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Reflection-Like Maps in High-Dimensional Euclidean Space

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Abstract: In this paper, we introduce reflection-like maps in n-dimensional Euclidean spaces, which are affinely conjugated to $\theta : (x_1, x_2, \ldots, x_n) \to \left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$. We shall prove that reflection-like maps are line-to-line, cross ratios preserving on lines and quadrics preserving. The goal of this article was to consider the rigidity of line-to-line maps on the local domain of $\mathbb{R}^n$ by using reflection-like maps. We mainly prove that a line-to-line map η on any convex domain satisfying $\eta^{\circ 2} = id$ and fixing any points in a super-plane is a reflection or a reflection-like map. By considering the hyperbolic isometry in the Klein Model, we also prove that any line-to-line bijection $f : \mathbb{D}^n \mapsto \mathbb{D}^n$ is either an orthogonal transformation, or a composition of an orthogonal transformation and a reflection-like map, from which we can find that reflection-like maps are important elements and instruments to consider the rigidity of line-to-line maps.

Keywords: line-to-line maps; reflection-like maps; affine transformations

MSC: 51F15; 30C35

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1. Introduction

The research of rigidity of line-to-line maps has a long history (see Reference [1–5], etc.) from different perspectives. We say that a map $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is line-to-line, if $f(l)$ is contained in some line for any line l in $\mathbb{R}^n$. Similarly, we say that a circle in Möbious space $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ (or a geodesic in hyperbolic space $\mathbb{H}^n = \{(x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n | x_n > 0\}$) is a line. For example, in Reference [4], J. Jeffers proves that a circle-to-circle bijection $f : \hat{\mathbb{R}}^n \mapsto \hat{\mathbb{R}}^n$ is a Möbious transformation, a geodesic-to-geodesic bijection $f : \mathbb{H}^n \mapsto \mathbb{H}^n$ is a hyperbolic isometry and a line-to-line bijection $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is an affine transformation. Various geometries are considered in mathematical researches of different transformations, such as complex curves, were studied using Laguerre planes and Grünwald planes in Reference [6].

It is well known that any Möbious transformation is a composition of finite inversions in $n-$dimensional spherical space $\hat{\mathbb{R}}^n$ (see Reference [7] for details). We can say that inversions are basic elements of Möbious transformations. Let

$$\mathbb{H}^{n+1} = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} > 0\}$$

be $(n+1)-$dimensional hyperbolic space with hyperbolic metric $\rho_H = \frac{1}{x_{n+1}}$. A reflection on $\mathbb{H}^{n+1}$ is an isometry which fixes an $n-$hyperplane in $\mathbb{H}^{n+1}$ and any hyperbolic isometry is a composition of finite reflections in $\mathbb{H}^{n+1}$. We can say that reflections are basic elements of hyperbolic isometries. Similarly,

$$\mathbb{S}^n_+ = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \dots + x_n^2 + x_{n+1}^2 = 1, x_{n+1} > 0\} \quad (1)$$

can be seen as an $n-$dimensional hyperbolic subspace of $\mathbb{H}^{n+1}$. Let

$$\mathbb{D}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_1^2 + x_2^2 + \dots + x_n^2 < 1\} \quad (2)$$

be the Klein Model of hyperbolic space defined by the natural projection

$$\begin{aligned} \tau : \mathbb{S}^n_+ &\mapsto \mathbb{D}^n \\ (x_1, \dots, x_n, x_{n+1}) &\to (x_1, \dots, x_n). \end{aligned} \quad (3)$$

Obviously, a map $F : \mathbb{S}^n_+ \mapsto \mathbb{S}^n_+$ is a hyperbolic isometry, if and only if the transformation $f = \tau \circ F \circ \tau^{-1} : \mathbb{D}^n \mapsto D^n$ is a hyperbolic isometry in Klein Model $\mathbb{D}^n$ in the following commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n_+ & \xrightarrow{F} & \mathbb{S}^n_+ \\ \tau \downarrow & & \downarrow \tau \\ \mathbb{D}^n & \xrightarrow{f} & \mathbb{D}^n \end{array} . \quad (4)$$

A geodesic in Klein Model $\mathbb{D}^n$ is a segment which is the projection of a geodesic in $\mathbb{S}^n_+$ under τ, since any geodesic in $\mathbb{S}^n_+$ is an arc perpendicular to $\partial\mathbb{S}^n_+ \subset \partial\mathbb{H}^{n+1}$.

For any subset $\Omega \subset \mathbb{R}^n$, we call L a line in Ω, if there exists a line l in $\mathbb{R}^n$, such that $L = l \cap \Omega$. We say that two lines L_1, L_2 in Ω are parallel, if l_1, l_2 are parallel. We say that three lines L_1, L_2, L_3 in Ω are concurrent, if l_1, l_2, l_3 have a common point in $\mathbb{R}^n$. We say that a map $f : \Omega \mapsto \mathbb{R}^n$ is *line-to-line*, if the image points of any collinear points are collinear and $f : \Omega \mapsto \Omega'$ is *line-onto-line*, if $f(L)$ is a line in $\Omega' \subset \mathbb{R}^n$ for any line L in Ω.

One can find that f is a line-to-line bijection in $\mathbb{D}^n$ because the isometry F is a geodesic-to-geodesic bijection in $\mathbb{S}^n_+$ in diagram (4). Especially, if the isometry $F : \mathbb{S}^n_+ \mapsto \mathbb{S}^n_+$ is a reflection, then the line-to-line map $f : \mathbb{D}^n \mapsto \mathbb{D}^n$ satisfies $f^{\circ 2} = id$ and its fixed-points set is an $(n-1)-$dimensional superplane in $\mathbb{D}^n$. Obviously, f may not be an affine transformation. This is the origin of reflection-like maps considered in this paper. We shall show that reflection-like maps are basic elements and instruments to consider the rigidity of line-to-line maps.

