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Digital Topological Properties of an Alignment of Fixed Point Sets

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Abstract: The present paper investigates digital topological properties of an alignment of fixed point sets which can play an important role in fixed point theory from the viewpoints of computational or digital topology. In digital topology-based fixed point theory, for a digital image (X,k), let $F(X)$ be the set of cardinalities of the fixed point sets of all k-continuous self-maps of (X,k) (see Definition 4). In this paper we call it an alignment of fixed point sets of (X,k). Then we have the following unsolved problem. How many components are there in $F(X)$ up to 2-connectedness? In particular, let $C_k^{n,l}$ be a simple closed k-curve with l elements in $\mathbb{Z}^n$ and $X := C_k^{n,l_1} \vee C_k^{n,l_2}$ be a digital wedge of C_k^{n,l_1} and C_k^{n,l_2} in $\mathbb{Z}^n$. Then we need to explore both the number of components of $F(X)$ up to digital 2-connectivity (see Definition 4) and perfectness of $F(X)$ (see Definition 5). The present paper addresses these issues and, furthermore, solves several problems related to the main issues. Indeed, it turns out that the three models $C_{2n}^{n,4}$, $C_{3^n-1}^{n,4}$, and $C_k^{n,6}$ play important roles in studying these topics because the digital fundamental groups of them have strong relationships with alignments of fixed point sets of them. Moreover, we correct some errors stated by Boxer et al. in their recent work and improve them (see Remark 3). This approach can facilitate the studies of pure and applied topologies, digital geometry, mathematical morphology, and image processing and image classification in computer science. The present paper only deals with k-connected spaces in *DTC*. Moreover, we will mainly deal with a set X such that $X^\sharp \geq 2$.

Keywords: digital image; digital wedge; normal adjacency; k-homotopy; k-contractibility; alignment; fixed point set; digital topology; fixed point property

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1. Introduction

Let $\mathbb{Z}$ (*resp.* $\mathbb{N}$) be the set of integers (*resp.* natural numbers) and $\mathbb{Z}^n$ be the n times Cartesian products of $\mathbb{Z}$. Moreover, we denote the set of even natural numbers with $\mathbb{N}_0$. A paper [1] initially raised the following query involving a fixed point set and a homotopy fixed point set from the viewpoint of typical fixed point theory.

(♠) How do the fixed points of a continuous self-map of an ordinary topological space depend on the given self-map?

It can also be of interest in digital topology-based fixed point theory. Therefore, a paper [2] partially studied this issue in a Rosenfeld's digital topological setting, the so-called "digital fixed point property" and "digital homotopy fixed point property".

Specifically, how do the fixed points of a digitally continuous self-map of a digital image (X,k) depend on the given self-map?

Indeed, this approach can be so natural and meaningful. Unfortunately, since the paper [2] contains many errors, a paper [3] corrected many things and improved them. In a similar way as the above, given a digital image (X,k), a paper [4] studied the set of cardinalities of fixed point sets of all k-continuous self-maps of (X,k), denoted by $F(X)$, and explored some features of it and the set of cardinalities of fixed point sets of all continuous self-maps g of (X,k) which are k-homotopic to a given continuous self-map f of (X,k). Then the authors of [4] used the term, the so-called "fixed point spectrum" and "homotopy fixed point spectrum". Indeed, we may follow the term "spectrum" because such a name can be taken by an author. However, for a given digital image (X,k), since each of the above quantities need not be 2-connected, the present paper will take another term to exactly characterize the "set of the cardinalities of fixed point sets of k-continuous self-maps of it", i.e., the so-called "an alignment of fixed point sets" exactly defining the above set in mathematical sense and further, we denote it with $F(X)$ as referred to in [4] (see Definition 4). A paper [5] explored some alignments of fixed point sets which are 2-connected because an existence of a 2-connected alignment of fixed point sets depends on the situation. At the moment, given a digital image (X,k), we need to examine if an alignment of fixed point sets of (X,k) is connected up to 2-connectedness, which can play an important role in digital topology and many areas in applied sciences [6,7].

In this paper, we investigate various properties of $F(X)$ in the category DTC, where DTC is the category consisting of the set of digital images and the set of digitally continuous maps (see Section 2 for details). In DTC, a recent paper [5] confirmed that a digital k-isomorphism preserves a k-homotopy. Based on this approach, after considering a digital wedge $C_k^{n,l} \vee C_k^{n,4}$ whose digital k-fundamental group is an infinite cyclic group or a trivial group, a recent paper [5] partially examined if $F(C_k^{n,l} \vee C_k^{n,4})$ is 2-connected and perfect (see Definition 5), where the term "perfect" means that for a digital image (X,k), $F(X) = [0, X^\sharp]_{\mathbb{Z}}$, e.g., $F(C_k^{n,l} \vee C_k^{n,4}) = [0, l+3]_{\mathbb{Z}}$. Motivated by the approach, in the Rosenfeld's digital topological setting for fixed point theory, regarding $F(X)$ of (X,k), we may raise the following queries.

(Q1) Given a digital image (X,k), does $F(X)$ always have $X^\sharp - 1$?
Indeed, this issue was partially studied in [4].
(Q2) Given a digital image (X,k), under what condition can we have $F(X)$ that is perfect?
(Q3) Given a digital image (X,k), how can we explore perfectness of $F(X)$?
(Q4) For a digital wedge $C_k^{n,l_1} \vee C_k^{n,l_2}$, assume $l_1 \geq l_2$. If $l_2 = 4$, then is $F(C_k^{n,l_1} \vee C_k^{n,4})$ 2-connected?
(Q5) For a digital wedge $C_k^{n,l_1} \vee C_k^{n,l_2}$, assume $l_1 \geq l_2$. If $l_2 = 6$, then how can we characterize $F(C_k^{n,l} \vee C_k^{n,6})$?
(Q6) Is $F(C_k^{n,l} \vee C_k^{n,6})$ perfect?
(Q7) Under what condition is $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$ perfect?

After developing many new tools, we will address all these queries.

The remainder of the paper is organized as follows: Section 2 investigates various concepts related to digital topological spaces and deals with some properties of them. Section 3 explores various properties related to 2-connectedness of an alignment of fixed point sets. In particular, we intensively explore some conditions of which $F(C_k^{n,l_1} \vee C_k^{n,4})$ is 2-connected, which addresses the main issues of the present paper. Indeed, we can recognize a certain special role of $C_k^{n,4}$ that makes $F((X,k) \vee C_k^{n,4})$ perfect. Section 4 intensively investigates some properties of $F(C_k^{n,l_1} \vee C_k^{n,6})$ that are involved in the 2-connectedness or the perfectness of it. Section 5 concludes the paper with some remarks and a further work. In this paper, we will often use the symbol ":=" to introduce a new term. In addition, we will denote the cardinality of a set X with $X^\sharp$. This paper corrects and improves some results stated by Boxer et al. in [4] (see Remark 3) and corrects several incorrect citations in [4] related to a digital wedge (see Section 2) and a generalization of digital k-connectivity and a normal adjacency for a digital product.

