Case 1. Assume that $I_f \cong K_3$. Then $|f(A)| = 2$ and $\rho_f = \{B, C, [s,t]\}$. The subgraphs of DF_n induced by $B \cup C$, $B \cup \{s,t\}$, $C \cup \{s,t\}$ have no isolated vertices. Then $f \in lEnd(DF_n)$. There are $8(n-1)$ endomorphisms of DF_n by Lemma 4.

Case 2. Assume that $I_f \cong F_3$. Denote $B_1 = \{4m+1 \in B | where\ m = 0,1,2,\cdots\}$ and $B_2 = \{4m+3 \in B | where\ m = 0,1,2,\cdots\}$. Then it is easy to see that $f \in lEnd(DF_n)$ if and only if n is odd and $\rho_f = \{B_1, B_2, C, [s,t]\}$. Now $V(I_f) = \{i,j,s,t\}$ for some $i,j \in A$ such that $\{i,j\} \in E(DF_n)$, or $V(I_f) = \{i,j,k,m\}$ for some $i,j,k \in A$ and $m \in \{s,t\}$ such that $\{i,j\} \in E(DF_n)$ and $\{j,k\} \in E(DF_n)$. If $V(I_f) = \{i,j,s,t\}$, then there are $n-1$ endomorphism images. Note that there are two ways to map $\{B_1, B_2\}$ to $\{s,t\}$ and there are two ways to map $\{C, \{s,t\}\}$ to $\{i,j\}$. Thus there are $4(n-1)$ locally strong endomorphisms. If $V(I_f) = \{i,j,k,m\}$, then there are $2(n-2)$ endomorphism images. Now $f(s) = f(t) = j$ and $f(C) = m$. Note that there are two ways to map $\{B_1, B_2\}$ to $\{i,k\}$. Thus, there are $4(n-2)$ locally strong endomorphisms. Therefore, there are $8n-12$ locally strong endomorphisms in this case.

Case 3. Assume that $I_f \cong DF_3$. Then f is not locally strong.

Finally, suppose that $f(s) = i$ and $f(t) = j$ for some $i,j \in A$. By Lemma 2, $|i-j| = 2$, $I_f \cong F_3$ or $I_f \cong DF_3$.

(1) If $I_f \cong F_3$, it is easy to check that $\rho_f = \{B, C, s, t\}$. Note that the subgraphs of DF_n induced by $B \cup C$, $B \cup \{s\}$, $C \cup \{s\}$, $B \cup \{t\}$, $C \cup \{t\}$ have no isolated vertices. Then $f \in lEnd(DF_n)$. By Lemma 4, There are $8(n-2)$ endomorphisms in this case.

(2) If $I_f \cong DF_3$, then $f \notin lEnd(DF_3)$ for any $n \geq 4$. Hence there are no locally endomorphisms in this case.

From the discussion above, we deduce Lemma 8 to finish our proof. □

Lemma 9. *$lEnd(DF_n) = qEnd(DF_n)$ if and only if $n = 2,3,4$.*

Proof. Necessity. We only need to show that $lEnd(DF_n) \neq qEnd(DF_n)$ for any $n \geq 5$. Define a mapping f on $V(DF_n)$ by

$$f(x) = \begin{cases} 1, & x \in B, \\ 2, & x \in C, \\ x, & otherwise. \end{cases}$$

It is not difficult to show that $f \in lEnd(DF_n) \setminus qEnd(DF_n)$. Therefore, $lEnd(DF_n) \neq qEnd(DF_n)$.