In Reference [8], B. Li et al., introduce $g-$reflection maps in $\mathbb{R}^2$, which are affinely conjugated to the map

$$(x, y) \to \left(-\frac{x}{1+x}, \frac{y}{1+x}\right) \quad (5)$$

for any point in $\{(x, y) \in \mathbb{R}^2 | x \neq -1\}$ and give the following result.

Theorem 1 ([8]). *Suppose that $\mathcal{D} \subset \mathbb{R}^2$ is a convex domain and a map $f : \mathcal{D} \mapsto \mathcal{D}$ is line-to-line and satisfies $f^{\circ 2} = id$. If f is not the restriction to $\mathcal{D}$ of an affine transformation of $\mathbb{R}^2$, then f is a restriction of $g-$reflection map to $\mathcal{D}$.*

In Reference [9], B. Li et al., use $g-$reflection maps on the rigidity of line-to-line maps in the upper plane $\mathbb{H} \subset \mathbb{R}^2$ and prove that

Theorem 2 ([9]). *Suppose that $f : \mathbb{H} \mapsto \mathbb{H}$ is a line-to-line surjection. Then, f is either an affine transformation, or a composition of an affine transformation and a g-reflection map.*

In Reference [10], B. Li et al., prove that any $g-$refection map preserves the cross ratios

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

of any four collinear points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ and the following result.

Theorem 3 ([10], Theorem 3.6)**.** *Suppose that $\mathcal{D} \subset \mathbb{R}^2$ is a domain and a line-to-line map $f : \mathcal{D} \mapsto \mathbb{R}^2$ is injective and non-degenerate. Then, f is either an affine transformation, or a composition of a g-reflection map and an affine transformation.*

Here, a line-to-line map $f : \mathcal{D} \mapsto \mathbb{R}^2$ is *degenerate* (see Reference [11]), if the image space $f(\mathcal{D})$ is contained in some line (otherwise, it is *non-degenerate*).

The goal of this article is to consider the rigidity of line-to-line maps on local domains in $\mathbb{R}^n$. We shall introduce the case in $n-$dimensional space $\mathbb{R}^n$ of $g-$reflection maps, named reflection-like maps in this paper, and prove the following main results.

Theorem 4. *Suppose that Ω is any convex domain in $\mathbb{R}^n$ and $\mathcal{A}^\eta$ is a super-plane such that $\Omega \cap \mathcal{A}^\eta \neq \emptyset$. A line-to-line map $\eta : \Omega \mapsto \Omega$ satisfies $\eta^{\circ 2} = id$ and $\eta(P) = P$ for any $P \in \Omega \cap \mathcal{A}^\eta$. Then, η is a reflection or a reflection-like map.*

Theorem 5. *Suppose that $\mathbb{D}^n \subset \mathbb{R}^n$ is a Klein Model of $n-$dimensional hyperbolic space and a map $f : \mathbb{D}^n \mapsto \mathbb{D}^n$ is a hyperbolic isometry. Then, f is either an orthogonal transformation, or a composition of an orthogonal transformation and a reflection-like map.*

In the next sections, we shall prove that reflection-like maps are line-to-line and linearly conjugated to each other. Moreover, the image of three parallel lines under reflection-like maps are parallel or concurrent. The absolute cross ratios may not be preserved by reflection-like maps. But, we shall prove that refection-like maps preserve the absolute cross ratios of any four distinct collinear points, something like projective maps preserve the cross ratios of four points in a projective line in projective geometries. We shall also prove that refection-like maps transfer spheres to quadrics, from which we can obtain that they map quadrics to quadrics. Especially, if the image of a sphere is a sphere, then it is invariant.

2. Reflection-Like Maps in High Dimension Space $\mathbb{R}^n$

In this section, we shall give the definition of reflection-like maps firstly and prove invariant properties under affine conjugation. We mainly prove Theorem 4, the rigidity of reflection-like maps in local domain of $n-$dimensional space.

Denote points in $\mathbb{R}^n$ by $X(x_1, x_2, \cdots, x_n)$, $Y(y_1, y_2, \cdots, y_n)$ and the line passing through X, Y by L_{XY}, the Euclidean distance between X, Y by $|X - Y|$. Denote the vector from X to Y by $\overrightarrow{XY}$.

Let $\mathcal{A}, \mathcal{B}$ be two $(n-1)-$dimensional planes (superplanes) in $\mathbb{R}^n$ and $\mathcal{P}$ be a point

$$\begin{aligned} &\mathcal{A} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1 = 1\}, \\ &\mathcal{B} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | x_1 = 0\}, \\ &\mathcal{P} = (-1, 0, \ldots, 0). \end{aligned} \tag{6}$$

Obviously, $\mathcal{P}$ and $\mathcal{A}$ have equal Euclidean distances to $\mathcal{B}$. The map

$$\begin{aligned} &\theta : \mathbb{R}^n \backslash \mathcal{B} \mapsto \mathbb{R}^n \backslash \mathcal{B} \\ &\quad (x_1, x_2, \ldots, x_n) \to \Big(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\Big) \end{aligned} \tag{7}$$

satisfies $\theta^{\circ 2} = id$. Moreover, $\{\mathcal{P}\} \cup \mathcal{A}$ is the fixed-point set of θ and the two components of $\mathbb{R}^n$ divided by $\mathcal{B}$ are invariant under θ.

Definition 1. *We say that a map η is a reflection-like map in $\mathbb{R}^n$, if it is affinely conjugated to θ. That is, one can find an affine transformation $g : \mathbb{R}^n \mapsto \mathbb{R}^n$, such that $\eta = g \circ \theta \circ g^{-1} : \mathbb{R}^n \backslash g(\mathcal{B}) \mapsto \mathbb{R}^n \backslash g(\mathcal{B})$.*

Obviously, θ defined in (7) is a refection-like map in $\mathbb{R}^n$.