2. Preliminaries

The papers [8,9] called a set $X \subset \mathbb{Z}^n$ with digital k-connectivity (or a k-adjacency) a *digital image* denoted by (X, k), $n \in \{1, 2, 3\}$. Thanks to these k-adjacency relations for digital images in $\mathbb{Z}^n, n \leq 3$, it turns out that a digital image (X, k) is a digital space [6,10]. Motivated by this approach, a paper [11] initially developed the generalized version of the adjacency relations of $\mathbb{Z}^3$ for high-dimensional digital images $X \subset \mathbb{Z}^n$, the so-called k-(or $k(t, n)$-)adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$ (see below). Indeed, this approach was initially taken in [11], which can play an important role in studying many structures in digital topology such as digital products with normal adjacencies [11] and their applications. To study digital images $X \subset \mathbb{Z}^n$, $n \in \mathbb{N}$, we now recall the following [11] (see also [12,13]):

For a natural number t, $1 \leq t \leq n$, distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n,$$

are $k(t, n)$-adjacent

$$\text{if at most } t \text{ of their coordinates differ by } \pm 1\text{, and the others coincide.} \tag{1}$$

According to the statement of (1), the $k(t, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, were initially formulated [11] (see also [12,13]) as follows:

$$k := k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \tag{2}$$

For instance,

$$(n, t, k) \in \left\{ \begin{array}{l} (1, 1, 2) \\ (2, 1, 4), (2, 2, 8) \\ (3, 1, 6), (3, 2, 18), (3, 3, 26) \\ (4, 1, 8), (4, 2, 32), (4, 3, 64), (4, 4, 80); \text{and} \\ (5, 1, 10), (5, 2, 50), (5, 3, 130), (5, 4, 210), (5, 5, 242). \end{array} \right\}$$

For $c, d \in \mathbb{Z}$ with $c \precneqq d$, the set $[c, d]_{\mathbb{Z}} = \{n \in \mathbb{Z} \,|\, c \leq n \leq d\}$ with 2-adjacency is called a digital interval [14].

Let us now recall some terminology and notions [8,9,11,15] which will be used in this paper.

- A digital image (X, k) is said to be k-disconnected [15] if there are non-empty sets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and further, there are no points $x_1 \in X_1$ and $x_2 \in X_2$ which are k-adjacent. Using this approach, a digital image (X, k) is said to be k-connected (or k-path connected) if it is not k-disconnected.
- A k-connected digital image (X, k) in $\mathbb{Z}^n$ whose cardinality is greater than 1, the so-called k-path with $l + 1$ elements in X is assumed to be the finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset X$ such that x_i and x_j are k-adjacent if $|\, i - j \,| = 1$ [14]. Then we call the number l the lenth of this k-path.
- We say that a simple k-path is the finite set $(x_i)_{i \in [0,m]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j \,| = 1$ [6]. In case $x_0 = x$ and $x_m = y$, we denote the length of the simple k-path with $l_k(x, y) := m$.
- A simple closed k-curve (or simple k-cycle) with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}$ [11,14], $l \geq 4, l \in \mathbb{N}_0 \setminus \{2\}$, means the finite set $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j \,| = \pm 1 (mod\, l)$. Owing to this notion, it is obvious that the number of l should be even. In particular, in the present paper we will use the notation $C_k^{n,l}$ to abbreviate $SC_k^{n,l}$. At the moment, we need to remind that in some papers $C_k^{n,l}$ is used for a notation of a closed k-curve

with l elements in $\mathbb{Z}^n$. However, in this paper since we will not deal with such curves, we may use the notation $C_k^{n,l}$ for only a simple closed k-curve with l elements in $\mathbb{Z}^n$.

To fix the incorrect citation of [4] related to the notion of digital wedge, we now recall the concept with originality.

- The notion of digital wedge (or one point union of two digital images) was initially proposed in [11,16]. To be precise, two given digital images (X, k) and (Y, k), a *digital wedge* [11,16], denoted by $(X \vee Y, k)$, is defined as the union of the digital images (X', k) and (Y', k), where

  (1) $X' \cap Y'$ is a singleton, say $\{p\}$.
  (2) $X' \setminus \{p\}$ and $Y' \setminus \{p\}$ are not k-adjacent, where the two subsets A and B of (X, k) are said to be k-adjacent [14] if $A \cap B = \emptyset$ and there are at least two points $a \in A$ and $b \in B$ such that a is k-adjacent to b.
  (3) (X', k) is k-isomorphic to (X, k) and (Y', k) is k-isomorphic to (Y, k) (see Definition 1).

In digital topology, we are strongly required to follow this digital wedge $(X \vee Y, k)$. If we do not follow this approach, we will have some big difficulties in proceeding with further works in digital topology. Meanwhile, in the case (X, k_0) and (Y, k_1) such that $k_0 \neq k_1$, the compatible k-adjacency of $(X \vee Y, k)$ was also established in [16].

- Motivated by the strong graph adjacency of a product of two typical graphs [17], its digital version was initially developed in [11]. Indeed, this notion plays an important role in calculating digital k-fundamental group of digital products [12,18–22].

For a digital image (X, k), $X \subset \mathbb{Z}^n$, we may consider it with two kinds of aspects: First, a graph theoretic approach with the above k-adjacency relation of (2), e.g., digital k-graphs [8,9,23,24], can be considered. Second, based on a discrete topological subspace induced by the n-dimensional usual topological space, further consider it with one of the above k-adjacency relation of (2). Indeed, they are eventually equivalent to each other. For a k-connected digital image (X, k), the paper [11] already established a certain metric on (X, k). To be precise, a metric function, say d_k, on a k-connected digital image (X, k) was established, as follows:

$$d_k(x, x') := \begin{Bmatrix} l_k(x, x'), \text{ if } x \neq x' \text{ in } X; \\ 0, \text{ if } x = x'. \end{Bmatrix} \tag{3}$$

Namely,

$$d_k(x, x') := min\{l \mid l \text{ is a length of a } k\text{-path beteween } x \text{ and } x'.\}[11].$$