Sufficiency. If $n = 2, 3$, then $lEnd(DF_n) = qEnd(DF_n)$ by the proof of Lemma 3. If $n = 4$, then there exist only two positive integers 1 and 3 such that $1|n-1$, $3|n-1$. Let $f \in lEnd(DF_4)$, then $\rho_f = \{[1], [2], [3], [4], [s], [t]\}$, or $\rho_f = \{[1], [2], [3], [4], [s,t]\}$, or $\rho_f = \{[1,3], [2,4], [s], [t]\}$ or $\rho_f = \{[1,3], [2,4], [s,t]\}$. If $\rho_f = \{[1], [2], [3], [4], [s], [t]\}$, then $f \in Aut(DF_4)$. If $\rho_f = \{[1], [2], [3], [4], [s,t]\}$, then $I_f \cong F_4$. It is easy to check that $f \in qEnd(DF_4)$. If $\rho_f = \{[1,3], [2,4], [s,t]\}$ or $\rho_f = \{[1,3], [2,4], [s], [t]\}$, it is a routine matter to check that $f \in qEnd(DF_4)$. Therefore, we have $lEnd(DF_n) = qEnd(DF_n)$. □

Lemma 10. *$qEnd(DF_n) = sEnd(DF_n)$ if and only if $n \neq 4$.*

Proof. Necessity. We show that $qEnd(DF_4) \neq sEnd(DF_4)$. Define a mapping f on $V(DF_4)$ by

$$f(x) = \begin{cases} 1, & x \in \{1,3\}, \\ 2, & x \in \{2,4\}, \\ x, & x \in \{s,t\}. \end{cases}$$

It is easy to check that $f \in qEnd(DF_4)$, $f^{-1}(1) \in \{1,3\}, f^{-1}(2) \in \{2,4\}$ and $\{1,4\} \notin E(DF_4)$. Then $f \notin sEnd(DF_4)$.

Sufficiency. If $n = 2, 3$, then $End(DF_n) = sEnd(DF_n)$ by the proof of Lemma 3. In the following, we suppose that $n \geq 5$. Let $f \in DF_n$.

(1) If $I_f \cong K_3$, then $\rho_f = \{B, C, [s,t]\}$. Since $n \geq 5$, $|B| \geq 3$. Note that $f \in qEnd(DF_n)$ and $\{f(B), f(C)\} \in E(I_f)$. Then there exists $d \in C$ adjacent to every vertex of B. This is impossible since each vertex of A is adjacent to at most two vertices of A.

(2) If I_f is not isomorphic to K_3. Then $f(A)$ contains at least 3 vertices. Since $f \notin Aut(DF_n)$, there exist $i, j \in A$ such that $f(i) = f(j)$. Suppose there exist $f(a), f(b) \in f(A)$ such that $\{f(i), f(a)\} \in E(I_f)$ and $\{f(i), f(b)\} \in E(I_f)$. Then there exists a preimage a' of $f(a)$ such that a' is adjacent to both i and j. Similarly, there exists $b' \in f^{-1}(f(b))$ such that b' is adjacent to both i and j. Thus $\{i, j, a', b'\}$ forms a cycle C_4. This is impossible since the subgraph of DF_n induced by A is a path. Suppose there exists only one vertex $f(c) \in f(A)$ such that $\{f(i), f(c)\} \in E(I_f)$. Then there exists $f(d) \in f(A)$ such that $\{f(c), f(d)\} \in E(I_f)$ since $|f(A)| \geq 3$. Now there exists $c' \in f^{-1}(f(c))$ such that c' is adjacent to both i and j, and there exists $d' \in f^{-1}(f(d))$ such that d' is adjacent to c'. Thus c' is adjacent to 3 vertices of A. This is a contradiction since each vertex of A is adjacent to at most 2 vertices in A. Consequently, $qEnd(DF_n) = sEnd(DF_n)$ for $n \geq 5$. □

Lemma 11. *$sEnd(DF_n) \neq Aut(DF_n)$ for any $n \geq 2$.*

Proof. Define a mapping f on $V(DF_n)$ by

$$f(x) = \begin{cases} t, & x = s, \\ x, & otherwise. \end{cases}$$

It is not hard to check that $f \in sEnd(DF_n)$, but $f \notin Aut(DF_n)$. Therefore, $sEnd(DF_n) \neq Aut(DF_n)$. □

Lemma 12.

