mathematics

MDPI

Article

Rational Limit Cycles on Abel Polynomial Equations

Claudia Valls

Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal; cvalls@math.tecnico.ulisboa.pt

Received: 14 April 2020; Accepted: 21 May 2020; Published: 1 June 2020

Abstract: In this paper we deal with Abel equations of the form $dy/dx = A_1(x)y + A_2(x)y^2 + A_3(x)y^3$, where $A_1(x)$, $A_2(x)$ and $A_3(x)$ are real polynomials and $A_3 \not\equiv 0$. We prove that these Abel equations can have at most two rational (non-polynomial) limit cycles when $A_1 \not\equiv 0$ and three rational (non-polynomial) limit cycles when $A_1 \equiv 0$. Moreover, we show that these upper bounds are sharp. We show that the general Abel equations can always be reduced to this one.

Keywords: algebraic limit cycles; rational limit cycles; abel equations

MSC: 34C05; 34C07; 34C08

1. Introduction and Statement of the Results

In this paper we study the existence of rational (non-polynomial) limit cycles of the Abel polynomial equations.

The Abel polynomial equations are equations of the form

$$\frac{dy}{dx} = A_1(x)y + A_2(x)y^2 + A_3(x)y^3, \tag{1}$$

where x, y are real variables and $A_1(x)$, $A_2(x)$ and $A_3(x)$ are polynomials with $A_3 \not\equiv 0$.

A *periodic solution* of Equation (1) is a solution $y(x)$ defined in the closed interval $[0, 1]$ such that $y(0) = y(1)$. We say that a *limit cycle* is a periodic solution isolated in the set of periodic solutions of a differential Equation (1). Without loss of generality we will assume that the period is 1. The limit cycle is called a *polynomial limit cycle* if the periodic solution $y(x)$ is a polynomial in the variable x.

The polynomial limit cycles of these equations have been intensively investigated (see for instance [1,2]). The problem of finding solutions for polynomial equations of these type have attracted the attention of many authors. See for instance [3–27] and the references therein. Here, we are interested in the rational limit cycles of Equation (1) (when the functions $A_i(x)$ are polynomials).

In particular, the authors in [16] proved that any polynomial limit cycle of system (1) is of the form $y = c$ with $c \in \mathbb{R}$, and that if a polynomial limit cycle exists with $c \neq 0$, then no other polynomial limit cycles can exist. So, in the present paper we will focus on the case in which the limit cycles are non-polynomials.

The objective of this paper is to consider the existence of *rational limit cycles* for system (1), i.e., we want to consider limit cycles of the form $y(x) = q(x)/p(x)$ where $p, q \in \mathbb{R}[x]$ with $p \notin \mathbb{R}$ and $(p(x), q(x)) = 1$. As usual $\mathbb{R}[x]$ denotes the set of all real polynomials in the variable x. Note that we will study only the rational limit cycles that are not polynomial limit cycles. We will also provide examples of differential equations (1) having the prescribed number of rational limit cycles.

We recall that if we have a general Abel equation of the form

$$y' = A_0(x) + A_1(x)y + A_2(x)y^2 + A_3(x)y^3 \tag{2}$$

with $A_i(x)$ being polynomials for $i = 0, \ldots, 3$ and $A_3 \not\equiv 0$, then with the change of variables $w = y - y_0(x)$, being $y_0(x)$ a solution of (2), we get

$$\begin{aligned} w' = y' - y_0' &= A_0(x) + A_1(x)y + A_2(x)y^2 + A_3(x)y^3 \\ &- A_0(x) - A_1(x)y_0 + A_2(x)y_0^2 + A_3(x)y_0^3 \\ &= A_1(x)(w + y_0(x)) + A_2(x)(w(x)^2 + 2w(x)y_0(x) + y_0(x)^2) \\ &+ A_3(x)(w(x)^3 + 3w(x)^2y_0(x) + 3w(x)y_0(x)^2 + y_0(x)^3) \\ &- A_1(x)y_0 - A_2(x)y_0^2(x) - A_3(x)y_0^3(x) \\ &= \tilde{A}_1(x)w(x) + \tilde{A}_2(x)w(x)^2 + A_3(x)w(x)^3 \end{aligned}$$

where

$$\begin{aligned} \tilde{A}_1(x) &= A_1(x) + 2A_2(x)y_0(x) + 3A_3y_0(x)^2, \\ \tilde{A}_2(x) &= A_2(x) + 3y_0(x)A_3(x), \end{aligned}$$

and so

$$w'(x) = \tilde{A}_1(x)w(x) + \tilde{A}_2(x)w(x) + A_3(x)w(x)^3.$$

In short we can always work with Equation (1).

The case $A_1(x) \equiv 0$ was studied in [28]. In particular the authors proved the following theorem.

Theorem 1. *System (1) with $A_1 \equiv 0$ has at most three rational (non-polynomial) limit cycles, and there are examples with three rational limit cycles.*

In this paper we will focus in the case in which $A_1 \not\equiv 0$. We note that with these two theorems we cover all the rational limit cycles in the Abel Equation (1) (and as explained above in Equation (2)).

Our main theorem is the following one.

Theorem 2. *System (1) with $A_1(x) \not\equiv 0$ has at most two rational (non-polynomial) limit cycles, and there are examples with two rational limit cycles.*

The proof of Theorem 2 and the example are given in Section 2.

2. Proof of Theorem 2

We start the proof with an auxiliary lemma.

Lemma 1. *The rational function $y = q(x)/p(x)$ with $p(x)$ non-constant is a periodic solution of system (1) if and only if $q(x) = c \in \mathbb{R} \setminus \{0\}$, $p(0) = p(1)$ and $p(x)$ has no zero in $[0,1]$ and*

$$p(x)A_2(x) + \frac{p'(x)p(x)}{c} + \frac{A_1(x)p^2(x)}{c} + A_3(x)c = 0. \tag{3}$$

Proof. For the reverse implication we note that if $q(x) = c \in \mathbb{R} \setminus \{0\}$, $p(0) = p(1)$, $p(x)$ has no zero in $[0,1]$ and equality (3) holds then it is clear that the rational function $y = c/p(x)$ is a periodic solution of system (1).

