mathematics

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A Quadratic Diophantine Equation Involving Generalized Fibonacci Numbers

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Received: 21 April 2020; Accepted: 18 June 2020; Published: 20 June 2020

Abstract: The sequence of the k-generalized Fibonacci numbers $(F_n^{(k)})_n$ is defined by the recurrence $F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)}$ beginning with the k terms $0, \ldots, 0, 1$. In this paper, we shall solve the Diophantine equation $F_n^{(k)} = (F_m^{(l)})^2 + 1$, in positive integers m, n, k and l.

Keywords: Fibonacci number; recurrence sequence; linear form in logarithms; reduction method

MSC: primary 11B39; secondary 11J86

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1. Introduction

We recall the Fibonacci sequence $(F_n)_n$ which is defined by the recurrent relation $F_{n+2} = F_{n+1} + F_n$, where $F_j = j$, for $j \in \{0, 1\}$. The Fibonacci numbers have been the main object of many studies (see, for example, [1–6] and references therein).

Like any very studied object in Mathematics, the Fibonacci sequence admits many generalizations (in several distinct ways). Among these generalizations, we are interested in the *k-generalized Fibonacci sequence* $(F_n^{(k)})_{n \geq -(k-2)}$ which is defined by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)},$$

with initial values $F_j^{(k)} = 0$ (for $j \in [-(k-2), 0]$) and $F_1^{(k)} = 1$. For instance, if $k = 2$, we have the usual Fibonacci numbers $(F_n^{(2)})_n$, for $k = 3$, $(F_n^{(3)})_n$ the sequence is called the Tribonacci sequence and so on (Kessler and Schiff [7] remarked the appearance of these numbers in probability theory and in certain sorting algorithms).

In the past few years, k-Fibonacci numbers are in the mainstream of many works. For example, in 2013, two related conjectures were proved. The first one (proposed by Marques [8]) was proved by Bravo and Luca [9] and is related to *repdigits* among k-Fibonacci numbers. The second conjecture (proposed by Noe and Post [10]) concerns the intersection between these sequences, and was solved (independently) by Marques [11] and Bravo, Luca [12]. In addition, Chaves and Marques [13] solved the equation $(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(k)}$ and then Bednařík et al. [14] generalized this study to the equation $(F_n^{(k)})^2 + (F_{n+1}^{(k)})^2 = F_m^{(l)}$. In 2019, Trojovský [15] proved that the Diophantine equation $F_m^{(k)} = m^t$, with $t > 1$ and $m > k + 1$, has only the solutions $F_{12}^{(2)} = 12^2$ and $F_9^{(3)} = 9^2$.

We remark that the problem of determining all the perfect powers among Fibonacci numbers was settled in a seminal work due to Bugeaud, Mignotte, and Siksek [16]. However, the problem of solving completely the equation $F_n^{(k)} = y^t$, for $k > 2$ and $t > 1$, is still far from being solved. Indeed, the particular case $(k,t) = (3,2)$ (i.e., to find all Tribonacci numbers which are perfect squares) is a known open problem which appeared as Problem 1 in a paper due to Pethő [17].

In this paper, we are interested in this kind of problem. Indeed, our goal is to study when a term of a k-generalized Fibonacci sequence is near to a perfect square, whose basis is also a generalized Fibonacci number (possibly of another order). More precisely, we have the Diophantine equation

$$F_n^{(k)} = (F_m^{(l)})^2 + c. \tag{1}$$

Thus, in this paper, we shall solve this equation for $c = 1$ by proving that

Theorem 1. *The solutions of Equation (1), for $c = 1$, in m, n, k and l, with $\min\{m,n\} \geq 1$ and $\min\{k,l\} \geq 2$, are*

$$(n,m,k,l) \in \{(3,1,k,l),(3,2,k,l),(5,3,2,l)\}.$$

Remark 1. *We point out that the method presented here can be used to obtain all solutions of Equation (1), for any previously fixed value of c (the choice of $c = 1$ has nothing of special). See a more detailed discussion (on this fact) in Section 8. In addition, we remark that it is well-known that (for any given c) this equation has only finitely many solutions (by a result of Nemes and Pethő [18]).*

Remark 2. *The Mandelbrot set is the set of complex numbers c for which the sequence $(z_n)_n$ defined by a nonlinear recurrence $z_{n+1} = z_n^2 + c$, with $z_0 = 0$, does not diverge. Thus, the problem of solving the Diophantine Equation (1) can be rephrased as: For which values of $t > 0$, a pair of consecutive z_t's belongs to $(F_n^{(k)}) \times (F_m^{(l)})$?*

For proving our main result, we shall apply *Baker's theory*, a Dujella–Pethö reduction method, some key arguments due to Bravo-Luca, and a combinatorial lemma to deal with an extremal case.