It obviously follows from (3) that this metric d_k is different from the typical Euclidean metric on $X \subset \mathbb{Z}^n$. Thus, for a certain $\varepsilon \in \mathbb{N}$, we defined [11,15]

$$N_k(x_0, \varepsilon) := \{x \in X \mid d_k(x_0, x) \leq \varepsilon\}, \tag{4}$$

which is called a digital k-neighborhood of x_0 with radius ε. Indeed, the notion of (4) is another representation of the typical one established in [11]. The paper [10] defined a digital space. At the moment, we should remind that a digital space need not be a topological space, e.g., a digital image $X(\subset \mathbb{Z}^n)$ with digital k-connectivity in terms of the Rosenfeld's model [8,9]. For instance, to wit that a digital image (X, k) is a digital space, consider two distinct points $p, q \in X$. Then we say that p is k-adjacent to q if

$$p \in N_k(q, 1) \text{ or } q \in N_k(p, 1).$$

Using this approach, we see that a digital image (X,k) on $\mathbb{Z}^n$ is one of the digital spaces. Indeed, there are countably many digital spaces in $\mathbb{Z}^n$ [25] including Khalimsky [26] and Marcus-Wyse topological spaces [27], Alexandroff space [28], and so on.

The notion of digital continuity of a map $f : (X,k_0) \to (Y,k_1)$ was initially defined by Rosenfeld [9] by saying that f maps every k_0-connected subset of (X,k_0) into a k_1-connected subset of (Y,k_1). Motivated by this approach, using the set in (4), we can represent the digital continuity of a map between digital images using a digital k-neighborhood (see Proposition 1 below).

Proposition 1. *[23,29] Let (X,k_0) and (Y,k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : (X,k_0) \to (Y,k_1)$ is (digitally) (k_0,k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x,1)) \subset N_{k_1}(f(x),1)$.*

In Proposition 1, in the case $n_0 = n_1$ and $k := k_0 = k_1$, the map f is called a 'k-continuous' map to abbreviate the (k,k)-continuity of Proposition 1. Due to this approach, we can have big advantages of calculating the digital fundamental groups of digital images (X,k) using the unique digital lifting theorem [11], the digital homotopy lifting theorem [29], a radius 2-(k_0,k_1)-isomorphism and its applications [29], the multiplicative properties of a digital fundamental group [12,21,22], a Cartesian product of the covering spaces [22], and so on.

To make the present paper self-contained, we need to recall the category *DTC* consisting of the following two pieces of data [11] (see also [23]), called the digital topological category.

- The set of (X,k), where $X \subset \mathbb{Z}^n$, as objects of *DTC*, denoted by $Ob(DTC)$;
- For every ordered pair of objects $(X_i,k_i), i \in \{0,1\}$, the set of all (k_0,k_1)-continuous maps between them as morphisms of *DTC*, denoted by $Mor(DTC)$.

In *DTC*, in the case of $k := k_0 = k_1$, we will particularly use the notation *DTC(k)* [20].

To classify digital images (X,k) [23,24]), we prefer the term a (k_0,k_1)-isomorphism (or k-isomorphism) as in [23] to a (k_0,k_1)-homeomorphism (or k-homeomorphism) as in [30].

Definition 1. *[23] (see also a (k_0,k_1)-homeomorphism in [30]) Consider two digital images (Z,k_0) and (W,k_1) in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Then a map $h : Z \to W$ is called a (k_0,k_1)-isomorphism if h is a (k_0,k_1)-continuous bijection and further, $h^{-1} : W \to Z$ is (k_1,k_0)-continuous. Then we use the notation $Z \approx_{(k_0,k_1)} W$. In the case $k := k_0 = k_1$, we use the notation $Z \approx_k W$ to abbreviate $Z \approx_{(k_0,k_1)} W$.*

Based on the pointed digital homotopy in [26,31,32] (see also [30]), the following notion of k-homotopy relative to a subset $A \subset X$ is often used in studying k-homotopic properties of digital images (X,k) in $\mathbb{Z}^n$. For a digital image (X,k) and $A \subset X$, we often call $((X,A),k)$ a digital image pair.

Definition 2. *[11,30,33] Let $((X,A),k_0)$ and (Y,k_1) be a digital image pair and a digital image in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $f,g : X \to Y$ be (k_0,k_1)-continuous functions. Suppose there exist $m \in \mathbb{N}$ and a function $H : X \times [0,m]_{\mathbb{Z}} \to Y$ such that*

($\bullet$1) *for all $x \in X$, $H(x,0) = f(x)$ and $H(x,m) = g(x)$;*

($\bullet$2) *for all $t \in [0,m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ given by $H_t(x) = H(x,t)$ for all $x \in X$ is (k_0,k_1)-continuous.*

($\bullet$3) *for all $x \in X$, the induced function $H_x : [0,m]_{\mathbb{Z}} \to Y$ given by $H_x(t) = H(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2,k_1)$-continuous; Then we say that H is a (k_0,k_1)-homotopy between f and g [30].*

($\bullet$4) *Furthermore, for all $t \in [0,m]_{\mathbb{Z}}$, assume that the induced map H_t on A is a constant which follows the prescribed function from A to Y [11] (see also [33]). To be precise, $H_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0,m]_{\mathbb{Z}}$.*

Then we call H a (k_0, k_1)-homotopy relative to A between f and g, and we say that f and g are (k_0, k_1)-homotopic relative to A in Y, $f \simeq_{(k_0,k_1)rel.A} g$ in symbols [33].

In Definition 2, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1)-homotopy at $\{x_0\}$ [30]. When f and g are pointed (k_0, k_1)-homotopic in Y, we use the notation $f \simeq_{(k_0,k_1)} g$. In the case $k := k_0 = k_1$ and $n_0 = n_1$, f and g are said to be pointed k-homotopic in Y and we use the notation $f \simeq_k g$ to abbreviate $f \simeq_{(k_0,k_1)} g$. If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map in the space X relative to $\{x_0\}$, then we say that (X, x_0) is *pointed k-contractible* [30]. Indeed, motivated by this approach, the notion of strong k-deformation retract was developed in [18,33].

Based on this k-homotopy, the notion of digital homotopy equivalence initially introduced in [34] (see also [24]), as follows:

Definition 3. *[34] (see also [24]) For two digital images (Z, k) and (W, k) in $\mathbb{Z}^n$, if there are k-continuous maps $h : Z \to W$ and $l : W \to Z$ such that the composite $l \circ h$ is k-homotopic to 1_Z and the composite $h \circ l$ is k-homotopic to 1_W, then the map $h : Z \to W$ is called a k-homotopy equivalence and is denoted by $Z \simeq_{k \cdot h \cdot e} W$. Moreover, we say that (Z, k) is k-homotopy equivalent to (W, k). In the case that the identity map 1_Z is k-homotopy equivalent to a certain constant map $c_{\{z_0\}}, z_0 \in Z$, we say that (Z, k) is k-contractible.*

In Definition 3, in the case of $Z \simeq_{k \cdot h \cdot e} W$, we call that (Z, k) is the same k-homotopy type with (W, k). In view of Definitions 2 and 3, it is easy to see that the pointed k-contractibility implies the k-contractibility, but the converse does not hold.