For the direct implication we note that if $y(x) = q(x)/p(x)$ is a periodic solution of system (1) then $p(x) \neq 0$ for $x \in [0,1]$. Let $g(x,y) = p(x)y - q(x)$. Then

$$\begin{aligned} 0 = \frac{dg(x,y)}{dx}\Big|_{g(x,y)=0} &= p'(x)y + p(x)\frac{dy}{dx} - q'(x) \\ &= p'(x)y + p(x)(A_1(x)y + A_2(x)y^2 + A_3(x)y^3) - q'(x). \end{aligned}$$

Note that $g(x,y)$ is irreducible, so there exists a polynomial $k(x,y)$ so that

$$p'(x)y + p(x)(A_1(x)y + A_2(x)y^2 + A_3(x)y^3) - q'(x) = k(x,y)g(x,y). \tag{4}$$

Since the highest degree in y in the left-hand side is 3 and the highest degree in y in $g(x,y)$ is 1 we get that the highest degree in y in $k(x,y)$ is 2 and so it can be written as $k(x,y) = k_0(x) + k_1(x)y + k_2(x)y^2$, where $k_0, k_1, k_2 \in \mathbb{R}[x]$. Comparing the coefficients of y^0, y^1, y^2 and y^3 in (4) we get

$$\begin{aligned} q'(x) &= k_0(x)q(x), \\ p'(x) + A_1(x)p(x) &= k_0(x)p(x) - k_1(x)q(x), \\ p(x)A_2(x) &= k_1(x)p(x) - k_2(x)q(x), \\ p(x)A_3(x) &= k_2(x)p(x). \end{aligned} \tag{5}$$

From the first relation we get that $q(x)|q'(x)$. This implies that $q(x)$ is a constant that we denote by c, that is, $q(x) = c \in \mathbb{R}$. If $c = 0$ then $y = q(x)/p(x) = 0$. This is not possible and so $c \neq 0$. Moreover, $y = q(x)/p(x) = c/p(x)$ is a periodic solution, then $p(0) = p(1)$. From the second relation we get that $k_1(x) = -p'(x)/c - A_1(x)p(x)/c$ and from the fourth relation we obtain $k_2(x) = A_3(x)$. Substituting them in the third relation we get (3) and the direct inclusion is proved. □

In view of Lemma 1 it is not restrictive to take $c = 1$ and consider all rational limit cycles of the form $y = 1/p(x)$ with $p(x)$ satisfying $p(0) = p(1)$ with $p(x)$ having no zero in $[0,1]$ and satisfying (3).

From (3) we must have that $A_3(x)$ is multiple of $p(x)$ and so $A_3(x) = p(x)r(x)$ for some polynomial $r(x)$. Therefore, (3) becomes

$$A_2(x) + p'(x) + A_1(x)p(x) + r(x) = 0. \tag{6}$$

Assume that Equation (1) has two rational limit cycles, $y(x) = 1/p_1(x)$ and $y(x) = 1/p_2(x)$ with $p_1(x), p_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. Denote by $q(x) = (p_1(x), p_2(x))$, i.e., the maximum common divisor of the polynomials $p_1(x)$ and $p_2(x)$, and consequently

$$p_1(x) = q(x)s_1(x), \quad p_2(x) = q(x)s_2(x) \tag{7}$$

with $q(x), s_i(x) \in \mathbb{R}[x]$ and $(s_1(x), s_2(x)) = 1$. Note that in view of the above observation we must have that

$$A_3(x) = q(x)s_1(x)s_2(x)s_3(x) \tag{8}$$

for some $s_3(x) \in \mathbb{R}[x]$.

Lemma 2. *The following equalities hold*

$$s_3(x) = q'(x) + A_1(x)q(x) \quad \textit{and} \quad s_1(x) - s_2(x) = c \in \mathbb{R}. \tag{9}$$

Proof. Note that in view of (6) we have

$$\begin{aligned} q'(x)s_1(x) + q(x)s_1'(x) &= -A_1(x)q(x)s_1(x) - A_2(x) - s_2(x)s_3(x), \\ q'(x)s_2(x) + q(x)s_2'(x) &= -A_1(x)q(x)s_2(x) - A_2(x) - s_1(x)s_3(x), \end{aligned}$$

and so

$$\begin{aligned} &q'(x)s_1(x) + q(x)s_1'(x) - q'(x)s_2(x) + q(x)s_2'(x) \\ &= -s_2(x)s_3(x) + s_1(x)s_3(x) - A_1(x)q(x)(s_1(x) - s_2(x)) \\ &= (s_1(x) - s_2(x))(s_3(x) - A_1(x)q(x)), \end{aligned}$$

which gives

$$q'(x)(s_1(x)-s_2(x))+q(x)(s_1(x)-s_2(x))' = (s_1(x)-s_2(x))(s_3(x)-A_1(x)q(x)),$$

that is

$$q(x)(s_1(x)-s_2(x))' = (s_1(x)-s_2(x))(s_3(x)-A_1(x)q(x)-q'(x)).$$

Hence

$$\frac{(s_1(x)-s_2(x))'}{(s_1(x)-s_2(x))} = \frac{s_3(x)-A_1(x)q(x)}{q(x)} - \frac{q'(x)}{q(x)}.$$

Therefore

$$s_1(x)-s_2(x) = \kappa_2 \frac{1}{q(x)} \exp\left(\int \frac{s_3(s)-A_1(s)q(s)}{q(s)}\, ds\right), \tag{10}$$

for some $\kappa_2 \in \mathbb{R}$.

If $\deg(s_3(x)) \leq \deg(q(x))$ then we get

$$s_1(x)-s_2(x) = \kappa_2 \frac{1}{q(x)} \exp\left(\int -A_1(x)\, dx\right) \exp\left(\int \frac{s_3(x)}{q(x)}\, dx\right). \tag{11}$$

The first factor in (11) cannot cancel with the second factor of (11) and this gives a contradiction because $s_1(x)-s_2(x)$ is a polynomial. So, we must have $\deg(s_3(x)) > \deg(q(x))$. Then we make the Euclidean division and we get

$$s_3(x) = s_4(x)q(x) + s_5(x)$$

where $\deg(s_5(x)) < \deg(q(x))$. Therefore we have

$$\frac{s_3(x)}{q(x)} - A_1(x) = s_4(x) - A_1(x) + \frac{s_5(x)}{q(x)}.$$

Integrating we get

$$s_1(x)-s_2(x) = \kappa_2 \frac{1}{q(x)} \exp\left(\int s_4(x)-A_1(x)\, dx\right) \exp\left(\int \frac{s_5(x)}{q(x)}\, dx\right). \tag{12}$$

The first factor in (12) cannot cancel with the second factor of (12) and this gives a contradiction because $s_1(x)-s_2(x)$ is a polynomial. So, we must have $s_4(x) = A_1(x)$. Now we write $H(x) = (s_5(x), q(x))$. Then

$$\frac{s_5(x)}{q(x)} = \frac{H(x)\bar{s}_5(x)}{H(x)\bar{q}(x)} = \frac{\bar{s}_5(x)}{\bar{q}(x)}.$$