2. Auxiliary Results

It is known that the characteristic polynomial of $(F_n^{(k)})_n$ is

$$\psi_k(x) := x^k - x^{k-1} - \cdots - x - 1.$$

This polynomial has only one root outside the unit circle (indeed this zero is a *Pisot number*, i.e., all the other zeros have absolute value strictly smaller than 1). In addition, this zero is simple and lies in the interval $(2(1-2^{-k}), 2)$ (see [19]). Furthermore, Bravo and Luca [12] (Lemma 1) provided the estimates

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \tag{2}$$

for all $n \geq 0$, where α is the root of $\psi_k(x)$ with largest absolute value.

There are many closed (non-recurrent) formulas for the nth term of $(F_n^{(k)})_n$ (see [20–23]). However, we are interested in the undermentioned consequence of the simplified "Binet-like" formula due to Dresden and Du [24] (Thm 2):

$$F_n^{(k)} = g\alpha^{n-1} + E_n(k), \tag{3}$$

with $|E_n(k)| < 1/2$, for all n, where $g := g(\alpha,k)$, for $g(x,y) := (x-1)/(1+(y+1)(x-2))$. Moreover, it is known that $g \in (1/2, 3/4)$ and a useful fact from [13] is that $g\alpha > 1$.

As mentioned before, we also shall use lower bounds for linear forms in logarithms. Among the several results on this topic, we decided to use one due to Bugeaud, Mignotte and Siksek [16] (Theorem 9.4).

Lemma 1. *Let $\gamma_1, \ldots, \gamma_t$ be nonzero real algebraic numbers and let $b_1, \ldots, b_t$ be nonzero integers. Let $D = [\mathbb{Q}[\gamma_1, \ldots, \gamma_t] : \mathbb{Q}]$ and let A_j be a real number satisfying*

$$A_j \geq \max\{D \cdot \mathrm{h}(\alpha_j), |\log \alpha_j|, 0.16\}, \textit{for } j \in [1, t].$$

Take

$$B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

If $|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1|$ *is nonzero, then*

$$|\gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1| \geq \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 \cdot (1 + \log D)(1 + \log B) A_1 \cdots A_t).$$

In the previous lemma, the *logarithmic height* of an n-degree algebraic number γ is defined as

$$\mathrm{h}(\gamma) = \frac{1}{n}(\log |a| + \sum_{j=1}^{n} \log \max\{1, |\gamma^{(j)}|\}),$$

where a is the leading coefficient of the minimal polynomial of γ and $(\gamma^{(j)})_{1 \leq j \leq n}$ are the algebraic conjugates of γ.

Some basic properties of the logarithmic height are:

i. $\mathrm{h}(\alpha\beta) \leq \mathrm{h}(\alpha) + \mathrm{h}(\beta)$;
ii. $\mathrm{h}(\alpha^r) = |r| \cdot \mathrm{h}(\alpha)$, for all $r \in \mathbb{Q}^*$ (nonzero rational numbers) and $\alpha \in \overline{\mathbb{Q}}$ (algebraic numbers);
iii. $\mathrm{h}(1/\alpha) = \mathrm{h}(\alpha)$.

After establishing an upper bound for one of our variables (which is in general too large to perform the necessary computations), the next step makes it substantially smaller. For this purpose, our next ingredient is a theorem due to Dujella and Pethö [25]. Recall that, for a real number x, the *Nint* function at x is $\| x \| := \min\{|x - n| : n \in \mathbb{Z}\}$.