For a given digital image (X, k), by using several notions such as digital k-homotopy class [26,31,32], Khalimsky operation of two k-homotopy classes [31], trivial extension [30], the paper [30] defined the digital k-fundamental group, denoted by $\pi^k(X, x_0), x_0 \in X$. Indeed, in digital topology there are several kinds of digital fundamental groups [19]. Also, we have the following: If X is pointed k-contractible, then $\pi^k(X, x_0)$ is a trivial group [30]. Hereafter, we shall assume that each digital image (X, k) is k-connected.

Using the unique digital lifting theorem [11] and the homotopy lifting theorem [29] in digital covering theory [11,12,16,18–22,33,35], for a non-k-contractible space $C_k^{n,l}$, we obtain the following:

Theorem 1. *[11] (1) For a non-k-contractible $C_k^{n,l}$, $\pi^k(C_k^{n,l})$ is an infinite cyclic group. Indeed, $\pi^k(C_k^{n,l})$ is trivial if and only if $l = 4$.*
(2) For two non-k-contractible digital images $C_k^{n,l_i}, i \in \{1, 2\}$, $\pi^k(C_k^{n,l_1} \vee C_k^{n,l_2})$ is a free group with two generators with infinite orders.

This result will be essential in characterizing an alignment of fixed point sets in Sections 3 and 4. By Theorem 1, $C_{3^n-1}^{n,4}$ has the trivial group, $n \geq 2$ [11,29,30] and further, $C_4^{2,4}$ also has the trivial group, i.e., $\pi^k(C_k^{n,4})$ is trivial if $k = 3^n - 1, n \in \mathbb{N} \setminus \{1\}$, and $\pi^4(C_4^{2,4})$ is also trivial.

Remark 1. *[9,15,35] Only a singleton digital image has the FPP.*

3. Digital Topological Properties of $F(C_k^{n,l} \vee C_k^{n,4})$ Up to 2-Connectedness and Perfectness of $F(C_k^{n,l} \vee C_k^{n,4})$

As mentioned above, for $C_k^{n,l}$, in this paper we need to assume $l \in \mathbb{N}_0 \setminus \{2\}$ in nature. The study of digital topological properties of digital wedges $C_k^{n,l} \vee C_k^{n,4}$, $C_k^{n,l} \vee C_k^{n,6}$ and in general $C_k^{n,l_1} \vee C_k^{n,l_2}$ plays an important role in computational or digital topology. This section counts on a certain role of $C_k^{n,4}$ that is involved in the perfectness of $F(C_k^{n,l} \vee C_k^{n,4})$ (see Definition 5) with some hypothesis. Indeed, the study of $F(C_k^{n,l} \vee C_k^{n,4})$, $F(C_k^{n,l} \vee C_k^{n,6})$, and $F(C_k^{n,l_1} \vee C_k^{n,l_2})$, and so on still remains open. Up to now, it turns out that digital fundamental groups of $C_k^{n,l} \vee C_k^{n,4}$, $C_k^{n,l} \vee C_k^{n,6}$ and $C_k^{n,l_1} \vee C_k^{n,l_2}$ are strongly related to the digital topological features of alignments of fixed point sets of them.

Definition 4. *[4] Given a digital image (X, k),*

$$F(X) := \{Fix(f)^\sharp \mid f \text{ is a } k\text{-continuous self-map of } X, i.e., f \in Mor(DTC(k))\}.$$

As mentioned in Section 1, this paper calls this $F(X)$ of Definition 4 an "alignment of fixed points sets of (X, k)". Indeed, this notion can be used to recognize the cardinalities of fixed point sets of all k-continuous self-maps $f \in Mor(DTC(k))$ of (X, k). It is obvious that $2 \leq F(X)^\sharp \leq X^\sharp$ (see Remark 1) because each identity map of a given digital image (X, k) is a k-continuous map, we obtain $X^\sharp \in F(X)$. Namely, for a digital image (X, k) with $X^\sharp \geq 2$, we see $\{0, X^\sharp\} \subset F(X)$.

In relation to the study of $F(X)$, we need to define the following:

Definition 5. *Given a digital image (X, k), if $F(X) = [0, X^\sharp]$, then we say that $F(X)$ is perfect.*

In particular, given a digital image (X, k) in the case that $F(X)$ is not perfect, we need a certain method of establishing (X', k) including (X, k) to make $F(X')$ perfect. Therefore, we intensively explore n alignment of fixed point sets of several digital wedges.

A recent paper [5] studied $F(C_k^{n,l} \vee C_k^{n,4})$. Thus, the study of $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$, $F(C_k^{n,l} \vee C_k^{n,6})$, $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,6})$ still remains open. Indeed, the study of these cases strongly plays an important role in computational or digital topology (see Theorems 2, 3, 4, and 5) because $C_k^{n,4}$ is a minimal simple closed k-curve in $\mathbb{Z}^n$. Indeed, since $C_k^{n,4}$ is k-contractible [11,30], its usage is very wide in digital and computational topology.

Furthermore, unlike the existence of $C_k^{n,6}$ in DTC, it is well known that there is no simple closed digital curve with six elements in $\mathbb{Z}^2$ (*resp.* $\mathbb{Z}^n$) in the setting of Marcus-Wyse (*resp.* Khalimsky) topology [5,27]. Thus, in this section, we firstly study some properties of the perfectness of $F(C_k^{n,l} \vee C_k^{n,4})$, $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$, and so on. Next, in Section 4 will investigate some properties of $F(C_k^{n,l} \vee C_k^{n,6})$, $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$, and so on. Moreover, it also develops a certain method of making $F(C_k^{n,l} \vee C_k^{n,6})$ and $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$ perfect depending on the number l.

Hereafter, as usual, we say that a digital topological invariant is a property of a digital image (X, k) which is invariant under a digital k-isomorphism. In other words, a property of a digital image is digital topological property if whenever a digital image (X, k) possesses that property, then every digital image k-isomorphic to (X, k) also has that property.