Assume first that $\bar{q}(x)$ is not-square free. Using the affine transformation $x \mapsto x+\alpha$ with $\alpha \in \mathbb{C}$ (if necessary) we can write $\bar{q}(x) = x^\mu r(x)$ where $\mu > 1$ and $r(0) \neq 0$. Moreover, $\bar{s}_3(0) \neq 0$ because $\bar{s}_3(x)$ and $\bar{q}(x)$ are coprime. If we develop $\bar{s}_3(x)/\bar{q}(x)$ in simple fractions of x we obtain

$$\frac{\bar{s}_5(x)}{\bar{q}(x)} = \frac{c_\mu}{x^\mu} + \frac{c_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{c_1}{x} + \frac{\alpha(x)}{r(x)}$$

where $\alpha(x)$ is a polynomial with $\deg(\alpha(x)) < \deg(r(x))$ and $c_i \in \mathbb{C}$ for $i = 1, \ldots, \mu$. Note that $c_\mu = \bar{s}_3(0)/\bar{q}(0) \neq 0$. Integrating, we get

$$\begin{aligned} s_1(x)-s_2(x) = \kappa_2 \frac{1}{q(x)} &\exp\left(\frac{c_\mu}{1-\mu}\frac{1}{x^{\mu-1}}\right) \\ &\times \exp\left(\int \left(\frac{c_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{c_1}{x} + \frac{\alpha(x)}{r(x)}\right) dx\right). \end{aligned} \tag{13}$$

The first exponential factor cannot cancel with any part of the second exponential factor and we get to a contradiction with the fact that $s_1(x) - s_2(x)$ is a polynomial. So, $\bar{q}(x)$ is square-free. Then we have that

$$\int \frac{\bar{s}_5(s)}{\bar{q}(s)}\, ds = \log h(x), \quad h(x) \in \mathbb{R}[x] \setminus \{0\}. \tag{14}$$

Therefore,

$$\frac{\bar{s}_5(x)}{\bar{q}(x)} = \frac{h'(x)}{h(x)},$$

where $h(x)$ is square-free and so $h'(x)$ and $h(x)$ are coprime. Hence $\bar{q}(x) = h(x)$ and $\bar{s}_5(x) = \kappa_3 \bar{q}'(x)$. From (10) and (14) we have

$$s_1(x) - s_2(x) = \kappa_2 \frac{1}{H(x)}$$

Since $s_1(x) - s_2(x)$ must be a polynomial, it follows that $H(x)$ can be one. Hence,

$$s_5(x) = \kappa_3 q'(x) \quad \text{and} \quad s_1(x) - s_2(x) = \kappa_2 q(x)^{\kappa_3 - 1}. \tag{15}$$

On the other hand, doing a change of variables of the form $Y = \beta y$ where $\beta^2 = \text{sign}(\kappa_3)\kappa_3$, the Abel Equation (1) becomes

$$\frac{dY}{dx} = \frac{A_1(x)}{\beta^2} Y + \frac{A_2(x)}{\beta} Y^2 + \frac{A_3(x)}{\beta^2} Y^3 = \overline{A}_1(x) Y + \overline{A}_2(x) Y^2 + \overline{A}_3(x) Y^3. \tag{16}$$

Since $A_3(x) = q(x)s_1(x)s_2(x)\kappa_3 q'(x)$, then $\overline{A}_3(x) = \pm q(x)s_1(x)s_2(x)q'(x)$. In what follows we shall work with the Abel Equation (16).

Repeating the previous computations starting with the Abel Equation (16) we will arrive to Equation (15) which now writes

$$s_3(x) = A_1(x)q(x) + q'(x) \quad \text{and} \quad s_1(x) - s_2(x) = \kappa_2,$$

because $\kappa_3 = \pm 1$ and only can be one. This concludes the proof of the lemma. □

Note that from (7), (8) and Lemma 2 we have that

$$A_3(x) = q(x)s_1(x)s_2(x)(q'(x) + A_1(x)q(x)). \tag{17}$$

Proof of Theorem 2. Assume that Equation (1) has three rational limit cycles, $y = 1/p_1(x)$ and $y = 1/p_2(x)$ and $y_3 = 1/p_3(x)$ with $p_1, p_2, p_3 \in \mathbb{R}[x] \setminus \mathbb{R}$. Denote by $q_1(x) = (p_1(x), p_2(x))$, $q_2(x) = (p_1(x), p_3(x))$ and $q_3(x) = (p_2(x), p_3(x))$. In view of Lemma 2 we have

$$\begin{aligned} p_1(x) &= q_1(x)s_1(x) = q_2(x)s_2(x), \\ p_2(x) &= q_1(x)(s_1(x) + c_1) = q_3(x)s_3(x), \\ p_3(x) &= q_2(x)(s_2(x) + c_2) = q_3(x)(s_3(x) + c_3), \end{aligned} \tag{18}$$

for some polynomials $s_1(x), s_2(x), s_3(x)$ and constants $c_1, c_2, c_3 \in \mathbb{R} \setminus \{0\}$ (we recall that the polynomials $s_1(x)$, $s_2(x)$ in (18) need not be the same as the ones in (17). In fact, the polynomial $q(x)$ in Equation (17) will be $q_1(x)$, $q_2(x)$ and $q_3(x)$ in formula (18) (or other polynomials that will appear in the paper) and polynomials $s_1(x)$ and $s_2(x)$ in formula (17) will be $s_1(x)$, $s_2(x)$ and $s_3(x)$ in formula (18) (or other forms that will appear along the paper, when appropriate). Hence, we get

$$\begin{aligned} p_2(x) - p_1(x) &= q_1(x)c_1, \; c_1 \in \mathbb{R}, \\ p_3(x) - p_1(x) &= q_2(x)c_2, \; c_2 \in \mathbb{R}, \\ p_3(x) - p_2(x) &= q_3(x)c_3, \; c_3 \in \mathbb{R}, \end{aligned}$$

and so

$$q_2(x)c_2 = q_1(x)c_1 + q_3(x)c_3. \tag{19}$$

We consider two situations.

Case 1. $q_1(x)$ and $s_2(x)$ are coprime. *Note that from (18) we have that $q_1(x)s_1(x) = q_2(x)s_2(x)$, and then from (19) we get*

$$\frac{q_1(x)s_1(x)c_2}{s_2(x)} = q_2(x)c_2 = q_1(x)c_1 + q_3(x)c_3.$$

In particular there exists $T(x) \in \mathbb{R}[x]$ so that

$$q_3(x) = q_1(x)T(x),$$

and consequently

$$\frac{s_1(x)c_2}{s_2(x)} = c_1 + T(x)c_3,$$

which yields

$$s_1(x) = \frac{s_2(x)}{c_2}(c_1 + T(x)c_3).$$

Therefore from (18) we get

$$q_2(x)s_2(x) = q_1(x)s_1(x) = q_1(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3),$$

and so

$$q_2(x) = q_1(x)\frac{c_1 + T(x)c_3}{c_2}.$$

Hence we have

$$\begin{aligned} p_1(x) &= q_1(x)s_1(x) = q_1(x)\frac{s_2(x)}{c_2}(c_1 + T(x)c_3), \\ p_2(x) &= q_1(x)(s_1(x) + c_1) = q_1(x)\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right), \\ p_3(x) &= q_2(x)(s_2(x) + c_2) = q_1(x)\frac{c_1 + T(x)c_3}{c_2}(s_2(x) + c_2). \end{aligned} \tag{20}$$

We consider two subcases.