Lemma 2. *Let* $M \in \mathbb{Z}_{>0}$ *and let* $\gamma, \mu \in \mathbb{R}$*, such that* γ *is irrational. Let* p/q *be a convergent of the continued fraction expansion* γ *with* $q > 6M$*, and let* A, B *be real numbers with* $A > 0$ *and* $B > 1$*. If* $\epsilon := \| \mu q \| - M \| \gamma q \|$ *is positive, then the Diophantine inequality*

$$0 < m\gamma - n + \mu < A \cdot B^{-k}$$

does not have solution in integers m, n*, and* k *with*

$$m \leq M \textit{ and } k \geq \frac{\log (Aq/\epsilon)}{\log B}.$$

Our last ingredient is a combinatorial argument which will be essential to deal with the extremal case $n = 2m - 2$.

Lemma 3. *Let* k*,* m *be any integers. For all* $k \geq 2$ *and* $m > k + 1$*, we have*

$$F_{2m-2}^{(k)} < (F_m^{(k)})^2 + 1.$$

Proof. It is well-known that $F_{r+1}^{(k)}$ counts the tiling of an $(1 \times r)$-board by tiles of lengths up to k. Thus, we have $F_{2m-2}^{(k)}$ tiles of an $(1 \times (2m - 3))$-board. On the other hand, we can see what happens at the mark $m - 1$ (from left to right and we have it as a boundary point). For coverings by tiles which do not intersect the position $(m - 1, m)$, we have $F_{m-1}^{(k)} F_m^{(k)}$ possible configurations. Now, for the intersecting case, we can have the apparition of a part of length t in the right part of the original board (for $t \in \{1, \ldots, k - 1\}$, since $k - 1 < m - 2$). This gives at most $F_{m-1-t}^{(k)} F_m^{(k)}$ possible configurations. Thus, the total number of configurations is at most $F_{m-1}^{(k)} F_m^{(k)} + \left(\sum_{t=1}^{k-1} F_{m-1-t}^{(k)}\right) F_m^{(k)} = (F_m^{(k)})^2$. In conclusion, $F_{2m-2}^{(k)} \leq (F_m^{(k)})^2$ which completes the proof. □

Now, we are ready to start the proof of our main result. We shall split it in some sections in order to make the text more readable.

3. An Inequality for m in Terms of l

Our goal is to solve the Diophantine equation

$$F_n^{(k)} = (F_m^{(l)})^2 + 1. \tag{4}$$

To avoid unnecessary repetitions, we shall consider $l > k$ (the case $l \le k$ can be handled in the same way). By the auxiliary results and Dresden–Du formula, we can rewrite (4) as

$$h^2\beta^{2m-2} - g\alpha^{n-1} = E_n(k) - 2h\beta^{m-1}E_m(l) - (E_m(l))^2 - 1,$$

where $g := g(\alpha, k)$ and $h := g(\beta, l)$ and both $|E_n(k)|$ and $|E_m(l)|$ are smaller than $1/2$, for all positive integers m and n. Thus,

$$|h^2\beta^{2m-2} - g\alpha^{n-1}| < 1/2 + h\beta^{m-1} + (1/2)^2 + 1 = h\beta^{m-1} + 7/4, \tag{5}$$

and, dividing by $h^2\beta^{2m-2}$, we get

$$|(g/h^2)\beta^{-2m+2}\alpha^{n-1} - 1| < \left(\frac{h\beta^{m-1} + 7/4}{h^2\beta^{2m-2}}\right) < \frac{9}{\beta^{m-1}}, \tag{6}$$

where we used $h\beta > 1$.

Note that, if we put $\Gamma_1 := (g/h^2)\beta^{-2m+2}\alpha^{n-1} - 1$, then $\Gamma_1 \ne 0$. Indeed, suppose, towards a contradiction, that $h^2\beta^{2m-2} = g\alpha^{n-1}$. Since α and β have degree k and l respectively, with $l > k$, then there exist σ_i and σ_j, $i \ne j$, embeddings of $\mathbb{Q}(\beta)$ into $\mathbb{C}$, satisfying $\sigma_i(\alpha) = \sigma_j(\alpha)$, which gives $(\sigma_j^{-1} \circ \sigma_i)(\alpha) = \alpha$, and $(\sigma_j^{-1} \circ \sigma_i)(\beta) = \beta' \ne \beta$, where β' is one of the conjugates of β. Therefore, by applying $(\sigma_j^{-1} \circ \sigma_i)$, one can get the following contradiction:

$$1 > (h(\beta', l)\beta')^{2m-2} = g\alpha^{n-1} > \left(\frac{7}{4}\right)^{n-2}$$

which is absurd for $n \ge 2$. Thus, in order to apply Lemma 1, we choose

$$\gamma_1 := g/h^2,\ \gamma_2 := \alpha,\ \gamma_3 := \beta,\ b_1 := 1,\ b_2 := n-1,\ b_3 := -2m+2.$$