Proposition 2. *[4,5] An alignment of fixed point sets of (X, k) is a digital topological invariant.*

This section explores some conditions supporting the perfectness of an alignment of fixed point sets. For $C_k^{n,l} \in DTC(k)$, it is easy to see that $F(C_k^{n,l}) = [0, \frac{l}{2} + 1]_{\mathbb{Z}} \cup \{l\}$ [4]. Since only a singleton set has the fixed point property in $DTC(k)$ (see Remark 1) [9,35], any digital image (X, k) with $X^\sharp \geq 2$ in DTC has the property $0 \in F(X)$ which is a little bit different feature compared to other digital space in applied topological space (see [5] for details). Indeed, it is clear that $F(C_4^{2,4}) = [0, 4]_{\mathbb{Z}} = F(C_{3^n-1}^{n,4})$ [4]. The following lemma characterizes a certain relationship between contractibility and perfectness of $C_k^{n,l}$.

Lemma 1. *(1) $C_k^{n,l}$ is k-contractible if and only if $F(C_k^{n,l})$ is perfect.*
(2) $C_k^{n,l}$ is not k-contractible if and only if $F(C_k^{n,l})$ is not perfect.

Proof. The k-contractibility of $C_k^{n,l}$ depends on the number l.

(1) In the case that $C_k^{n,l}$ is k-contractible, we obviously obtain $l = 4$ (see Theorem 1(1)). Hence $F(C_k^{n,4}) = [0, 4]_{\mathbb{Z}}$. Conversely, since $F(C_k^{n,l}) = [0, \frac{l}{2} + 1]_{\mathbb{Z}} \cup \{l\}$ [4], by the hypothesis, we obtain that $l = 4$, which implies the k-contractibility of $C_k^{n,l}$.

(2) In the case $l \geq 6$, $C_k^{n,l}$ is not k-contractible (see Theorem 1). Since $F(C_k^{n,l}) = [0, \frac{l}{2}+1]_{\mathbb{Z}} \cup \{l\}$, we clearly obtain $l-1 \notin F(C_k^{n,l})$. By using a method similar to the proof of (1), the converse is proved.

□

In view of Theorem 1 and Lemma 1, it appears that $F(C_k^{n,l})$ is perfect if and only if $l = 4$ and further, $F(C_k^{n,l})$ is not perfect if and only if $l \geq 6$. When the paper [4] proved the non-perfectness of the fixed point sets of the 3-dimensional digital cube $I^3 := [0,1]^3_{\mathbb{Z}}$, the author of [4] used the 6-contractibility of it. However, we need to point out that the proof is already done in [35]. Regarding Lemma 1, we also need to remind that not every k-contractible digital image (X, k) has a perfect $F(X)$ [4]. However, in the case of $C_k^{n,l}$, Lemma 1 always holds. Namely, it turns out that the k-contractibility of $C_k^{n,l}$ implies the perfectness of $F(C_k^{n,l})$ and the converse also holds. This section, hereafter, mainly investigates some digital topological properties of $F(C_k^{n,l} \vee C_k^{n,4})$ and perfectness of it depending on the number l, where $l \geq 4$. One important thing to note is that we need to remind that the triviality of the digital k-fundamental group of $C_k^{n,l} \vee C_k^{n,4}$ depends on the number l (see Theorem 1). Specifically, $C_k^{n,4} \vee C_k^{n,4}$ is k-contractible [11], and if $l \geq 6$, then $C_k^{n,l} \vee C_k^{n,4}$ is not k-contractible [11] (see also Theorem 1). Based on this observation, we obtain the following:

Lemma 2. *[5] $F(C_k^{n,l} \vee C_k^{n,4})$ is perfect if and only if $l \in \{4, 6, 8, 10\}$.*

Using Figure 1(1–3) and Figure 2(2), we can confirm Lemma 2.

Based on this approach, we now examine if $F(C_k^{n,l} \vee C_k^{n,4})$ is perfect depending on the number l.

Theorem 2. *For a digital wedge $C_k^{n,l} \vee C_k^{n,4}$, assume $l \geq 12$. Then $F(C_k^{n,l} \vee C_k^{n,4})$ has two components up to 2-connectivity such that $l+2 \in F(C_k^{n,l} \vee C_k^{n,4})$ and $l-1 \notin F(C_k^{n,l} \vee C_k^{n,4})$. Specifically, $F(C_k^{n,l} \vee C_k^{n,4})$ is not perfect if and only if $l \geq 12$.*

Proof. Let us consider any k-continuous self-map f of $C_k^{n,l} \vee C_k^{n,4}$. Indeed, though there are many kinds of k-continuous self-maps f of it, regarding $F(C_k^{n,l} \vee C_k^{n,4})$, it is sufficient to consider only the following k-continuous maps f such that

(a) $f|_{C_k^{n,4}}(x) = x$; or

(b) $f|_{C_k^{n,l}}(x) = x$; or

(c) $f(C_k^{n,l}) \subsetneq C_k^{n,l}$ and $f(C_k^{n,4}) \subsetneq C_k^{n,4}$ (see Figure 1(1–4)); or

(d) f does not support any fixed point of it, i.e., there is no point $x \in C_k^{n,l} \vee C_k^{n,4}$ such that $f(x) = x$,

where $f|_A$ means the restriction function f to the given set A. Regarding our consideration with (a)–(d) above, we may take another cases similar to the cases of (a)–(d) only exhausting $F(C_k^{n,l} \vee C_k^{n,4})$. Now, we can confirm that the case (a) or (b) is related to the identity map $1_{C_k^{n,l} \vee C_k^{n,4}}$, $f(C_k^{n,l}) \subset C_k^{n,l} \vee C_k^{n,4}$ or $f(C_k^{n,4}) \subset C_k^{n,l} \vee C_k^{n,4}$. Thus, they obviously produce the element $l+3 \in F(C_k^{n,l} \vee C_k^{n,4})$, where $l+3$ is equal to $(C_k^{n,l} \vee C_k^{n,4})^\sharp$. Next, the case (c) covers all k-continuous self-maps of $C_k^{n,l} \vee C_k^{n,4}$ whose images by the maps f are simple k-paths on $C_k^{n,l} \vee C_k^{n,4}$. Finally, the case (d) supports the property $0 \in F(C_k^{n,l} \vee C_k^{n,4})$.

Let us now investigate the set of cardinalities of fixed point sets of the maps relagted to the cases (a)–(d) above.

First, from (a) above, we obtain

$$[4, 4+\frac{l}{2}]_{\mathbb{Z}} \cup \{l+3\} \subset F(C_k^{n,l} \vee C_k^{n,4}). \tag{5}$$

Second, based on (b) above, we have

$$[l, l+3]_{\mathbb{Z}} \subset F(C_k^{n,l} \vee C_k^{n,4}). \tag{6}$$

Third, owing to (c) and (d) above, we obtain

$$[0, \frac{l}{2}+3]_{\mathbb{Z}} \subset F(C_k^{n,l} \vee C_k^{n,4}). \tag{7}$$

After comparing the four numbers $\frac{l}{2}+3$, l, $l+3$, and $4+\frac{l}{2}$ from (5)–(7), with the hypothesis of "$l \geq 12$", we conclude that

$$F(C_k^{n,l} \vee C_k^{n,4}) = [0, 4+\frac{l}{2}]_{\mathbb{Z}} \cup [l, l+3]_{\mathbb{Z}}, \tag{8}$$

which deduces that there is at least an element $l-1$ which do not belong to $F(C_k^{n,l} \vee C_k^{n,4})$ if $l \geq 12$.