Subcase 1.1: Assume that $T(x)$ and $s_2(x) + c_2$ are coprime. *Then the maximum common divisor between $p_2(x)$ and $p_3(x)$ is $q_1(x)$. Indeed, we will show that*

$$r_1(x) = \frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1$$

and

$$r_2(x) = (c_1 + T(x)c_3)(s_2(x) + c_2)$$

are coprime. Note that if x^* is a zero of $c_1 + T(x)c_3$ then we have that $r_2(x^*) = 0$ but $r_1(x^*) = c_1 \neq 0$. Moreover, if $\hat{x}$ is a solution of $s_2(x) + c_2 = 0$ then $r_2(\hat{x}) = 0$ but $r_1(\hat{x}) = -(c_1 + T(\hat{x})c_3) + c_1 = T(\hat{x})c_3 \neq 0$. Therefore, using $p_1(x)$ and $p_2(x)$ from (17) and (20) we can write

$$A_3(x) = q_1(x)(q_1'(x) + A_1(x)q_1(x))\frac{s_2(x)}{c_2}(c_1 + T(x)c_3)\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right),$$

and from $p_1(x)$ *and* $p_3(x)$ *we can write*

$$A_3(x) = q_1(x)(q_1'(x) + A_1(x)q_1(x))\left(\frac{s_2(x)}{c_2}(c_1 + T(x)c_3) + c_1\right)\frac{c_1 + T(x)c_3}{c_2} \cdot (s_2(x) + c_2),$$

and so

$$s_2(x) = s_2(x) + c_2$$

which is not possible because $c_2 \neq 0$.

Subcase 1.2: Assume that $T(x)$ and $s_2(x) + c_2$ are not coprime. *Write*

$$T(x) = \alpha_1(x)\alpha_2(x), \quad s_2(x) + c_2 = \alpha_1(x)\alpha_3(x),$$

where $\alpha_2, \alpha_3 \in \mathbb{R}[x]$ *and* $\alpha_1(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. *Then*

$$p_3(x) = q_1(x)\alpha_1(x)\alpha_3(x)\frac{c_1 + T(x)c_3}{c_2},$$
$$p_2(x) = q_1(x)\frac{\alpha_1(x)}{c_2}(c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3).$$

We first note that the maximum common divisor between $p_2(x)$ *and* $p_3(x)$ *is* $q_1(x)\alpha_1(x)$. *To do so, we will show that*

$$r_3(x) = \alpha_3(x)(c_1 + T(x)c_3) \quad \text{and} \quad r_4(x) = c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3$$

are coprime. If x^* *is a zero of* $\alpha_3(x)$ *then* $r_3(x^*) = 0$ *but* $r_4(x^*) = s_2(x^*)\alpha_2(x^*)c_3 = -c_2\alpha_2(x^*)c_3$. *Since* $\alpha_2(x)$ *and* $\alpha_3(x)$ *are coprime, we get that* $\alpha_2(x^*) \neq 0$, *and then* $r_4(x^*) \neq 0$. *Moreover, if* $c_1 + T(\hat{x})c_3 = 0$ *then* $r_4(\hat{x}) = c_1 \neq 0$. *So* $r_3(x)$ *and* $r_4(x)$ *are coprime.*

From $p_1(x)$, $p_2(x)$, (17) *and* (20) *we get*

$$A_3(x) = q_1(x)(q_1'(x) + A_1(x)q_1(x))\frac{s_2(x)}{c_2}(c_1 + T(x)c_3)\frac{\alpha_1(x)}{c_2} \cdot (c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3). \tag{21}$$

Note that from $p_2(x)$, $p_3(x)$, (17) *and* (20) *we have*

$$A_3(x) = \frac{q_1(x)}{c_2^2}\alpha_1(x)\big((q_1(x)\alpha_1(x))' + A_1(x)q_1(x)\alpha_1(x)\big)\alpha_3(x)(c_1 + T(x)c_3) \cdot (c_1\alpha_3(x) + s_2(x)\alpha_2(x)c_3). \tag{22}$$

Comparing (21) *with* (22) *we obtain*

$$\alpha_3(x)\big((q_1(x)\alpha_1(x))' + A_1(x)q_1(x)\alpha_1(x)\big) = \big(q_1'(x) + A_1(x)q_1(x)\big)(\alpha_1(x)\alpha_3(x) - c_2),$$

i.e.,

$$-c_2q_1'(x) = -q_1(x)(\alpha_3(x)\alpha_1'(x) + c_2A_1(x)),$$

which is not possible because the left-hand side of this equality has less degree than the right-hand side and $c_2 \neq 0$ *and* $q_1'(x) \neq 0$ *(otherwise would be constant a contradiction). In short, Case 1 is not possible.*

Case 2. $q_1(x)$ and $s_2(x)$ are not coprime *We write*

$$q_1(x) = R_1(x)R_2(x), \quad s_2(x) = R_1(x)R_3(x)$$

with $R_1(x), R_2(x), R_3(x) \in \mathbb{R}[x]$ *and* $R_1(x) \notin \mathbb{R}$.

We consider two different subcases.

Subcase 2.1: $R_3(x) = R \in \mathbb{R}$. *So* $s_2(x) = R_1(x)R$ *and* $q_1(x) = R_2(x)s_2(x)/R$.

We also consider two cases

2.1.1: $R_2(x) = R_2 \in \mathbb{R}$. *From* (7) *we have* $q_1(x)s_1(x) = q_2(x)s_2(x)$ *and so* $q_2(x) = R_2 s_1(x)/R$. *Then*

$$\begin{aligned} p_1(x) &= \frac{R_2}{R} s_1(x)s_2(x), \\ p_2(x) &= \frac{R_2}{R} s_2(x)(s_1(x)+c_1), \\ p_3(x) &= \frac{R_2}{R} s_1(x)(s_2(x)+c_2). \end{aligned}$$

Note that taking $\hat{s}_1 = s_1/c_1$ *we can write*

$$\begin{aligned} p_1(x) &= \frac{R_2 c_1}{R} \hat{s}_1(x)s_2(x), \\ p_2(x) &= \frac{R_2 c_1}{R} s_2(x)(\hat{s}_1(x)+1), \\ p_3(x) &= \frac{R_2 c_1}{R} \hat{s}_1(x)(s_2(x)+c_2). \end{aligned}$$

From $p_1(x)$, $p_2(x)$, (17) *and* (20) *we get*

$$A_3(x) = \left(\frac{R_2 c_1}{R}\right)^2 s_2(x)s_1(x)(s_1(x)+c_1)(s_2'(x) + A_2(x)s_2(x)),$$

and from $p_1(x)$, $p_3(x)$, (17) *and* (20) *we obtain*

$$A_3(x) = \left(\frac{R_2 c_1}{R}\right)^2 s_2(x)\hat{s}_1(x)(\hat{s}_1'(x) + A_1(x)\hat{s}_1(x))(s_2(x)+c_2),$$

and so

$$(s_2'(x) + A_1(x)s_2(x))(\hat{s}_1(x)+1) = (\hat{s}_1'(x) + A_1(x)\hat{s}_1(x))(s_2(x)+c_2),$$

which yields

$$\frac{s_2'(x)}{s_2(x)+c_2} - \frac{\hat{s}_1'(x)}{\hat{s}_1(x)+1} = \frac{A_1(x)(\hat{s}_1(x)c_2 - s_2(x)c_1)}{(s_2(x)+c_2)(\hat{s}_1(x)+1)}.$$