We have that $\mathrm{h}(\gamma_2) = \mathrm{h}(\alpha) = (\log\alpha)/k$, $\mathrm{h}(\gamma_3) = \mathrm{h}(\beta) = (\log\beta)/l$, and, by the mentioned properties of the logarithmic height, we obtain

$$\mathrm{h}(\gamma_1) = \mathrm{h}(g/h^2) \le \mathrm{h}(g) + 2\mathrm{h}(h) \le 3\log k + 6\log l < 9\log l.$$

Since $D = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] \le lk < l^2$, we can choose $A_1 := 9l^2\log l$, $A_2 := l\log\alpha$ and $A_3 := l\log\beta$. In addition, the following inequalities hold:

$$2^{\frac{n-2}{2}} < \alpha^{n-2} \le F_n^{(k)} = (F_m^{(l)})^2 + 1 < \beta^{2(m-1)+1} < 2^{2m-1},\ \text{ and}$$

$$\alpha^{n-1} \ge F_n^{(k)} = (F_m^{(l)})^2 + 1 \ge \beta^{2m-4} + 1 \ge \beta^{2m-5} > \alpha^{2m-5}.$$

This implies that $2m - 4 \le n - 1 < 4m - 1$. However, note that, by Lemma 3, the cases in which $n - 1 \in \{2m-4, 2m-3\}$ can not happen, since

$$F_{2m-3}^{(k)} < F_{2m-2}^{(k)} < (F_m^{(k)})^2 + 1 < (F_m^{(l)})^2 + 1.$$

Thus, $2m-2 \le n-1$, and hence $\max\{1, n-1, 2m-2\} = n-1$. Thus, we choose $B := n-1$. Therefore, the conditions to apply Lemma 1 are fulfilled, and since

$$1.4 \times 30^{3+3} \times 3^{4.5} \times (lk)^2 \times (1+\log(lk)) \le 4.1 \times 10^{11} l^4 \log l$$

holds for $l \ge 4$, then

$$\begin{aligned} |\Gamma_1| &> \exp\left(-4.1 \times 10^{11} l^4 \log l (1+\log(n-1))(9l^2 \log l)(l \log \alpha)(l \log \beta)\right) \\ &> \exp\left(-1.8 \times 10^{12} \times l^8 (\log l)^2 (1+\log(n-1))\right). \end{aligned} \quad (7)$$

Now, since $n-1 < 4m$ and, by combining (6) and (7), we have

$$\frac{9}{\beta^{m-1}} > |\Gamma_1| > \exp\left(-4.5 \times 10^{12} \times l^8 (\log l)^2 \log m\right)$$

and so

$$\frac{m}{\log m} < 8.1 \times 10^{12} \times l^8 (\log l)^2.$$

Hence, from the useful fact that

$$\frac{x}{\log x} < A \Rightarrow x < 2A \log A\,, \quad (8)$$

whenever $x > e$ and $A \ge 3$, we get the following upper bound for m in terms of l

$$m < 4.9 \times 10^{14} \times l^8 (\log l)^3\,. \quad (9)$$

4. The Case of Small l

Next, we treat the cases when $l \in [3, 238]$. In this case, $k < l \le 238$, and inequality (9) implies that $m < 8.27 \times 10^{35}$ and $n < 4m < 3.31 \times 10^{36}$. Now, write

$$\Lambda_1 := (n-1)\log\alpha - (2m-2)\log\beta + \log(g/h^2).$$

Suppose $\Lambda_1 > 0$ (the other case can be handled in the same way). Then, $0 < \Lambda_1 < e^{\Lambda_1} - 1 = (g/h^2)\beta^{-2m+2}\alpha^{n-1} - 1 < 9 \times \beta^{-(m-1)}$. Thus, we have

$$0 < (n-1)\log\alpha - (2m-2)\log\beta + \log(g/h^2) < 9 \times \beta^{-(m-1)}\,.$$

By dividing the above inequality by $\log\beta$, we get

$$0 < (n-1)\gamma - (2m-2) + \mu < 16.1 \times \beta^{-(m-1)}, \quad (10)$$

where the numbers γ and μ are defined as $\gamma := \gamma_{k,l} = \log\alpha / \log\beta$ and $\mu := \mu_{k,l} = \log(g/h^2)/\log\beta$.