Specifically, it appears that there is at least an element $l-1 \in [0, l+3]_{\mathbb{Z}}$ such that

$$l-1 \notin F(C_k^{n,l} \vee C_k^{n,4}).$$

Indeed, depending on the number l, the volume of the part

$$[0, l+3]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,4}) \tag{9}$$

is determined. To be specific, according to equality of (8), with the hypothesis of $l \geq 12$, let us precisely examine $F(C_k^{n,l} \vee C_k^{n,4})$ (see Figure 1(4)).

(Case 1) If $l = 12$, then we see that the number 11 does not belong to $F(C_k^{n,12} \vee C_k^{n,4})$. To be specific, we obtain

$$F(C_k^{n,12} \vee C_k^{n,4}) = [0, 10]_{\mathbb{Z}} \cup [12, 15]_{\mathbb{Z}}.$$

(Case 2) If $l = 14$, then owing to (8), it appears that the elements $12, 13$ do not belong to $F(C_k^{n,14} \vee C_k^{n,4})$. Namely,

$$F(C_k^{n,14} \vee C_k^{n,4}) = [0, 11]_{\mathbb{Z}} \cup [14, 17]_{\mathbb{Z}}.$$

(Case 3) If $l = 16$, then we observe that the numbers 13, 14, and 15 do not belong to $F(C_k^{n,16} \vee C_k^{n,4})$, i.e.,

$$F(C_k^{n,16} \vee C_k^{n,4}) = [0, 12]_{\mathbb{Z}} \cup [16, 19]_{\mathbb{Z}}.$$

(Case 4) If $l = 18$, then owing to (8), we obtain that the elements 14, 15, 16, and 17 do not belong to $F(C_k^{n,18} \vee C_k^{n,4})$, i.e.,

$$F(C_k^{n,18} \vee C_k^{n,4}) = [0, 13]_{\mathbb{Z}} \cup [18, 21]_{\mathbb{Z}}.$$

(Case 5) If $l = 20$, then we see that the numbers 15, 16, 17, 18, and 19 do not belong to $F(C_k^{n,20} \vee C_k^{n,4})$, i.e.,

$$F(C_k^{n,l} \vee C_k^{n,4}) = [0, 14]_{\mathbb{Z}} \cup [20, 23]_{\mathbb{Z}}.$$

(Case 6) If $l = 22$, then it appears that the elements $16, 17, 18, 19, 20$, and 21 do not belong to $F(C_k^{n,22} \vee C_k^{n,4})$, i.e.,

$$F(C_k^{n,l} \vee C_k^{n,4}) = [0, 15]_{\mathbb{Z}} \cup [22, 25]_{\mathbb{Z}}.$$

Now, we strongly count on the set

$$[0, l+3]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,4})$$

which is one component up to 2-connectedness, which deduces that $F(C_k^{n,l} \vee C_k^{n,4})$ has only two components up to 2-connectedness if $l \geq 12$.

For instance, though there are countably many cases as a generalization of the above six cases, just see each of the six cases above according to the formula of (8). Furthermore, the volume of the set of (9) depends on the number l of $F(C_k^{n,l} \vee C_k^{n,4})$ (see the six cases above). In view of the equality of (8), in the case $l \geq 12$, we confirmed that at least the element $l-1$ does not belong to $F(C_k^{n,l} \vee C_k^{n,4})$, which implies that $F(C_k^{n,l} \vee C_k^{n,4})$ is not 2-connected. Hence it is not perfect since $F(C_k^{n,l} \vee C_k^{n,4})$ has two components up to 2-connectedness. Eventually, if $l \geq 12$, it turns out that $F(C_k^{n,l} \vee C_k^{n,4})$ is not perfect.

In addition, one important thing to note is that the number $l+2$ is equal to $(C_k^{n,l} \vee C_k^{n,4})^\sharp - 1$ which is quite different feature compared to the feature of the case $F(C_k^{n,l}) = [0, \frac{l}{2}+1]_{\mathbb{Z}} \cup \{l\}, l \geq 6$. $\square$

In the following Figure 1, the dotted arrows indicate 8-continuous mappings. Moreover, all points are on $\mathbb{Z}^2$ formulating certain $C_8^{2,l}$ depending on the element l of $C_8^{2,l}$.