We consider two cases: if $deg(A_1(x)(\hat{s}_1(x)c_2 - s_2(x)c_1)) > deg((s_2(x)+c_2)(\hat{s}_1(x)+1))$ *then we write*

$$A_1(x)(\hat{s}_1(x)c_2 - s_2(x)c_1) = \hat{A}_1(x)(s_2(x)+c_2)(\hat{s}_1(x)+1) + A_1^*(x)$$

where $deg(A_1^*(x)) < deg(s_2(x)+c_2)(\hat{s}_1(x)+1)$. *Integrating we get*

$$\begin{aligned} (s_2(x)+c_2) = (\hat{s}_1(x)+1) \exp\left(\kappa + \int \hat{A}_1(x)\,dx\right) \\ \cdot \exp\left(\int \frac{A_1^*(x)}{(s_2(x)+c_2)(\hat{s}_1(x)+1)}\right), \end{aligned} \tag{23}$$

where κ is the constant of integration. Note that the first factor in (23) cannot cancel with the second factor of (23) and this gives a contradiction because $s_2(x)+c_2$ and $\hat{s}_1(x)+1$ are polynomials. So, we must have that $deg(A_1(x)(\hat{s}_1(x)c_2 - s_2(x)c_1)) \le deg((s_2(x)+c_2)(\hat{s}_1(x)+1))$. Then we introduce the notation

$$B(x) = A_1(x)(\hat{s}_1(x)c_2 - s_2(x)), \quad C(x) = (s_2(x)+c_2)(\hat{s}_1(x)+1).$$

Let $D(x) = (B(x), C(x))$ and write $B(x) = \bar{B}(x)D(x)$, $C(x) = \bar{C}(x)D(x)$. Then if $\bar{C}(x)$ is not square-free with an affine change of variables we can write $\bar{C}(x) = x^{\mu} r(x)$ with $r(0) \neq 0$. Moreover, $\bar{B}(0) \neq 0$ because $\bar{C}(x)$ and $\bar{B}(x)$ are coprime. Therefore, if we develop $\bar{B}(x)(x)/\bar{C}(x)$ in simple fractions of x we obtain

$$\frac{\bar{B}(x)}{\bar{C}(x)} = \frac{d_\mu}{x^\mu} + \frac{d_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{d_1}{x} + \frac{\alpha(x)}{r(x)}$$

where $\alpha(x)$ is a polynomial with $deg(\alpha(x)) < deg(r(x))$ and $d_i \in \mathbb{C}$ for $i = 1, \ldots, \mu$. Note that $d_\mu = \bar{B}(0)/\bar{C}(0) \neq 0$. Integrating we get

$$(s_2(x)+c_2) = (\hat{s}_1(x)+1)\exp\left(\kappa + \frac{d_\mu}{1-\mu}\frac{1}{x^{\mu-1}}\right) \times \exp\left(\int\left(\frac{d_{\mu-1}}{x^{\mu-1}} + \cdots + \frac{d_1}{x} + \frac{\alpha(x)}{r(x)}\right)dx\right), \tag{24}$$

where κ is the constant of integration. The first exponential factor cannot cancel with any part of the second exponential factor and we get to a contradiction with the fact that $s_1(x) - s_2(x)$ is a polynomial. So, $\bar{C}(x)$ is square-free. Then we have that

$$\int \frac{\bar{B}(s)}{\bar{C}(s)}\, ds = \log h(x), \quad h(x) \in \mathbb{R}[x] \setminus \{0\}. \tag{25}$$

Therefore,

$$\frac{\bar{B}(x)}{\bar{C}(x)} = \frac{h'(x)}{h(x)},$$

where $h(x)$ is square-free and so $h'(x)$ and $h(x)$ are coprime. Hence $\bar{C}(x) = h(x)$ and $\bar{B}(x) = \bar{C}'(x)$. From (10) and (25) we have

$$(s_2(x)+c_2) = (\hat{s}_1(x)+1)\left(\frac{(s_2(x)+c_2)(\hat{s}_1(x)+1)}{D(x)}\right).$$

Therefore, we have that

$$D(x) = (\hat{s}_1+1)^2.$$

So,

$$(s_2+c_2) = \bar{C}(x)(\hat{s}_1+1)^2, \quad A_1(x)(\hat{s}_1(x)c_2 - s_2(x)) = (\hat{s}_1+1)^2\bar{C}(x)' \tag{26}$$

which yields

$$A_1(x)((\hat{s}_1(x)+1)c_2 - (s_2(x)+c_2)) = A_1(x)(c_2 - \bar{C}(x)) = (\hat{s}_1+1)\bar{C}(x)'$$

Solving this last linear equation we get

$$\bar{C}(x) = \exp\left(-\int \frac{A_1(x)}{\hat{s}_1(x)+1}\, dx\right)\kappa + c_2,$$

where $\kappa \in \mathbb{R}$ is the constant of integration. Since $\bar{C}(x)$ must be a polynomial if we write $S_1(x) = (A_1(x), \hat{s}_1(x)+1)$ so that $A_1(x) = S_1(x)S_2(x)$ and $\hat{s}_1(x)+1 = S_1(x)S_3(x)$ then proceeding as above we get that deg$(S_2(x)) <$ deg$(S_3(x))$ and $S_3(x)$ must be square-free and then

$$-c_1 \int \frac{A_1(x)}{\hat{s}_1(x)+1}\, dx = -c_1 \int \frac{S_2(x)}{S_3(x)}\, dx = \log r(x), \quad r \in \mathbb{R}[x]$$

yielding $S_3(x) = r(x)$ and $S_2(x) = -r'(x)$. Hence,

$$\bar{C}(x) = \kappa S_3(x) + c_2, \quad \kappa \in \mathbb{R} \setminus \{0\}.$$

Note that if $\kappa = 0$ then $\bar{C}(x) = c_2$ which yields $A_1(x)((\hat{s}_1(x)+1)c_2 - (s_2(x)+c_2)) = 0$. Since $A_1(x) \neq 0$ we must have

$$c_2(\hat{s}_1+1) = s_2 + c_2 = \bar{C}(x)(\hat{s}_1+1)^2 = c_2(\hat{s}_1+1)^2$$

and so

$$1 = \hat{s}_1 + 1 \quad \text{that is} \quad \hat{s}_1 = 0,$$

which is not possible.