Moreover, we say that $\mathcal{A}$ defined in (6) is Axis, $\mathcal{B}$ is Boundary, and $\mathcal{P}$ is Base point of the refection-like map θ.

Generally, given any affine transformation g, the reflection-like map $\eta = g \circ \theta \circ g^{-1} : \mathbb{R}^n \backslash g(\mathcal{B}) \mapsto \mathbb{R}^n \backslash g(\mathcal{B})$ has Boundary $\mathcal{B}^\eta = g(\mathcal{B})$, Axis $\mathcal{A}^\eta = g(\mathcal{A})$, and Base point $\mathcal{P}^\eta = g(\mathcal{P})$. Obviously, $\eta^{\circ 2} = id$, $\{\mathcal{P}^\eta\} \cup \mathcal{A}^\eta$ is the fixed-point set of η, $\mathcal{B}^\eta$ is parallel to $\mathcal{A}^\eta$, and $\mathcal{P}^\eta$ and $\mathcal{A}^\eta$ have equal Euclidean distances to $\mathcal{B}^\eta$. Moreover, the two components of $\mathbb{R}^n$ divided by $\mathcal{B}^\eta$ are invariant under η.

Definition 2. *We call L a line in $\mathbb{R}^n \backslash \mathcal{B}$, if there exists a line l in $\mathbb{R}^n$, such that $L = l \cap \mathbb{R}^n \backslash \mathcal{B}$. If $l \cap \mathcal{B} = \{\tilde{P}\}$, then we say that L has boundary point $\tilde{P}$.*

Proposition 1. *The reflection-like map $\theta : \mathbb{R}^n \backslash \mathcal{B} \mapsto \mathbb{R}^n \backslash \mathcal{B}$ is a line-onto-line bijection.*

Proof. Let us prove that f is line-to-line in $\mathbb{R}^n \backslash \mathcal{B}$ firstly. That is, for any three collinear points $X(x_1, x_2, \cdots, x_n)$, $Y(y_1, y_2, \cdots, y_n)$, $Z(z_1, z_2, \cdots, z_n)$, their image points $X'(x'_1, x'_2, \cdots, x'_n)$, $Y'(y'_1, y'_2, \cdots, y'_n)$, $Z'(z'_1, z'_2, \cdots, z'_n)$ are collinear. There exists some $\lambda \in \mathbb{R} \backslash \{0, 1\}$, such that $Z = \lambda X + (1 - \lambda)Y$. That is, $z_i = \lambda x_i + (1 - \lambda) y_i$, for any $i = 1, 2, \cdots, n$. We have $x'_1 = \frac{1}{x_1}$, $y'_1 = \frac{1}{y_1}$, and

$$
\begin{aligned}
z'_1 = \frac{1}{z_1} &= \frac{1}{\lambda x_1 + (1-\lambda) y_1} \\
&= \frac{x'_1 y'_1}{\lambda y'_1 + (1-\lambda) x'_1} \\
&= \frac{\lambda y'_1}{\lambda y'_1 + (1-\lambda) x'_1} x'_1 + \frac{(1-\lambda) x'_1}{\lambda y'_1 + (1-\lambda) x'_1} y'_1.
\end{aligned}
$$

Let $\lambda' = \frac{\lambda y'_1}{\lambda y'_1 + (1-\lambda) x'_1}$, and then $z'_1 = \lambda' x'_1 + (1 - \lambda') y'_1$. Meanwhile, $x'_i = \frac{x_i}{x_1}$, $y'_i = \frac{y_i}{y_1}$ for any $i = 2, 3, \ldots, n$ and

$$
\begin{aligned}
z'_i = \frac{z_i}{z_1} &= \frac{\lambda x_i + (1-\lambda) y_i}{\lambda x_1 + (1-\lambda) y_1} \\
&= \frac{\lambda y'_1}{\lambda y'_1 + (1-\lambda) x'_1} x'_i + \frac{(1-\lambda) x'_1}{\lambda y'_1 + (1-\lambda) x'_1} y'_i \\
&= \lambda' x'_i + (1-\lambda') y'_i.
\end{aligned}
$$

Thus, $Z' = \lambda' X' + (1 - \lambda') Y'$, which follows that X', Y', Z' are collinear. Hence, θ is line-to-line. Moreover, one can find that θ is bijective and $\theta(L)$ is a line in $\mathbb{R}^n \backslash \mathcal{B}$ for any line L in $\mathbb{R}^n \backslash \mathcal{B}$, since $\theta^{\circ 2} = id$. That is, θ is a line-onto-line bijection and the proof is completed. □

Proposition 2. *For any line L in $\mathbb{R}^n \backslash \mathcal{B}$, the reflection-like map $\theta : \mathbb{R}^n \backslash \mathcal{B} \mapsto \mathbb{R}^n \backslash \mathcal{B}$ satisfies the following.*

(i) *If $L \not\subset \mathcal{A}$, then $\theta(L) = L$, if and only if Base point $\mathcal{P} \in L$;*
(ii) *If $\theta(L) \neq L$, then $\theta(L)$ is parallel to L, if and only if L is parallel to Axis $\mathcal{A}$.*

Proof. (i). We only need to prove that $\mathcal{P} \in L_{XX'}$ for any point $X \in \mathbb{R}^n \backslash \mathcal{B}$ and $X' = \theta(X) \neq X$, since θ is line-to-line and satisfies $\theta^{\circ 2} = id$.