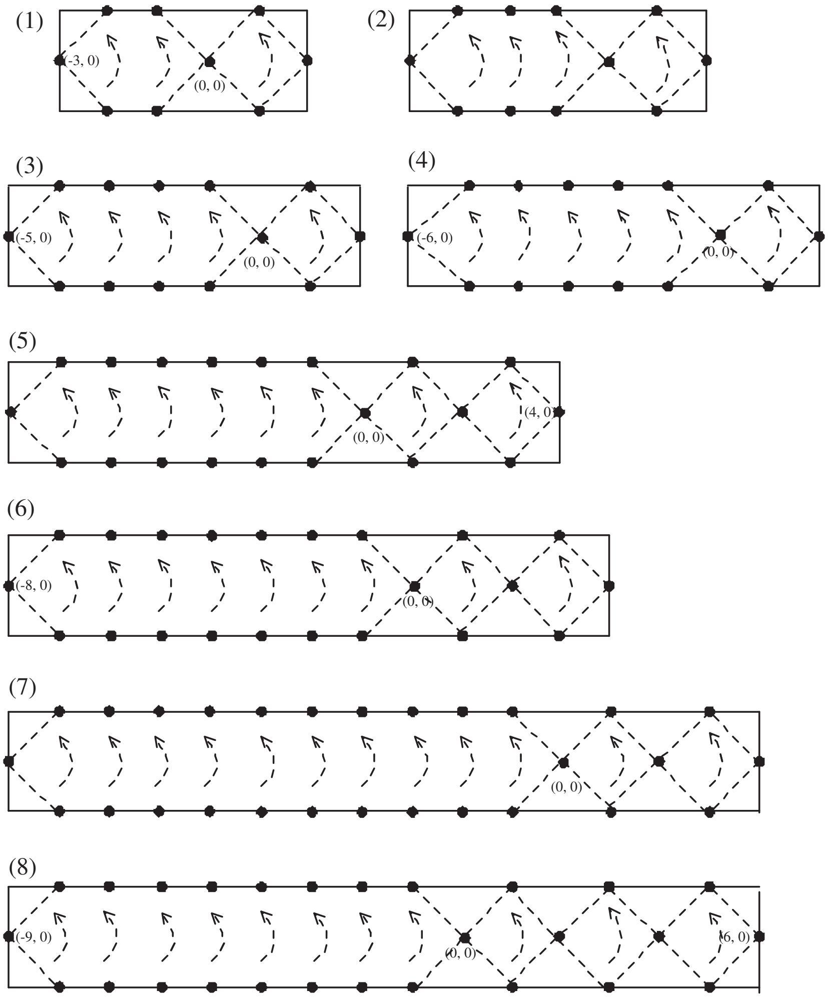

Figure 1. (1) $F(C_8^{2,6} \vee C_8^{2,4}) = [0,9]_{\mathbb{Z}}$; (2) $F(C_8^{2,8} \vee C_8^{2,4}) = [0,11]_{\mathbb{Z}}$; (3) $F(C_8^{2,10} \vee C_8^{2,4}) = [0,13]_{\mathbb{Z}}$; (4) $F(C_8^{2,12} \vee C_8^{2,4}) = [0,15]_{\mathbb{Z}} \setminus \{11\}$; (5) $F(C_8^{2,14} \vee C_8^{2,4} \vee C_8^{2,4}) = [0,20]_{\mathbb{Z}}$; (6) $F(C_8^{2,16} \vee C_8^{2,4} \vee C_8^{2,4}) = [0,22]_{\mathbb{Z}}$; (7) $F(C_8^{2,22} \vee C_8^{2,4} \vee C_8^{2,4}) = [0,28]_{\mathbb{Z}} \setminus \{19,20,21\}$; and (8) $F(C_8^{2,18} \vee C_8^{2,4} \vee C_8^{2,4} \vee C_8^{2,4}) = [0,27]_{\mathbb{Z}}$.

Indeed, the above two cases such as (Case 4) and (Case 6) in the proof of Theorem 2 will be used in further studying $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$, $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$, and so on (see Theorem 4). In view of Theorem 2, we obtain the following:

Remark 2. *As a generalization of* $F(C_k^{n,l} \vee C_k^{n,4})$*, for* $C_k^{n,l_1} \vee C_k^{n,l_2}$*, we can have a certain formula similar to that of (8) depending on the numbers* l_1 *and* l_2*. The current formula of (8) only for the case* $F(C_k^{n,l} \vee C_k^{n,4})$*. In the case that the number* 4 *of* $C_k^{n,4}$ *is changed into another, the formula of (8) is also changed depending on the number* l *of* $C_k^{n,l}$ *(compare (8) with (18) of Section 4).*

Thanks to the formula in (8), the following is obtained, which will be used in Theorem 4.

Example 1. *(1)* $F(C_8^{2,24} \vee C_8^{2,4})$ *does not have the following elements*

$$17, 18, 19, 20, 21, 22, \textit{and } 23.$$

(2) $F(C_8^{2,30} \vee C_8^{2,4})$ *does not contain the following elements*

$$20, 21, 22, 23, 24, 25, 26, 27, 28, \textit{and } 29.$$

In view of Example 1, we see that $F(C_k^{n,24} \vee C_k^{n,4})$ (cf. see (1) of Example 1) and $F(C_k^{n,30} \vee C_k^{n,4})$ (cf. see (2) of Example 1) have the same features with respect to 2-connectedness. Indeed, the number $l \in \{18, 24, 30\}$ plays an important role in studying some properties of the perfectness of $F(C_k^{n,l} \vee C_k^{n,4})$. Using a method similar to the method suggested in Example 1, we may further consider many cases.

Remark 3. *Lemma 4.8 of [4] is incorrectly stated, as follows: To make the paper self-contained, we now write it as follows: "Let* X *be connected with* $n = X^\sharp$*. Then* $n - 1 \in F(X)$ *if and only if there are distinct points* $x_1, x_2 \in X$ *with* $N(x_1) \subset N^*(x_2)$*," where* $N(x_1) = \{y \in X \,|\, y$ *is k-adjacent to* $x.\}$ *and* $N^*(x_1) = N(x_1) \cup \{x_1\}$*. Indeed, for convenience of the reader, we need to understand this situation with a digital image* (X, k)*, i.e., we need to represent this situation with* $N_k(x_1)$ *and* $N_k^*(x_2)$*. To explain this lemma more precisely, let us consider the digital image* $(X, 8)$ *as an example, where* $X := \{0 := x_0 := (0,0), 1 := x_1 := (1,0), 2 := x_2 := (2,1), 3 := x_3 := (1,2), 4 := x_4 := (0,2), 5 := x_5 := (-1,1), p := (1,1)\}$ *(see Figure 2(1)). Then we clearly see that* $X \setminus \{p\}$ *is equal to* $C_8^{2,6}$*. To adapt this digital image* $(X, 8)$ *into Lemma 4.8 of [4], we can assume two distinct points* p *and* $q \in \{1, 2, 3\} \subset X$*. Then we see that* $N_8(q) \subset N_8^*(p)$ *and further, we obviously obtain* $F(X) = [0, 7]_{\mathbb{Z}}$ *(see the process given through Figure 1(a–d)) so that* $F(X)$ *is perfect, which satisfies Lemma 4.8 of [4].*

However, we now point out that the condition for the assertion need not be "if and only if" with the following counterexample.

Consider the digital image $C_8^{2,4} \vee C_8^{2,4}$ *(see Figure 2(2)). As described in Figure 2(2) via (a)–(c), we can see that* $F(C_8^{2,4} \vee C_8^{2,4})$ *is also perfect, i.e.,* $F(C_8^{2,4} \vee C_8^{2,4}) = [0, 7]_{\mathbb{Z}}$*. Despite this situation, we see that there are not any distinct points* x_1, x_2 *in* $C_8^{2,4} \vee C_8^{2,4}$ *satisfying the condition* $N_8(x_1) \subset N_8^*(x_2)$*, where* $N_8(x_1) := N(x_1)$ *and* $N_8^*(x_2) := N^*(x_2)$*. Indeed, for any distinct points* x_1, x_2 *in* $C_8^{2,4} \vee C_8^{2,4}$*, it is easy to see that* $N_8(x_1) \not\subseteq N_8^*(x_2)$*.*

As another counterexample, we may consider the case $F(C_8^{2,4}) = [0, 4]_{\mathbb{Z}}$ *and so on.*

In view of the previously-obtained results, since $C_k^{n,l}$, $C_k^{n,l_1} \vee C_k^{n,l_2}$, $C_k^{n,l_1} \vee C_k^{n,l_2} \vee C_k^{n,l_3}$ and so on play important roles in digital topology, let us now intensively explore an alignment of fixed point sets of them, e.g., $F(C_k^{n,l})$, $F(C_k^{n,l_1} \vee C_k^{n,l_2})$, and so forth.