In short, $\kappa \in \mathbb{R} \setminus \{0\}$ and

$$\begin{aligned} s_2(x) + c_2 &= (\kappa S_3(x) + c_2)S_1(x)^2 S_3(x)^2, \\ \hat{s}_1(x) + 1 &= S_1(x)S_3(x), \\ A_1(x) &= -S_1(x)S_3'(x). \end{aligned} \tag{27}$$

Hence

$$\begin{aligned} A_2(x) = {} & \left(\frac{R_2 c_1}{R}\right)^2 (\kappa S_3(x) + c_2)S_1(x)^2 S_3(x)^2 (S_1(x)S_3(x) - 1) \\ & \cdot \Big((\kappa S_3(x) + c_2)S_1(x)^2 S_2(x)^2 - c_2\Big) \\ & \cdot \Big(S_1'(x)S_3(x) + 2S_1(x)S_3'(x) - S_1(x)^2 S_3(x) S_3'(x)\Big) \end{aligned}$$

Doing the rescaling $Y = \beta y$, we can assume that the constant $(R_2 c_1/R)^2 = 1$.

In short

$$\begin{aligned} p_1(x) &= \hat{s}_1(x)s_2(x) = (S_1(x)S_3(x) - 1)\left((\kappa S_3(x) + c_2)S_1(x)^2 S_3(x)^2 - c_2\right), \\ p_2(x) &= s_2(x)(\hat{s}_1(x)+1) = S_1(x)S_3(x)\left((\kappa S_3(x) + c_2)S_1(x)^2 S_3(x)^2 - c_2\right), \\ p_3(x) &= \hat{s}_1(x)(s_2(x)+c_2) = (S_1(x)S_3(x) - 1)(\kappa S_3(x) + c_2)S_1(x)^2 S_3(x)^2. \end{aligned} \tag{28}$$

with

$$A_1(x) = -S_1(x)S_3'(x).$$

Note that $\hat{s}_1$ and s_2 are coprime. Indeed it follows from (27) that if $x*$ is such that $s_2(x^*) = \hat{s}_1(x^*) = 0$ then (since $S_1(x^*)S_3(x^*) = -1$) we get

$$c_2 = (\kappa S_3(x^*) + c_2)S_1(x^*)^2 S_3(x^*)^2 = \kappa S_3(x^*) + c_2$$

and so $S_3(x^*) = 0$ but then again from (27) we would have $s_2(x^*) = -c_2$ and $s_1(x^*) = -1$ which is not possible. So, $\hat{s}_1(x)$ and $s_2(x)$ are coprime. Then, it follows from

Note that $\hat{s}_1$ and s_2 are coprime. Indeed it follows from (27) that if $x*$ is such that $s_2(x^*) = \hat{s}_1(x^*) = 0$ then (since $S_1(x^*)S_3(x^*) = -1$) we get

$$c_2 = (\kappa S_3(x^*) + c_2)S_1(x^*)^2 S_3(x^*)^2 = \kappa S_3(x^*) + c_2$$

and so $S_3(x^*) = 0$ but then again from (27) we would have $s_2(x^*) = -c_2$ and $s_1(x^*) = -1$ which is not possible. So, $\hat{s}_1(x)$ and $s_2(x)$ are coprime. Then, it follows from the first and third relation in (28) (using (26)) implies

$$\begin{aligned} A_3(x) &= \hat{s}_1(x)s_2(x)(s_2(x) + c_2)(\hat{s}_1'(x) + A_1\hat{s}_1(x)) \\ &= \hat{s}_1(x)s_2(x)C(x)(\hat{s}_1(x) + 1)^2(\hat{s}_1'(x) + A_1(x)\hat{s}_1(x)) \end{aligned} \tag{29}$$

and from the second and third relation in (28) that

$$A_3(x) = \hat{s}_1(x)s_2(x)C(x)(\hat{s}_1(x) + 1)^2((\hat{s}_1(x) + 1)' + A_1(x)(\hat{s}_1(x) + 1)). \tag{30}$$

However, then it follows from (29) and (30) that

$$(\hat{s}_1'(x) + A_1(x)\hat{s}_1(x)) = (\hat{s}_1(x) + 1)' + A_1(x)(\hat{s}_1(x) + 1)$$

that is $A_1(x) = 0$, which is not possible. In short this case is not possible.

2.1.2: *$R_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. Since $R_3(x) = R$ we have $s_2(x) = R_1(x)R$ and $q_1(x) = R_2(x)s_2(x)/R$. From (19) we get*

$$\frac{R_2(x)s_1(x)c_2}{R} = R_2(x)s_2(x)c_1 + q_3(x)c_3(x),$$

and so

$$q_3(x) = \frac{R_2(x)}{c_3}\left(s_1(x)c_2 - \frac{c_1 s_2(x)}{R}\right).$$

Since $q_1(x)s_1(x) = q_2(x)s_2(x)$ we get $q_2(x) = R_2(x)s_1(x)/R$. In short

$$\begin{aligned} p_1(x) &= \frac{R_2(x)}{R}s_1(x)s_2(x), \\ p_2(x) &= \frac{R_2(x)}{R}s_2(x)(s_1(x) + c_1), \\ p_3(x) &= \frac{R_2(x)}{R}s_1(x)(s_2(x) + c_2). \end{aligned}$$

We consider two cases:

2.1.2.1: $s_1(x)$ and $s_2(x)$ are coprime. *In this case the maximum common divisor between $p_2(x)$ and $p_3(x)$ is $R_2(x)$ and so from (17) we get*

$$\begin{aligned} A_3(x) &= \frac{R_2(x)}{R^2}s_2(x)\big((R_2(x)s_2(x))' + A_1(x)R_2(x)s_2(x)\big)s_1(x)(s_1(x) + c_1) \\ &= \frac{R_2(x)}{R^2}s_2(x)s_1(x)(R_2'(x) + A_1(x)R_2(x))(s_1(x) + c_1)(s_2(x) + c_2), \end{aligned}$$

and so

$$(R_2(x)s_2(x))' + A_1(x)R_2(x)s_2(x) = (R_2'(x) + A_1(x)R_2(x))(s_2(x) + c_2),$$

that is

$$R_2'(x)c_2 = R_2(x)(s_2'(x) - A_1(x)c_2),$$

which is not possible because the left-hand side of the above expression has less degree than the right-hand side and $R_2'(x) \neq 0$ and $c_2 \neq 0$.