Let $X(x_1, x_2, \ldots, x_n)$, $X'\left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$ and $\lambda = \frac{1}{1 - x_1}$, then one can find that $\mathcal{P} = \lambda X + (1 - \lambda) X'$, which means $\mathcal{P} \in L_{XX'}$.

(ii). For two distinct points $X(x_1, x_2, \cdots, x_n)$, $Y(y_1, y_2, \cdots, y_n)$ in L, denote the image points under θ by $X'\Big(\frac{1}{x_1}, \frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}\Big)$, $Y'\Big(\frac{1}{y_1}, \frac{y_2}{y_1}, \cdots, \frac{y_n}{y_1}\Big)$. Obviously, L is parallel to Axis $\mathcal{A}$, if and only if $x_1 = y_1 (\neq 0)$. Then, $\theta(L)$ is parallel to L by

$$\overrightarrow{X'Y'} = \Big(0, \frac{y_2 - x_2}{y_1}, \cdots, \frac{y_n - x_n}{y_1}\Big) = \frac{1}{y_1}\overrightarrow{XY}.$$

On the other side, suppose that $\theta(L)$ is parallel to L and $x_1 \neq y_1$. We can obtain that $y_i = \frac{1+y_1}{1+x_1}x_i$, for any $i = 2, \cdots, n$ by $\overrightarrow{X'Y'}//\overrightarrow{XY}$. Let $\lambda = \frac{1+y_1}{y_1 - x_1}$, then $\mathcal{P} = \lambda X + (1-\lambda)Y$, which means that $\mathcal{P} \in L$, and $\theta(L) = L$ by the result of (i). This is a contradiction, and the proof is completed. □

Corollary 1. *The image of a parallelogram under a reflection-like map is a parallelogram, if and only if the parallelogram is parallel to Axis of the reflection-like map. Moreover, the image of a square is a square, if the square is parallel to Axis.*

Proposition 3. *For any two lines L_1, L_2 in $\mathbb{R}^n \backslash \mathcal{B}$, not parallel to $\mathcal{A}$, the reflection-like map $\theta : \mathbb{R}^n \backslash \mathcal{B} \mapsto \mathbb{R}^n \backslash \mathcal{B}$ satisfies the followings.*

(i) *$\theta(L_1)$ and $\theta(L_2)$ share a common boundary point if L_1 is parallel to L_2;*
(ii) *$\theta(L_1)$ is parallel to $\theta(L_2)$ if L_1 and L_2 share a common boundary point.*

Proof. We only need to prove that L_1 is a line passing through $\mathcal{P}$. From Proposition 2, we have $\theta(L_1) = L_1$. Denote the boundary point of L_1 by $\tilde{X}(0, x_2, \cdots, x_n)$, then the vector $\overrightarrow{\mathcal{P}\tilde{X}} = (1, x_2, \cdots, x_n) \subset L_1$.

(i) Suppose that L_2 is any line parallel to L_1. For any point $Y(y_1, y_2, \cdots, y_n)$ in L_2, one can obtain $L_2 = \{(y_1 + t, y_2 + tx_2, \cdots, y_n + tx_n) | t \in \mathbb{R}\backslash\{-y_1\}\}$ and

$$\theta(L_2) = \Big\{\Big(\frac{1}{y_1 + t}, \frac{y_2 + tx_2}{y_1 + t}, \cdots, \frac{y_n + tx_n}{y_1 + t}\Big) | t \in \mathbb{R}\backslash\{-y_1\}\Big\}.$$

It follows that $\tilde{X}$ is the limit point of $\theta(L_2)$ as t tends to ∞. That is, $\theta(L_2)$ and $\theta(L_1)$ share common boundary point if L_2 is parallel to L_1.

(ii) Suppose that L_2 is any line sharing common boundary point $\tilde{X}(0, x_2, \cdots, x_n)$ with L_1. For any point $Y(y_1, y_2, \cdots, y_n) \in L_2$, we can find that the vector

$$\overrightarrow{\tilde{X}Y} = (y_1, y_2 - x_2, \cdots, y_n - x_n) \subset L_2.$$

So, we have

$$L_2 = \{((1+t)y_1, y_2 + t(y_2 - x_2), \cdots, y_n + t(y_n - x_n)) | t \in \mathbb{R}\backslash\{-1\}\},$$

and

$$\theta(L_2) = \Big\{\Big(\frac{1}{(1+t)y_1}, \frac{y_2 + t(y_2 - x_2)}{(1+t)y_1}, \cdots, \frac{y_n + t(y_n - x_n)}{(1+t)y_1}\Big) | t \in \mathbb{R}\backslash\{-1\}\Big\}.$$

As t tends to ∞, we obtain its boundary point $\tilde{Y}\Big(0, \frac{y_2 - x_2}{y_1}, \cdots, \frac{y_n - x_n}{y_1}\Big)$.

Denote $\theta(Y) = Y'\Big(\frac{1}{y_1}, \frac{y_2}{y_1}, \cdots, \frac{y_n}{y_1}\Big) \in \theta(L_2)$, then the vector

$$\overrightarrow{\tilde{Y}Y'} = \Big(\frac{1}{y_1}, \frac{x_2}{y_1}, \cdots, \frac{x_n}{y_1}\Big) \subset \theta(L_2),$$

which follows that $\theta(L_2)$ is parallel to $\theta(L_1)$ for $\overrightarrow{\mathcal{P}\tilde{X}} = y_1\overrightarrow{\tilde{Y}Y'}$. $\square$

Moreover, we can have the following.