$$|Aut(DF_n)| = \begin{cases} 4\,, & n = 2, \\ 8, & n = 3, \\ 4, & n \geq 4. \end{cases}$$

Proof. Let $f \in Aut(DF_n)$. If $n = 2$, then $f(1), f(2) \in \{1,2\}$ and $f(s), f(t) \in \{s,t\}$. So $|Aut(DF_2)| = 4$. If $n = 3$, then $f(2) = 2$. Denote by Y the subgraph of DF_3 induced by $\{1,3,x,y\}$. Then $f|_Y \in Aut(Y)$. Thus $|Aut(DF_3)| = 8$. If $n \geq 4$, then $f(x), f(y) \in \{x,y\}$. If $f(x) = x$ and $f(y) = y$, then the number of automorphisms of DF_n is equal to the number of automorphisms of P_n. It is 2. And analogously for $f(x) = y$ and $f(y) = x$. Therefore $|Aut(DF_n)| = 4$. □

Lemma 13. *If $n \geq 4$, then $|sEnd(DF_n)| = 8$.*

Proof. Assume $n \geq 4$. Let $f \in sEnd(DF_n) \backslash Aut(DF_n)$. Then there exist $x, y \in V(DF_n)$ such that $f(x) = f(y)$. Then $N(x) = N(y)$. Thus $x, y \in \{s,t\}$, $f(x) = f(y) = x$ or $f(x) = f(y) = y$. If $f(x) = f(y) = x$, then $|sEnd(DF_n)| = |Aut(F_n)| = 2$. Analogously for $f(x) = f(y) = y$. Hence, the number of strong endomorphisms of DF_n which are not automorphism is 4. By Lemma 12, $|Aut(DF_4)| = 4$. Therefore, $|sEnd(DF_n)| = 8$. □

Now we obtain the main result in this paper.

Theorem 1. *Let DF_n be a double fan graph. Then*

$$Endospec(DF_n) = \begin{cases} (16,16,16,16,16,4), & n = 2, \\ (64,64,64,64,64,8), & n = 3, \\ (284,232,72,72,8,4), & n = 4, \\ (A_1(n), A_2(n), B_1(n), 8, 8, 4), & n \geq 5, \text{ and } n \text{ is odd}, \\ (C_1(n), C_2(n), B_2(n), 8, 8, 4), & n \geq 5, \text{ and } n \text{ is even}. \end{cases}$$

where

$$\begin{aligned} A_1(n) &= 4(n+1)2^{n-1} - 4(2n-1)\binom{n-1}{\frac{n-1}{2}} + 16n - 24 + (10n-12)(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4) \\ &\quad +8(n-2)\left[\left(2^{\frac{n+1}{2}} - 2\right)\left(2^{\frac{n-1}{2}} - 2\right)\right], \\ A_2(n) &= 4(n+1)2^{n-1} - 4(2n-1)\binom{n-1}{\frac{n-1}{2}} + 16n - 24 + (10n-12)\left(2^{\frac{n+1}{2}} + 2^{\frac{n-1}{2}} - 4\right) \\ &\quad +8(n-2)\left[\left(2^{\frac{n+1}{2}} - 2\right)\left(2^{\frac{n-1}{2}} - 2\right)\right] - 8(n-2)\left(1 + \sum_{i=1}^{\frac{n-5}{2}} a_i\right), \\ C_1(n) &= 4(n+1)2^{n-1} - 4n\binom{n}{\frac{n}{2}} + 16n - 24 + (16n-24)(2^{\frac{n}{2}} - 2) + 8(n-2)(2^{\frac{n}{2}} - 2)^2, \\ C_2(n) &= 4(n+1)2^{n-1} - 4n\binom{n}{\frac{n}{2}} + 16n - 24 + (16n-12)\left(2^{\frac{n}{2}} - 2\right) + 8(n-2)\left(2^{\frac{n}{2}} - 2\right)^2 \\ &\quad -8(n-2)\left(1 + \sum_{i=1}^{\frac{n-6}{2}} a_i + b_{\frac{n-4}{2}+1}\right), \end{aligned}$$