2.1.2.2: $s_1(x)$ and $s_2(x)$ are not coprime. *In this case we write*

$$s_1(x) = \kappa(x)\hat{s}_1(x), \quad s_2(x) = \kappa(x)\hat{s}_2(x),$$

with $\kappa(x), \hat{s}_1(x), \hat{s}_2(x) \in \mathbb{R}[x]$ *with* $\kappa(x) \notin \mathbb{R}$. *Then*

$$\begin{aligned} p_1(x) &= \frac{R_2(x)}{R}\kappa^2(x)\hat{s}_1(x)\hat{s}_2(x), \\ p_2(x) &= \frac{R_2(x)}{R}\kappa(x)\hat{s}_2(x)(\kappa(x)\hat{s}_1(x) + c_1), \\ p_3(x) &= \frac{R_2(x)}{R}\kappa(x)\hat{s}_1(x)(\kappa(x)\hat{s}_2(x) + c_2). \end{aligned}$$

Then

$$\begin{aligned} A_3(x) &= \frac{R_2(x)}{R^2}\kappa(x)\big((R_2(x)\kappa(x))' + A_1(x)R_2(x)\kappa(x)\big)\hat{s}_2(x)(\kappa(x)\hat{s}_1(x) + c_1) \\ &\cdot \hat{s}_1(x)(\kappa(x)\hat{s}_2(x) + c_2) \\ &= \frac{R_2(x)}{R^2}\kappa(x)\hat{s}_2(x)\big((R_2(x)\kappa(x)\hat{s}_2(x))' + A_1(x)R_2(x)\kappa(x)\hat{s}_2(x)\big)\hat{s}_1(x) \\ &\cdot (\kappa(x)\hat{s}_1(x) + c_1)\kappa(x), \end{aligned}$$

and so

$$\begin{aligned} &\big((R_2(x)\kappa(x))' + A_1(x)R_2(x)\kappa(x)\big)(\kappa(x)\hat{s}_2(x) + c_2) \\ &= \kappa(x)\big((R_2(x)\kappa(x)\hat{s}_2(x))' + A_1(x)R_2(x)\kappa(x)\hat{s}_2(x)\big), \end{aligned}$$

which yields

$$(R_2(x)\kappa(x))'c_2 = (R_2(x)\kappa(x))\kappa(x)(\hat{s}_2(x) - A_1(x)c_2).$$

This is not possible because the left hand side has less degree than the right hand side and $c_2(R_2(x)\kappa(x))' \neq 0$. *In summary, Subcase* 2.1.2 *is not possible.*

Subcase 2.2: $R_2(x) \in \mathbb{R}[x] \setminus \mathbb{R}$. *We have* $q_1(x) = R_1(x)R_2(x)$ *and* $s_2(x) = R_1(x)R_3(x)$. *Then*

$$\frac{R_2(x)s_1(x)c_2}{R_3(x)} = R_1(x)R_2(x)c_1 + q_3(x)c_3.$$

In particular there exists $T(x) \in \mathbb{R}[x]$ *so that*

$$q_3(x) = R_2(x)T(x),$$

and so

$$\frac{s_1(x)c_2}{R_3(x)} = R_1(x)c_1 + T(x)c_3,$$

which yields $s_1(x) = R_4(x)R_3(x)$. *Therefore, from* $p_1(x)$ *in* (7) *we get*

$$q_2(x)s_2(x) = q_1(x)s_1(x) = R_1(x)R_2(x)R_3(x)R_4(x) = q_2(x)R_1(x)R_3(x)$$

and so

$$q_2(x) = R_2(x)R_4(x).$$

Hence we have

$$\begin{aligned}
p_1(x) &= q_1(x)s_1(x) = R_1(x)R_2(x)R_3(x)R_4(x),\\
p_2(x) &= q_1(x)(s_1(x)+c_1) = R_1(x)R_2(x)(R_3(x)R_4(x)+c_1),\\
p_3(x) &= q_2(x)(s_2(x)+c_2) = R_2(x)R_4(x)(R_1(x)R_3(x)+c_2).
\end{aligned}$$

We consider two cases.

2.2.1: $R_1(x)$ and $R_4(x)$ are coprime. *We have*

$$\begin{aligned}
A_3(x) &= R_1(x)R_2(x)\big((R_1(x)R_2(x))' + A_1(x)R_1(x)R_2(x)\big)R_3(x)R_4(x)\\
&\quad \cdot (R_3(x)R_4(x)+c_1)\\
&= R_2(x)(R_2'(x)+A_1(x)R_2(x))R_1(x)R_4(x)(R_3(x)R_4(x)+c_1)\\
&\quad \cdot (R_1(x)R_3(x)+c_2),
\end{aligned}$$

and so

$$\begin{aligned}
&\big((R_1(x)R_2(x))' + A_1(x)R_1(x)R_2(x)\big)R_3(x)\\
&= (R_2'(x)+A_1(x)R_2(x))(R_1(x)R_3(x)+c_2),
\end{aligned}$$

which yields

$$R_2(x)(R_1'(x)R_3(x) - A_1(x)c_2) = c_2R_2'(x).$$

This is not possible because the right hand side has less degree than the left hand side and $c_2R_2'(x) \neq 0$.

2.2.2: $R_1(x)$ and $R_4(x)$ are not coprime. *We write*

$$R_1(x) = R(x)\hat{R}_1(x), \quad R_4(x) = R(x)\hat{R}_4(x)$$

where $R(x), \hat{R}_1(x), \hat{R}_4(x) \in \mathbb{R}[x]$ *with* $R(x) \notin \mathbb{R}$. *Note that*

$$\begin{aligned}
p_1(x) &= q_1(x)s_1(x) = R^2(x)\hat{R}_1(x)R_2(x)R_3(x)\hat{R}_4(x),\\
p_2(x) &= q_1(x)(s_1(x)+c_1) = R(x)\hat{R}_1(x)R_2(x)(R_3(x)R(x)\hat{R}_4(x)+c_1),\\
p_3(x) &= q_2(x)(s_2(x)+c_2) = R_2(x)R(x)\hat{R}_4(x)(R(x)\hat{R}_1(x)R_3(x)+c_2).
\end{aligned}$$

Then

$$\begin{aligned}
A_3(x) &= \big((R(x)\hat{R}_1(x)R_2(x))' + A_1(x)R(x)\hat{R}_1(x)R_2(x)\big)R(x)\hat{R}_1(x)R_2(x)R(x)\\
&\quad \cdot R_3(x)\hat{R}_4(x)(R_3(x)R(x)\hat{R}_4(x)+c_1)\\
&= R(x)R_2(x)\big((R(x)R_2(x))' + A_1(x)R(x)R_2(x)\big)\hat{R}_1(x)\hat{R}_4(x)\\
&\quad (R_3(x)R_4(x)+c_1)(R(x)\hat{R}_1(x)R_3(x)+c_2),
\end{aligned}$$

and so

$$\begin{aligned}
&\big((R(x)\hat{R}_1(x)R_2(x))' + A_1(x)\hat{R}_1(x)R_2(x)\big)R(x)R_3(x)\\
&= \big((R(x)R_2(x))' + A_1(x)R(x)R_2(x)\big)(R(x)\hat{R}_1(x)R_3(x)+c_2),
\end{aligned}$$

which yields

$$R(x)R_2(x)\big((\hat{R}_1(x))'R(x)R_3(x) - A_1(x)\big) = c_2(R(x)R_2(x))'.$$

This is not possible because the right hand side has less degree than the left hand side and $c_2(R(x)R_2(x))' \neq 0$. *So subcase* 2.2 *is not possible.*

In short we have proved that there are at most two rational limit cycles when $A_1 \neq 0$. This completes the proof of the theorem. □

Now we provide an Abel Equation (1) with two rational limit cycles. Take

$$A_1(x) = 1,$$
$$A_2(x) = -2 - 7x - x^2 - 2x^3 - 2x^4,$$
$$A_3(x) = (1 - x + x^2)(2 - x + x^2)(3 - x + x^2)(1 + x + x^2).$$

Then system (1) has the two rational solutions $y_i(x) = 1/p_i(x)$ for $i = 1, 2$ with

$$p_1(x) = (x^2 - x + 1)(x^2 - x + 2),$$
$$p_2(x) = (x^2 - x + 2)(x^2 - x + 3).$$

Note that $p_i(0) = p_i(1)$ for $i = 1, 2$ and $p_i(x) \neq 0$ for $x \in [0, 1]$. In short, the Abel system that we have constructed has two rational limit cycles.