Lemma 1. *Suppose that a reflection-like map η has the same Base point and Axis as θ. Then, $\eta = \theta$.*

Proof. We need only prove that the reflection-like map is uniquely determined by Base point $\mathcal{P}$ and Axis $\mathcal{A}$. One can know that Boundary $\mathcal{B}$ is parallel to $\mathcal{A}$ and lies between $\mathcal{P}$ and $\mathcal{A}$ with equal distances. For any point $X \in \mathbb{R}^n \backslash \mathcal{B}$, let L_1 be the line in $\mathbb{R}^n \backslash \mathcal{B}$ passing through X and $\mathcal{P}$, then $\eta(L_1) = L_1$ by Proposition 2. Choose any point $Y \in \mathcal{A} \backslash L_1$ and let L_2 be the line in $\mathbb{R}^n \backslash \mathcal{B}$ passing through X and Y, then it is easy to find $Y \in \eta(L_2)$. Let L_3 be the line passing through $\mathcal{P}$ and parallel to L_2, then $\eta(L_3) = L_3$. So $\eta(L_2), \eta(L_3)$ share common boundary point, denoted by $\tilde{Y}$ by Proposition 3. It follows that $\eta(L_2)$ is the line passing through Y and having boundary point $\tilde{Y}$. Then, $\eta(X) = L_1 \cap \eta(L_2)$ is determined uniquely. That is, the reflection-like map η is determined by $\mathcal{A}$ and $\mathcal{P}$. Therefore, we have $\eta = \theta$. $\square$

A transformation $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ is linear, if it is a composition of translations, scaling and orthogonal transformations on $\mathbb{R}^n$. We say that a reflection-like map η is linearly conjugated to θ, if one can find a linear map g, such that $\eta = g \circ \theta \circ g^{-1}$.

For any super-plane Π and a point $P \notin \Pi$, one can find a linear transformation g such that $g(\mathcal{A}) = \Pi$ and $g(\mathcal{P}) = P$. Then, $\eta = g \circ \theta \circ g^{-1}$ is a reflection-like map with Base point P and Axis Π. So, we can obtain the following by Lemma 1.

Theorem 6. *Any reflection-like map is linearly conjugated to θ.*

By conjugating affine transformation $g : (x_1, x_2, \cdots, x_n) \to (x_1 - 1, x_2, \cdots, x_n)$,

$$\theta' = g \circ \theta \circ g^{-1} : (x_1, x_2, \ldots, x_n) \to \left(-\frac{x_1}{1+x_1}, \frac{x_2}{1+x_1}, \cdots, \frac{x_n}{1+x_1} \right)$$

is the general form of the $g-$reflection map defined in (5) on $n-$dimensional space.

Proof of Theorem 4. Let $\mathcal{U}$ and $\mathcal{U}'$ be the two components of Ω divided by $\mathcal{A}^\eta$. We claim that there exist $P \in \mathcal{U}$ and $P' \in \mathcal{U}'$ such that $\eta(P) = P'$. Otherwise, suppose that we have $X, X' \in \mathcal{U}$, such that $\eta(X) = X'$ (as in the Figure 1a). For any $P' \in \mathcal{U}' \backslash L_{XX'}$, denote $Y_1 \in L_{XP'} \cap \mathcal{A}^\eta$, $Y_2 \in L_{X'P'} \cap \mathcal{A}^\eta$ and $P \in L_{XY_2} \cap L_{X'Y_1}$, then $P \in \mathcal{U}$ and $P' = \eta(P)$.

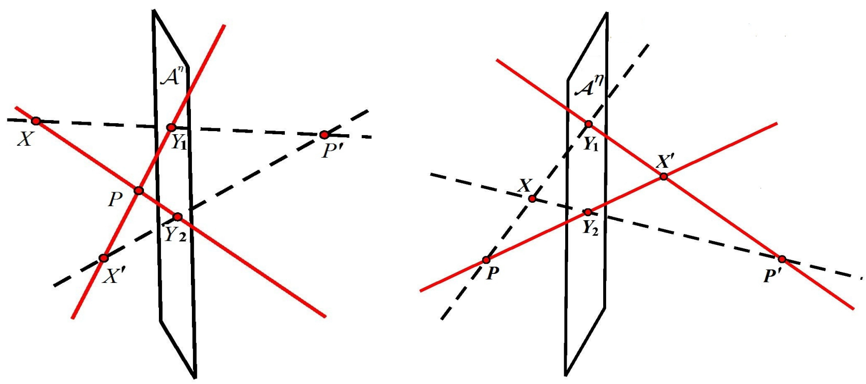

Figure 1. (**a**) Existence of P, P' in different sides. (**b**) Uniqueness determined by P, P'.

We shall prove that the line-to-line map is uniquely determined by P, P' and $\mathcal{A}^\eta \cap \Omega$. Let $\mathcal{V}$ denote the smallest convex domain containing P, P' and $\Omega \cap \mathcal{A}^\eta$. For any point $X \in \mathcal{V} \backslash L_{PP'}$

(as in the Figure 1b), let $Y_1 \in L_{XP} \cap \mathcal{A}^\eta$, $Y_2 \in L_{XP'} \cap \mathcal{A}^\eta$, we can find that $X' = \eta(X) \in L_{PY_2} \cap L_{P'Y_1}$ is unique. Moreover, the line-to-line map on Ω will be uniquely determined by the mapping on its sub-domain $\mathcal{V} \backslash L_{PP'}$.