We claim that γ is irrational. Indeed, if $\gamma = p/q$, for some $p, q \in \mathbb{Z}_{>0}$, we would obtain $\alpha^q = \beta^p$, which is impossible by using the same argument as for $\Gamma_1 \neq 0$. Let us denote $q_{(m,k,l)}$ by the denominator of the m-th convergent of the continued fraction expansion of $\gamma_{k,l}$.

By setting $M := 3.31 \times 10^{36}$, we use software *Mathematica*® (see book [26] and our codes of these computations in Appendix A) to get

$$\min_{\substack{3 \le k \le 237 \\ 4 \le l \le 238}} \{q_{(80,k,l)}\} > 2.1 \times 10^{47} > 6M,$$

and also, for $\epsilon_{(k,l)} := ||\mu \cdot q_{(80,k,l)}|| - M||\gamma \cdot q_{(80,k,l)}||$, we obtain that

$$\min_{\substack{3\leq k\leq 237\\ 4\leq l\leq 238}}\{\epsilon_{(k,l)}\} > 3.9\times 10^{-144}.$$

Note that the all conditions to use 2 are satisfied for the choice of $A = 16.1$ and $B = \beta$, and hence there is no integer solution to inequality (10) (and consequently no integer solution to Equation (4)) for n and m with

$$\frac{\log(A\cdot q_{(80,k,l)}/\epsilon_{(k,l)})}{\log(B)} \leq m-1 \text{ and } n-1\leq M,$$

for all $k \in [3, 237]$ and $l \in [4, 238]$. Since $n - 1 < M$, then, we have

$$m-1 < \frac{\log(16.1\times q_{(80,k,l)}/\epsilon_{(k,l)})}{\log(\beta)} < 973.$$

Therefore, $m \leq 973$ and so $n < 4m \leq 3892$. Now, we prepare a simple routine in *Mathematica*® which returns only the solutions

$$(n,m,k,l) \in \{(3,1,k,l),(3,2,k,l),(5,3,2,l)\}.$$

In conclusion, there are no solutions of (4) for $k < l \leq 238$ and $m > 3$ (and so for $n > 3$).

5. The Case $l \geq 239$

Now, we deal with the case $l \geq 239$. For that, the following holds:

$$m < 4.9\times 10^{14}\times l^8(\log l)^3 < 2^{l/2}.$$

By applying a very useful argument, due to Bravo and Luca [12] (pp. 2130–2132), we deduce that

$$|h\beta^{m-1} - 2^{m-2}| < \frac{2^{m-1}}{2^{l/2}} + \frac{2^m l}{2^l} + \frac{2^{m+1}l}{2^{3l/2}} < 4\times\frac{2^{m-2}}{2^{l/2}}, \tag{11}$$

where the last inequality of (11) holds, since $4l < 2^{l/2}$ and $8l < 2^l$ are true for $l \geq 11$. Now, by the Mean Value Theorem, we have

$$|h^2\beta^{2m-2} - 2^{2m-4}| \leq 2\max\{h\beta^{m-1}, 2^{m-2}\}|h\beta^{m-1} - 2^{m-2}| < 4\times\frac{2^{2m-2}}{2^{l/2}}, \tag{12}$$

where we used (11) together with $h\beta^{m-1} < 2^{m-1}$. Thus, by combining (5) and (12), we get

$$\begin{aligned} |2^{2m-4} - g\alpha^{n-1}| &\leq |h^2\beta^{2m-2} - 2^{2m-4}| + |g\alpha^{n-1} - h^2\beta^{2m-2}| \\ &< 4\times\frac{2^{2m-2}}{2^{l/2}} + h\beta^{m-1} + \frac{7}{4}. \end{aligned}$$

Therefore, after dividing the last inequality by 2^{2m-4}, we get

$$\left|\frac{g\alpha^{n-1}}{2^{2m-4}} - 1\right| < \frac{4}{2^{l/2-2}} + \frac{h\beta^{m-1}}{2^{2m-4}} + \frac{7}{2^{2m-2}} < \frac{4}{2^{l/2-2}} + \frac{1}{2^{m-3}} + \frac{7}{2^{2m-2}}. \tag{13}$$