Funding: This research was partially supported by FCT/Portugal through UID/MAT/04459/2019.

Acknowledgments: Partially supported by FCT/Portugal through UID/MAT/04459/2019.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Cima, A.; Gasull, A.; Manosas, F.M. On the number of polynomial solutions of Bernouilli and Abel polynomial differential equations. *J. Differ. Equ.* **2017**, *263*, 7099–7122. [CrossRef]
2. Gasull, A.; Torregrossa, J.; Zhang, X. The number of polynomial solutions of polynomial Riccati equations. *J. Differ. Equ.* **2016**, *261*, 5071–5093. [CrossRef]
3. Gasull, A.; Llibre, J. Limit cycles for a class of Abel equations. *SIAM J. Math. Anal.* **1990**, *21*, 1235–1244. [CrossRef]
4. Álvarez, A.; Bravo, J.L.; Fernández, M. The number of limit cycles for generalized Abel equations with periodic coefficients of definite sign. *Commun. Pure Appl. Anal.* **2009**, *8*, 1493–1501. [CrossRef]
5. Álvarez, M.J.; Bravo, J.L.; Fernández, M. Existence of non-trivial limit cycles in Abel equations with symmetries. *Nonlinear Anal.* **2013**, *84*, 18–28. [CrossRef]
6. Álvarez, A.; Bravo, J.L.; Fernández, M. Limit cycles of Abel equations of the first kind. *J. Math. Anal. Appl.* **2015**, *423*, 734–745. [CrossRef]
7. Álvarez, M.J.; Bravo, J.L.; Fernández, M.; Prohens, R. Centers and limit cycles for a family of Abel equations. *J. Math. Anal. Appl.* **2017**, *453*, 485–501. [CrossRef]
8. Álvarez, M.J.; Bravo, J.L.; Fernández, M.; Prohens, R. Alien limit cycles in Abel equations. *J. Math. Anal. Appl.* **2020**, *482*, 123525. [CrossRef]
9. Álvarez, M.J.; Gasull, A.; Yu, J. Lower bounds for the number of limit cycles of trigonometric Abel equations. *J. Math. Anal. Appl.* **2008**, *342*, 682–693. [CrossRef]
10. Bravo, J.L.; Fernández, M.; Gasull, A. Limit cycles for some Abel equations having coefficients without fixed signs. *Int. J. Bifur. Chaos* **2009**, *19*, 3869–3876. [CrossRef]
11. Bravo, J.L.; Fernández, M.; Gasull, A. Stability of singular limit cycles for Abel equations. *Discrete Contin. Dyn. Syst.* **2015**, *35*, 1873–1890. [CrossRef]
12. Fossas, E.; Olm, J.M.; Sira-Ramírez, H. Iterative approximation of limit cycles for a class of Abel equations. *Physics D* **2008**, *237*, 3159–3164. [CrossRef]
13. Françoise, J.P. *Local Bifurcations of Limit Cycles, Abel Equations and Liénard Systems. Normal Forms, Bifurcations and Finiteness Problems in Differential Equations*; NATO Sci. Ser. II Math. Phys. Chem., 137; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2004; pp. 187–209.
14. Françoise, J.P. *Integrability and Limit Cycles for Abel Equations. Algebraic Methods in Dynamical Systems*; Banach Center Publications: Warsaw, Poland, 2011; pp. 187–196.

15. Gasull, A. From Abel's differential equations to Hilbert's sixteenth problem. *Butl. Soc. Catalana Mat.* **2013**, *28*, 123–146. (In Catalan)
16. Giné, J.; Grau, M.; Llibre, J. On the polynomial limit cycles of polynomial differential equations. *Israel J. Math.* **2011**, *181*, 461–475. [CrossRef]
17. Huang, J.; Liang, H. Estimate for the number of limit cycles of Abel equation via a geometric criterion on three curves. *NoDEA Nonlinear Differ. Equ. Appl.* **2017**, *24*, 31. [CrossRef]
18. Ilyashenko, Y. Hilbert-type numbers for Abel equations, growth and zeros of holomorphic functions. *Nonlinearity* **2000**, *13*, 1337–1342. [CrossRef]
19. Liu, C.; Li, C.; Wang, X.; Wu, J. On the rational limit cycles of Abel equations. *Chaos Solitons Fractals* **2018**, *110*, 28–32. [CrossRef]
20. Lloyd, N.G. A note on the number of limit cycles in certain two-dimensional systems. *J. Lond. Math. Soc.* **1979**, *20*, 277–286. [CrossRef]
21. Neto, A.L. On the number of solutions of the equation $\frac{dx}{dt} = \sum_{j=0}^{n} a_j(t)x^j$, $0 \leq t \leq 1$, for which $x(0) = x(1)$. *Invent. Math.* **1980**, *59*, 67–76. [CrossRef]
22. Torres, P. Existence of closed solutions for a polynomial first order differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 1108–1116. [CrossRef]
23. Valls, C. Rational limit cycles on Bernouilli and Riccati equations. *J. Geom. Phys.*, to appear. [CrossRef]
24. Wang, G.D.; Chen, W.C. The number of closed solutions to the Abel equation and its application. *J. Syst. Sci. Math. Sci.* **2005**, *25*, 693–699. (In Chinese)
25. Xie, X.D.; Cai, S.L. The number of limit cycles for the Abel equation and its application. *Gaoxiao Yingyong Shuxue Xuebao Ser. A* **1994**, *9*, 266–274. (In Chinese)
26. Zhang, J.F. Limit cycles for a class of Abel equations with coefficients that change sign. *Ann. Math. Ser. A* **1997**, *18*, 271–278.
27. Zhang, J.F.; Chen, X.X. *Some criteria for limit Cycles of a Class of Abel Equations. J. Fuzhou Univ. Nat. Sci. Ed.* **1999**, *27*, 9–11. (In Chinese)
28. Llibre, J.; Valls, C. Rational limit cycles on Abel equations. 2020, preprint.