Next, we shall prove the existence of η. By conjugating some suitable affine transformation, we can suppose that $\mathcal{A}^\eta = \{(x_1, x_2, \cdots, x_n) | x_1 = 0\}$, $P = (-1, 0, \cdots, 0)$ and $P' = (k, 0, \cdots, 0)$ $(k > 0)$. If $k = 1$, then η is a reflection about $\mathcal{A}^\eta$

$$\eta : (x_1, x_2, \cdots, x_n) \to (-x_1, x_2, \cdots, x_n).$$

Otherwise, let $\mathcal{P}^\eta = (-\frac{2k}{k-1}, 0, \cdots, 0)$ and $K = \frac{k-1}{k}$, then

$$\eta : (x_1, x_2, \cdots, x_n) \to \Big(-\frac{x_1}{1 + Kx_1}, \frac{x_2}{1 + Kx_1}, \cdots, \frac{x_n}{1 + Kx_1} \Big)$$

is the reflection-like map with Axis $\mathcal{A}^\eta$ and Base point $\mathcal{P}^\eta$ such that $\eta(P) = P'$. □

Corollary 2. *Suppose that θ is the reflection-like map defined in (7). Given any positive integer $1 < r < n$, let Π be any $r-$dimensional plane in $\mathbb{R}^n \backslash \mathcal{B}$ passing through $\mathcal{P}$, then $\theta(\Pi) = \Pi$. Moreover, if $\Pi \cap \mathcal{A} \neq \emptyset$, then $\theta|_\Pi : \Pi \mapsto \Pi$ is a reflection-like map with Axis $\Pi \cap \mathcal{A}$ and Base point $\mathcal{P}$.*

Remark 1. *We give an example ($n = 3$) to show that Theorem 1* **A** *does not hold in the case of reflection-like maps in $\mathbb{R}^n$ ($n > 2$). That is, a line-to-line map $f : \Omega \mapsto \Omega$ on a convex domain $\Omega \subset \mathbb{R}^n$ satisfying $f^{\circ 2} = id$ may not be an affine transformation or a reflection-like map.*

Example 1. *Let $\mathcal{B} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 0\}$ and $f : \mathbb{R}^3 \backslash \mathcal{B} \mapsto \mathbb{R}^3 \backslash \mathcal{B}$ be defined as*

$$f : (x_1, x_2, x_3) \to \Big(\frac{1}{x_1}, \frac{x_2}{x_1}, -\frac{x_3}{x_1} \Big).$$

Obviously, $f^{\circ 2} = id$ and f is line-to-line, since f is a composition of an orthogonal transformation and a reflection-like map, while f cannot be a reflection-like map since its fixed-point set is $L_1 \cup L_2$, where $L_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = 1, x_3 = 0\}$ and $L_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 = -1, x_2 = 0\}$.

3. The Absolute Cross ratios in High Dimension Space $\mathbb{R}^n$

For any four distinct points $X(x_1, x_2, \cdots, x_n)$, $Y(y_1, y_2, \cdots, y_n)$, $Z(z_1, z_2, \cdots, z_n)$, $W(w_1, w_2, \cdots, w_n)$ in $\mathbb{R}^n$, the absolute cross ratio is defined as

$$|X, Y, Z, W| = \frac{|X - Z| \cdot |Y - W|}{|X - W| \cdot |Y - Z|}.$$

It is very important in high dimensional space. Especially, if $Z = \infty$, we can define it by the limit as Z tends to ∞

$$|X, Y, \infty, W| = \frac{|Y - W|}{|X - W|}.$$

It is well known that, for any subdomain $\Omega \subset \mathbb{R}^n$, a map $f : \Omega \mapsto \mathbb{R}^n$ is a Möbious transformation, if and only if f preserves the absolute cross ratios. The cross ratio is defined on four collinear points in projective geometry, and a projective transformation preserves cross ratios (see Reference [2,12] for details). While a reflection-like map considers one more dimension than a projectivity, it does not preserve absolute cross ratios.

For example (as in Figure 2), let $X(1,0)$, $Y(1,1)$, $Z(2,1)$ and $W(2,0) \in \mathbb{R}^2$, then $\theta(X) = X'(1,0)$, $\theta(Y) = Y'(1,1)$, $\theta(Z) = Z'\Big(\frac{1}{2}, \frac{1}{2}\Big)$ and $\theta(W) = W'\Big(\frac{1}{2}, 0\Big)$. We have that θ maps the square $XYZW$

to the quadrilateral $X'Y'Z'W'$ since θ is line-to-line. It is easy to calculate that $|X, Y, Z, W| = 2$ and $|X', Y', Z', W'| = \sqrt{5}$.

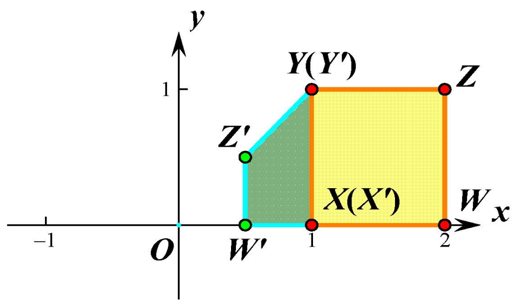

Figure 2. Reflection-like maps may not preserve absolute cross ratios.

In this section, we shall prove that reflection-like maps preserve the absolute cross ratios of any four collinear points. In fact, for any collinear points X, Y, Z, W, if $x_i \neq y_i$ for some $i = 1, \cdots, n$, then we can have

$$|X, Y, Z, W| = \frac{|x_i - z_i| \cdot |y_i - w_i|}{|x_i - w_i| \cdot |y_i - z_i|}.$$