If $m \leq l$, then $F_n^{(k)} = (F_m^{(l)})^2 + 1 = 2^{2m-4} + 1$. On the other hand, we can slightly modify the Bravo and Gomez's [27] argument to find k-Fibonacci numbers of the form $2^t - 1$, to work on the equation $F_n^{(k)} = 2^t + 1$. Thus, in our case, the only solution is $(k, m, n) = (2, 3, 2)$. Therefore, $m > l \Rightarrow m - 3 > l - 3 > l/2 - 2$, for all $l \geq 3$, and we can rewrite (13) as

$$\left| \frac{g\alpha^{n-1}}{2^{2m-4}} - 1 \right| < \frac{12}{2^{l/2-2}}. \tag{14}$$

Note that, if $\Gamma_2 := g\alpha^{n-1}2^{4-2m} - 1$, then $\Gamma_2 \neq 0$. In fact, we can proceed as before to conclude that, if $g\alpha^{n-1} = 2^{2m-4}$, then, by Galois conjugation, we arrive at an absurdity as $1 > 2^{2m-4}$. Thus, in order to apply Lemma 1 again, we consider

$$\beta_1 := g,\ \beta_2 := \alpha,\ \beta_3 := 2,\ c_1 := 1,\ c_2 := n - 1,\ c_3 := 4 - 2m.$$

Since $\mathrm{h}(\beta_1) = \mathrm{h}(g) \leq 3 \log k$, $\mathrm{h}(\beta_2) = \mathrm{h}(\alpha) = \log \alpha / k$, $\mathrm{h}(\beta_3) = \mathrm{h}(2) = \log 2$ and $D' = [\mathbb{Q}(\alpha) : \mathbb{Q}] = k$, then we can choose $A_1' := 3k \log k$, $A_2' := \log 2$ and $A_3' := k \log 2$. Again, we can take $B' := n - 1$, and thus the conditions to apply Lemma 1 are satisfied yielding

$$\begin{aligned} |\Gamma_2| &> \exp\left(-2.80 \times 10^{11} k^2 \log k (1 + \log(n-1))(3k \log k)(\log 2)(k \log 2)\right) \\ &> \exp\left(-4.79 \times 10^{11} k^4 \log k^2 \log(n-1)\right). \end{aligned} \tag{15}$$

Now, from (9) and the fact that $n < 4m$, we obtain $\log(n-1) < 13.6 \log(l)$, which holds for $l \geq 239$. Thus, using this inequality, we rewrite (15) as

$$|\Gamma_2| > \exp\left(-6.52 \times 10^{12} k^4 \log k^2 \log l\right).$$

We obtain, by combining the previous inequality with (14),

$$(l/2 - 2) \log 2 - \log 12 < 6.52 \times 10^{12} k^4 \log k^2 \log l$$

and so

$$\frac{l}{\log l} < 1.89 \times 10^{13} k^4 \log k^3.$$

Again, from inequality (8), we get

$$l < 1.21 \times 10^{15} k^4 (\log k)^3. \tag{16}$$

Now, by (9) and (16), we arrive at

$$n < 4.2 \times 10^{140} k^{32} (\log k)^{27}. \tag{17}$$

6. The Case of Small k

Now, we consider the cases where $k \in [3, 1782]$. By (16) and $k \leq 1782$, we have $l \leq 5.12 \times 10^{30}$. Write $\Lambda_2 := (n-1) \log \alpha - (2m - 4) \log 2 - \log g$. Suppose $\Lambda_2 > 0$ (again, the other case is completely similar). Then,

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = g\alpha^{n-1}2^{4-2m} - 1 < 12 \times 2^{-l/2+2}$$

and so

$$0 < (n-1) \log \alpha - (2m - 4) \log 2 + \log g < 12 \times 2^{-l/2+2}.$$

By dividing by $\log 2$, we get

$$0 < (n-1)\gamma' - (2m - 4) + \mu' < 69.25 \times 2^{-l/2}, \tag{18}$$

where $\gamma' := \gamma_k' = \log \alpha / \log 2$ and $\mu' := \mu_k' = \log g / \log 2$.