$$B_1(n) = 8\sum\nolimits_{l|(n-1)}(n-l) + 24n - 36,\ B_2(n) = 8\sum\nolimits_{l|(n-1)}(n-l) + 16n - 24,$$
$$a_m = b_1 2^{m-1} + b_2 2^{m-2} + \cdots + b_{m-2}2^2 + 5b_{m-1} + 2b_{m-2}\ \ (m \geq 3),$$

$$b_m = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right] \text{ is the Fibonacci sequence.}$$

Proof. This follows immediately from Lemmas 4, 5, 7–12. □

Theorem 2. *Let DF_n be a double fan graph. Then*

$$Endotype(DF_n) = \begin{cases} 16, & n = 2, \\ 16, & n = 3, \\ 27, & n = 4, \\ 23, & n \geq 5. \end{cases}$$

Proof. This follows directly from Lemmas 3, 5, 8–10. □

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References

1. Gu, R.; Hou, H. Unicyclic graphs whose completely regular endomorphisms form a monoid. *Mathematics* **2020**, *8*, 240. [CrossRef]
2. Hou, H.; Gu, R. Split graphs whose completely regular endomorphisms form a monoid. *ARS Comb.* **2016**, *127*, 79–88. [CrossRef]
3. Hou, H.; Luo, Y. Graphs whose endomorphism monoids are regular. *Discret. Math.* **2008**, *308*, 3888–3896. [CrossRef]
4. Kelarev, A.V.; Praeger, C.E. On transitive Cayley graphs of groups and semigroups. *Eur. J. Comb.* **2003**, *24*, 59–72. [CrossRef]
5. Kelarev, A.V.; Ryan, J.; Yearwood, J. Cayley graphs as classifiers for data mining: The influence of asymmetries. *Discret. Math.* **2009**, *309*, 5360–5369. [CrossRef]
6. Kelarev, A.V. Labelled Cayley graphs and minimal automata. *Australas. J. Comb.* **2004**, *30*, 95–101.
7. Böttcher, M.; Knauer, U. Endomorphism spectra of graphs. *Discret. Math.* **1992**, *109*, 45–57.
8. Fan, S. Endomopphism spectra of bipartite graph with dimater three and girth six. *Southeast Asian Bull. Math.* **2001**, *25*, 217–221. [CrossRef]
9. Hou, H.; Fan, X.; Luo, Y. Endomorphism types of generalized polygons. *Southeast Asian Bull. Math.* **2009**, *33*, 433–441.
10. Hou, H.; Luo, Y.; Cheng, Z. The Endomorphism monoid of $\overline{P_n}$. *Eur. J. Comb.* **2008**, *29*, 1173–1185. [CrossRef]
11. Hou, H.; Gu, R.; Song, Y. Endomorphism spectra of fan graphs. *Ars Comb.* **2019**, *142*, 89–99.
12. Godsil, C.; Royle, G. *Algebraic Graph Theory*; Springer: New York, NY, USA, 2000.
13. Kelarev, A.V. *Graph Algebras and Automata*; Marcel Dekker: New York, NY, USA, 2003.
14. Kelarev, A.V. *Ring Constructions and Applications*; World Scientific: River Edge, NJ, USA, 2002.
15. Knauer, U. *Algebraic Graph Theory: Morphisms, Monoids and Matrices*; De Gruyter: Berlin, Germany; Boston, MA, USA, 2011.
16. Li, W. Graphs with regular monoid. *Discrete Math.* **2003**, *265*, 105–118. [CrossRef]
17. Lin, Z.; Zeng, J. On the number of congruence classes of paths. *Discret. Math.* **2012**, *312*, 1300–1307. [CrossRef]
18. Arworn, S.; Knauer, U.; Leeratanavalee, S. Locally strong endomorphisms of paths. *Discret. Math.* **2008**, *308*, 2525–2532. [CrossRef]