Theorem 7. *Suppose that η is a reflection-like map with boundary $\mathcal{B}^\eta$. Then, for any four distinct collinear points X, Y, Z, W in $\mathbb{R}^n \backslash \mathcal{B}^\eta$, the absolute cross ratio $|X, Y, Z, W|$ is invariant under η. That is,*

$$|X, Y, Z, W| = |\eta(X), \eta(Y), \eta(Z), \eta(W)|.$$

Proof. By conjugating some suitable linear transformation, we can suppose that the reflection-like map is θ defined in (7). Then, we have $\theta(X) = X'\left(\frac{1}{x_1}, \frac{x_2}{x_1}, \cdots, \frac{x_n}{x_1}\right)$, $\theta(Y) = Y'\left(\frac{1}{y_1}, \frac{y_2}{y_1}, \cdots, \frac{y_n}{y_1}\right)$, $\theta(Z) = Z'\left(\frac{1}{z_1}, \frac{z_2}{z_1}, \cdots, \frac{z_n}{z_1}\right)$ and $\theta(W) = W'\left(\frac{1}{w_1}, \frac{w_2}{w_1}, \cdots, \frac{w_n}{w_1}\right)$ are collinear. If $x_1 \neq y_1$, we have $\frac{1}{x_1} \neq \frac{1}{y_1}$ and

$$\begin{aligned} |X', Y', Z', W'| &= \frac{\left|\frac{1}{x_1} - \frac{1}{z_1}\right| \cdot \left|\frac{1}{y_1} - \frac{1}{w_1}\right|}{\left|\frac{1}{x_1} - \frac{1}{w_1}\right| \cdot \left|\frac{1}{y_1} - \frac{1}{z_1}\right|} \\ &= \frac{|x_1 - z_1| \cdot |y_1 - w_1|}{|x_1 - w_1| \cdot |y_1 - z_1|} \\ &= |X, Y, Z, W|. \end{aligned}$$

If $x_1 = y_1$, then there exists some i, such that $x_i \neq y_i$. Thus, $\frac{x_i}{x_1} \neq \frac{y_i}{y_1}$ and

$$\begin{aligned} |X', Y', Z', W'| &= \frac{\left|\frac{x_i}{x_1} - \frac{z_i}{z_1}\right| \cdot \left|\frac{y_i}{y_1} - \frac{w_i}{w_1}\right|}{\left|\frac{x_i}{x_1} - \frac{w_i}{w_1}\right| \cdot \left|\frac{y_i}{y_1} - \frac{z_i}{z_1}\right|} \\ &= \frac{|x_i - z_i| \cdot |y_i - w_i|}{|x_i - w_i| \cdot |y_i - z_i|} \\ &= |X, Y, Z, W|. \end{aligned}$$

We complete the proof. □

4. Reflection-Like Maps and Quadrics

In this section, we shall prove that θ maps spheres to quadrics, from which we can obtain that reflection-like maps transfer quadrics to quadrics. Especially, if the image of a sphere is a sphere, then it is invariant.

Definition 3. *Given any reflection-like map, we say that the line passing its Base point and perpendicular to its Axis is its Equator.*

For example, the Equator of θ is
$$\mathcal{L} = \{(x_1, 0, \ldots, 0) | x_1 \in \mathbb{R}, x_1 \neq 0\}.$$
One can find that, given any affine transformation, the Equator of $\eta = g \cdot \theta \cdot g^{-1}$ may not be $g(\mathcal{L})$, while, if g is linear, the Equator of η is $g(\mathcal{L})$.

Theorem 8. *The reflection-like map θ maps any sphere to a quadric.*
If both $\mathbb{S}$ and $\theta(\mathbb{S})$ are $(n-1)-$dimensional spheres, then $\theta(\mathbb{S}) = \mathbb{S}$.
Moreover, if $\theta(\mathbb{S}) = \mathbb{S}$, then the center of $\mathbb{S}$ lies in the equator $\mathcal{L}$ of θ.
For any $P \in \mathcal{L}$, such that $P' = \theta(P) \neq P$, let $\mathbb{S}$ be the $(n-1)-$dimensional sphere with diameter PP', then $\theta(\mathbb{S}) = \mathbb{S}$.

Proof. Suppose that $\mathbb{S}$ is a sphere with radius r and center $C(c_1, c_2, \ldots, c_n)$. Then, any point $X(x_1, x_2, \ldots, x_n) \in \mathbb{S}$ satisfies

$$\mathbb{S} : (x_1 - c_1)^2 + (x_2 - c_2)^2 + \cdots + (x_n - c_n)^2 = r^2.$$

Denote the image point $\theta(X) = X'(x'_1, x'_2, \cdots, x'_n) \in \theta(\mathbb{S})$, then $\theta(X') = X$ since $\theta^{\circ 2} = id$. It follows $\theta(X') = \left(\frac{1}{x'_1}, \frac{x'_2}{x'_1}, \cdots, \frac{x'_n}{x'_1}\right) \in \mathbb{S}$, that is

$$\theta(\mathbb{S}) = \left\{(x'_1, x'_2, \cdots, x'_n) \Big| \left(\frac{1}{x'_1} - c_1\right)^2 + \left(\frac{x'_2}{x'_1} - c_2\right)^2 + \cdots + \left(\frac{x'_n}{x'_1} - c_n\right)^2 = r^2\right\}.$$

Obviously, it is a quadric

$$\theta(\mathbb{S}) : (1 - c_1 x'_1)^2 + (x'_2 - c_2 x'_1)^2 + \cdots + (x'_n - c_n x'_1)^2 = r^2 x_1^2.$$

Then, $\theta(\mathbb{S})$ is a sphere, if and only if $c_2 = c_3 = \cdots = c_n = 0$ and $c_1^2 - r^2 = 1$, since $-2c_i$ is the coefficient of the term $x'_1 x'_i$ $(i = 2, \cdots, n)$ and $c_1^2 - r^2$ is the coefficient of the term $x_1'^2$. It follows that, if $\theta(\mathbb{S})$ is also a sphere, then

$$\theta(\mathbb{S}) : (x'_1 - c_1)^2 + x_2'^2 + \cdots + x_n'^2 = r^2.$$

Thus, $\theta(\mathbb{S}) = \mathbb{S}$ and the center $C(c_1, 0, \cdots, 0) \in \mathcal{L}$ (as in Figure 3).