Note that γ' is irrational, since $0 < \log\alpha/\log 2 = p/q$, for p, q positive integers, gives, when taking conjugates, $\alpha^q = 2^p \Rightarrow 1 > (\alpha^{(i)})^q = 2^q$, which is absurd. Now, we define $q_{m,k}$ as the denominator of the m-th convergent of the continued fraction of γ'. By taking $M' := 1.81 \cdot 10^{268}$, we use *Mathematica*® again to get

$$\min_{3\le k\le 1782} q_{600,k} > 5.1 \times 10^{301} > 6M'.$$

We also have that $\epsilon'_k := ||\mu' q_{600,k}|| - M'||\gamma' q_{600,k}|| > 2.6 \times 10^{-236}$ for all $k \in [3, 1782]$. Since the assumptions of Lemma 2 are satisfied, for $A' = 69.25$ and $B' = 2$, we can conclude that there are no solutions of inequality (10) for n and k satisfying

$$n - 1 < M \;\text{ and }\; \frac{l}{2} > \frac{\log(A' \cdot q_{200,k}/\epsilon'_k)}{\log(B')}.$$

Thus, $l/2 < 3669$, and then $239 \le l \le 7339$. By using (9) together with $n < 4m$, we obtain $n < 1.17 \times 10^{49}$. By using the reduction method again (in inequality (18)), we get $l < 225$, which was already solved.

7. The Final Step

Now, we still have $l \ge 239$. Then, it remains to verify the cases when $k \ge 1783$. Thus, the following inequality holds:

$$n < 4.42 \times 10^{118} k^{32} (\log k)^{35} < 2^{k/2}.$$

Using again the argument due to Bravo and Luca, we obtain

$$|g\alpha^{n-1} - 2^{n-2}| < \frac{2^{n-1}}{2^{k/2}} + \frac{2^n k}{2^k} + \frac{2^{n+1}k}{2^{3k/2}} < 4 \times \frac{2^{n-2}}{2^{k/2}}, \tag{19}$$

where we used that $4k < 2^{k/2}$ and $8k < 2^k$ are true for $k \ge 11$. Combining (5), (12), and (19), we get

$$\begin{aligned} |2^{2m-4} - 2^{n-2}| &\le |h^2\beta^{2m-2} - 2^{2m-4}| + |g\alpha^{n-1} - h^2\beta^{2m-2}| + |g\alpha^{n-1} - 2^{n-2}| \\ &< 4 \times \frac{2^{2m-2}}{2^{l/2}} + h\beta^{m-1} + \frac{7}{4} + 4 \times \frac{2^{n-2}}{2^{k/2}}. \end{aligned} \tag{20}$$

If $n \le k$, then (4) becomes $2^{n-2} = F_n^{(k)} = (F_m^{(l)})^2 + 1$, which cannot happen for $n \ge 4$, since a square plus 1 is never divisible by 4. The remaining cases, $n \in \{2, 3\}$, give us the solutions already known. It follows that $n > k$.

Therefore, by dividing (20) by 2^{n-2} and using the inequalities $n - 2 \ge 2m - 3$, $(n-1)/2 \ge m - 1$, $n > k$ and $l > k$, we get

$$|2^{2m-n-2} - 1| < \frac{8}{2^{l/2}} + \frac{1}{2^{(n-3)/2}} + \frac{7}{2^n} + \frac{4}{2^{k/2}} < \frac{20}{2^{(k-3)/2}}.$$

Since $m > l > k$, as a consequence of Lemma 3, we have $n \ne 2m - 2$, and then $|2^{2m-n-2} - 1| > 1/2$, which, combined with the previous inequality, gives $k \le 13$, which is a contradiction. This completes our proof.

8. Further Comments: The Case of a General *c*

As mentioned in Remark 1, we only choose $c = 1$ in order to explicit all calculations. In the general case, the equation

$$F_n^{(k)} = (F_m^{(l)})^2 + c$$

has infinitely many solutions (n, k, m, l, c) (this follows, clearly, because the linear dependence of equation in the variable c). For this reason, the more interesting case happens when $c \ge 1$ is fixed.

In this case, it seems reasonable to expect to deal with the case of an upper bound for all other variables (i.e., n, m, k and l) in terms of c. In fact, the proof is completely similar until we arrive at the inequality (6), which would be

$$|(g/h^2)\beta^{-2m+2}\alpha^{n-1} - 1| < \frac{11c}{\beta^{m-1}}.$$

Now, we split the proof into two cases:

- If $\beta^{m/2} < c$. In this case, we get directly the bounds

$$l < m < 4\log c/\log 2 \text{ and } k < n < 2 + 2\log(2c^4)/\log 2,$$

where the last inequality is obtained from

$$(\sqrt{2})^{n-2} < F_n^{(k)} = (F_m^{(l)})^2 + c < \beta^{2m-2} + c < c^4 + c \leq 2c^4.$$

- If $\beta^{m/2} \geq c$. For this case, inequality (6) becomes

$$|(g/h^2)\beta^{-2m+2}\alpha^{n-1} - 1| < \frac{22}{\beta^{m/2}}$$

which does not depend on c and thus, from this point on, we simply mimic the proof of Theorem 1.

9. Conclusions

In this paper, we study a Diophantine problem related to a higher order generalization of the Fibonacci sequence. In fact, the k-generalized Fibonacci numbers, denoted by $(F_n^{(k)})_n$, are defined by the kth order recurrence $F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)}$ with initial values $0, \ldots, 0, 1$ (k terms), where $F_1^{(k)} = 1$. In particular, we solve completely the Diophantine equation $F_n^k = (F_m^{(l)})^2 + 1$ (which can be related to the problem of terms two (possibly distinct) generalized Fibonacci sequences as consecutive terms of an orbit in a quadratic dynamics related to the Mandelbrot set). The main tools in the proof are *Baker's theory*, reduction, and Bravo–Luca methods (combined with a combinatorial lemma and some *Mathematica*® routines).

Author Contributions: P.T. conceived the presented idea. Both authors have worked on the conceptualization, methodology, investigation, writing, reviewing, editing. The software was done by A.P.C. All authors have read and agreed to the published version of the manuscript.

Funding: Ana Paula Chaves is thankful for the support in part by grant CNPq Universal 01/2016 - 427722/2016-0. Pavel Trojovský was supported by Project of Excelence PrF UHK No. 2215/2020, University of Hradec Králové, Czech Republic.

Acknowledgments: The authors express their debt of gratitude to the editor and reviewers for their helpful and detailed comments in improving the presentation and quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Mathematica Commands

Below, we shall present the *Mathematica* commands used along the paper (the calculations in this paper took roughly four days on a 2.5 GHz Intel Core i5 4 GB Mac OSX.).

- The nth term of the k-generalized Fibonacci sequence $F_n^{(k)}$:

```
F[n_, k_] :=
SeriesCoefficient[Series[x/(1 - Sum[x^j, {j, 1, k}]), {x, 0, 1100}], n]
```

- The characteristic polynomial $\psi_k(x)$

```
s[x_, k_] := x^k - Sum[x^j, {j, 0, k - 1}]
```

- The dominant root α of $\psi_k(x)$:

```
alphasd[k_] := x /. Last[NSolve[s[x, k], x, 1400]]
```

- The function $g(\alpha, k)$:

```
gsd[k_] := (alphasd[k] - 1)/(2 + (k + 1)*(alphasd[k] - 2))
```

- The denominator of the nth convergent of the continued fraction of x:

```
DeFrac[x_, n_] := Last[Denominator[Convergents[x, n]]]
```

- The number $\gamma := \gamma_{k,l}$ in (10):

```
gama[k_,l_] := Log[alphasd[k]]/Log[alphasd[l]]
```

- The Nint function of x, i.e., $||x||$:

```
Near[x_] := Min[Abs[x - Floor[x]], Abs[Ceiling[x] - x]]
```

- The number $\mu := \mu_{k,l}$ in (10):

```
Mi[k_,l_] := Log[gsd[k]/gsd[l]^2]/Log[alphasd[l]]
```

- The number $\epsilon_{(k,l)} := ||\mu \cdot q_{(80,k,l)}|| - M||\gamma \cdot q_{(80,k,l)}||$:

```
e[k_, l_] :=   Near[Mi[k,l]*DeFrac[gama[k,l], 80]]
-3.31*10^(36)*Near[gama[k,l]*DeFrac[gama[k,l], 80]]
```

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