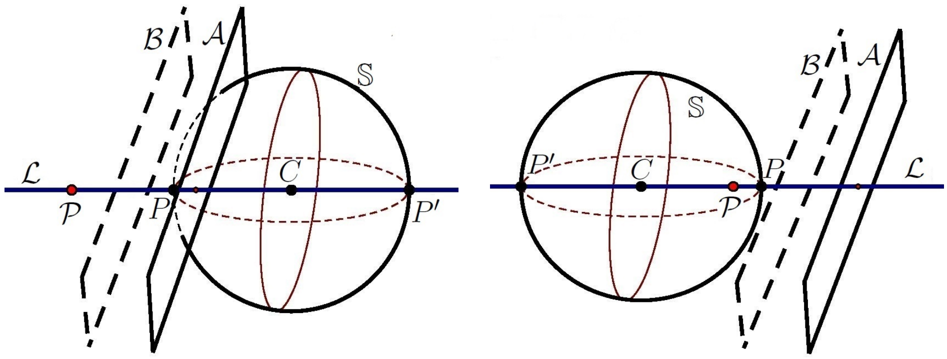

Figure 3. (**a**) Invariant sphere crossing Axis. (**b**) Invariant sphere surrounding Base point.

For any $P \in \mathcal{L}$ satisfying $P' = \theta(P) \neq P$, let $\mathbb{S}$ be the $(n-1)-$dimensional sphere with diameter PP' (as in Figure 3). Denote $P(p_1, \cdots, 0)$, $P'\Big(\frac{1}{p_1}, \cdots, 0\Big)$, $c_1 = \frac{1}{2}\Big(p_1 + \frac{1}{p_1}\Big)$ and $r = \frac{1}{2}\Big|p_1 - \frac{1}{p_1}\Big|$, then $\mathbb{S}$ has radius r and center $C(c_1, 0, \cdots, 0) \in \mathcal{L}$. One can find that $\theta(\mathbb{S}) = \mathbb{S}$ since $c_1^2 - r^2 = 1$. □

Obviously, the invariant sphere $\mathbb{S}$ lies in one component of $\mathbb{R}^n \backslash \mathcal{B}$ and the interior Ω of $\mathbb{S}$ is invariant under θ by the continuity of reflection-like maps, which shows that $\theta : \Omega \mapsto \Omega$ is a line-to-line bijection. Moveover, if Ω is a Klein Model of hyperbolic space, then $\theta : \Omega \mapsto \Omega$ is an isometry.

5. Reflection-Like Maps and Hyperbolic Isometries in Klein Model

In this section, we shall prove Theorem 5, the rigidity of line-to-line maps in a local domain of $\mathbb{R}^n$ by hyperbolic isometry on Klein Model defined by projection $\tau : \mathbb{S}^n_+ \mapsto \mathbb{D}^n$ as in Equations (1)–(3).

Lemma 2. *Suppose that $F : \mathbb{S}^n_+ \mapsto \mathbb{S}^n_+$ is a reflection. Then, $f = \tau \circ F \circ \tau^{-1} : \mathbb{D}^n \mapsto \mathbb{D}^n$ is a refection-like map or a reflection.*

Proof. Suppose that $F : \mathbb{S}^n_+ \mapsto \mathbb{S}^n_+$ is a reflection relative to $(n-1)-$hyperbolic plane $S \subset \mathbb{S}^n_+$. Then $F^{\circ 2} = id$ and $F(P) = P$ for any $P \in S$. It follows that $\tau(S)$ is an $(n-1)-$dimensional plane in $\mathbb{D}^n$ and $f = \tau \circ F \circ \tau^{-1} : \mathbb{D}^n \mapsto \mathbb{D}^n$ is a line-onto-line bijection, satisfying $f^{\circ 2} = id$ and $f(X) = X$ for any $X \in \tau(S)$. Then, f is the restriction of a refection-like map or a reflection by Theorem 4. Specifically, f is a reflection if the origin point $O \in \tau(S)$; otherwise, f is a reflection-like map. □

For any two distinct points $P, Q \in \mathbb{S}^n_+$, one can always get a unique reflection $F : \mathbb{S}^n_+ \mapsto \mathbb{S}^n_+$, satisfying that $F(P) = Q$. We can obtain the following Corollary.

Corollary 3. *For any point $X \in \mathbb{D}^n \backslash \{O\}$, there is a reflection-like map η satisfying that $\eta(\mathbb{D}^n) = \mathbb{D}^n$ and $\eta(O) = X$. Moveover, denote Axis of η by $\mathcal{A}^\eta$, then $\mathcal{A}^\eta \cap \mathbb{D}^n \neq \emptyset$.*

Proof of Theorem 5. If $f(O) = O$, then $f : \mathbb{D}^n \mapsto \mathbb{D}^n$ is the restriction to $\mathbb{D}^n$ of an orthogonal transformation on $\mathbb{R}^n$.

If $f(O) \neq O$, then there exists a reflection-like map η such that $\eta(\mathbb{D}^n) = \mathbb{D}^n$ and $\eta(O) = f^{-1}(O)$ by Corollary 3, which follows $g = f \circ \eta : \mathbb{D}^n \mapsto \mathbb{D}^n$ is a hyperbolic isometry satisfying $g(O) = O$. Thus, $g : \mathbb{D}^n \mapsto \mathbb{D}^n$ is the restriction to $\mathbb{D}^n$ of an orthogonal transformation on $\mathbb{R}^n$. It implies that $f = g \circ \eta$.

Above all, any hyperbolic isometry in Klein Model is either an orthogonal transformation, or a composition of an orthogonal transformation and a reflection-like map. □

From Theorem 5, one can deduce that any line-to-line bijection on $\mathbb{D}^n$ can be extended line-to-line to $\mathbb{R}^n$ (or except a superplane).

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