By Theorem 2, for $l \geq 12$, it turns out that $F(C_k^{n,l} \vee C_k^{n,4})$ is not 2-connected. Hereafter, let us now examine if $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect, according to the number l. Specifically, after joining another $C_k^{n,4}$ onto $C_k^{n,l} \vee C_k^{n,4}$ to establish the new digital wedge $C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}$ (see Figure 1(6,7)), we confirm the perfectness of $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ if $l \in \{12, 14, 16\}$, as follows:

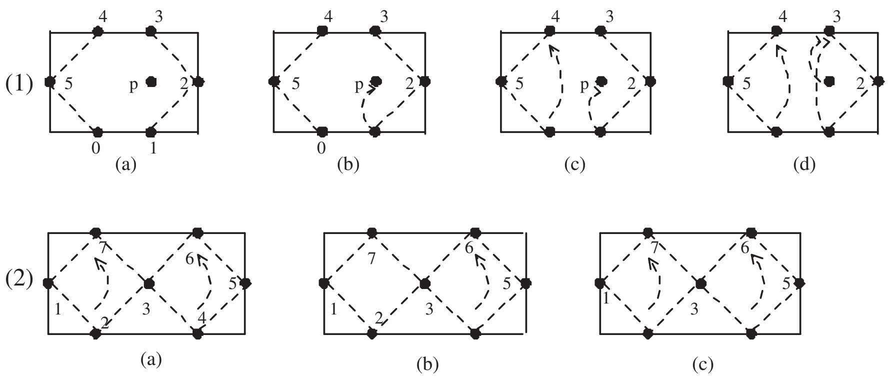

Figure 2. In these figures, the dotted arrows indicate 8-continuous mappings in the $DTC(8)$-setting. (1) Perfectness of $F(X, 8)$; (2) Perfectness of $F(C_8^{2,4} \vee C_8^{2,4})$.

Theorem 3. *If* $l \in \{12, 14, 16\}$, *then* $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ *is perfect.*

Proof. Let us consider any k-continuous self-map f of $C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}$. Indeed, though there are many kinds of k-continuous self-maps f of it, regarding $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$, it is sufficient to consider only the maps f such that

(a) $f|_{C_k^{n,4}}(x) = x$; or
(b) $f|_{C_k^{n,l}}(x) = x$; or
(c) $f|_{C_k^{n,l} \vee C_k^{n,4}}(x) = x$; or
(d) $f|_{C_k^{n,4} \vee C_k^{n,4}}(x) = x$; or
(e) $f(C_k^{n,l}) \subsetneq C_k^{n,l}$ and $f(C_k^{n,4}) \subsetneq C_k^{n,4}$; or
(f) f does not support any fixed point of it, i.e., there is no $x \in C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}$ such that $f(x) = x$.

Let us investigate the set of cardinalities of fixed point sets of k-continuous maps associated with the cases of (a)–(f) above.

First, from (a) above, we obtain

$$[4, 7 + \frac{l}{2}]_{\mathbb{Z}} \cup \{l + 6\} \subset F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}). \tag{10}$$

Second, from (b) above, we have

$$[l, l + 6]_{\mathbb{Z}} \subset F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}). \tag{11}$$

Third, from (c) above, we have

$$[l + 3, l + 6]_{\mathbb{Z}} \subset F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}). \tag{12}$$

Fourth, from (d) above, we have

$$[7, 7 + \frac{l}{2}]_{\mathbb{Z}} \cup \{l + 6\} \subset F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}). \tag{13}$$

Fifth, from (e)–(f), we obtain

$$[0, \frac{l}{2} + 5]_{\mathbb{Z}} \subset F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}). \tag{14}$$

After checking the sets from (10)–(14), we can obtain $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ according to the numbers $l \in \{12, 14, 16\}$, as follows:

According to these three cases, with the hypothesis of $l \in \{12, 14, 16\}$, let us precisely investigate $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$.

(Case 1) If $l = 12$, then we see that $F(C_k^{n,12} \vee C_k^{n,4} \vee C_k^{n,4}) = [0, 18]_{\mathbb{Z}}$.
(Case 2) If $l = 14$, then we obtain that $F(C_k^{n,14} \vee C_k^{n,4} \vee C_k^{n,4}) = [0, 20]_{\mathbb{Z}}$ (see Figure 1(5)).
(Case 3) If $l = 16$, then we observe that $F(C_k^{n,16} \vee C_k^{n,4} \vee C_k^{n,4}) = [0, 22]_{\mathbb{Z}}$ (see Figure 1(6)).

Hence, it turns out that $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect if $l \in \{12, 14, 16\}$. □

In view of Theorem 3, it turns out that if $l \leq 16$, then $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect.

After checking (10)–(14) in the proof of Theorem 3, for $l \in \{18, 20, 22\}$, let us precisely examine if $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect, as follows:

Theorem 4. *If $l \geq 18$, then*

(1) $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ *is not 2-connected.*
(2) $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ *has two components up to 2-connectedness.*

Proof. For $l \geq 18, l \in \mathbb{N}_0$, after checking the sets from (10)–(14) in the proof of Theorem 3, we obtain the following:

(Case 1) If $l = 18$, then we see that

$$F(C_k^{n,18} \vee C_k^{n,4} \vee C_k^{n,4}) = [0, 24]_{\mathbb{Z}} \setminus \{17\}.$$

(Case 2) If $l = 20$, then we see that

$$F(C_k^{n,20} \vee C_k^{n,4} \vee C_k^{n,4}) = [0, 26]_{\mathbb{Z}} \setminus \{18, 19\}.$$

(Case 3) If $l = 22$, then we observe that

$$F(C_k^{n,22} \vee C_k^{n,4} \vee C_k^{n,4}) = [0, 28]_{\mathbb{Z}} \setminus \{19, 20, 21\} (\textit{see Figure } 1(7)).$$

(Case 4) If $l \geq 24$, then it is obvious that $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is not 2-connected.

In view of this calculation, if $l \geq 18, l \in \mathbb{N}_0$, then

(1) $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is not perfect, and the proof is completed.
(2) Since the set $[0, l+6]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is 2-connected, it appears that $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ has only two components.

□

As proved in Theorem 2, in the case $l \geq 12$, it turns out that $F(C_k^{n,l} \vee C_k^{n,4})$ is not perfect. Then, after joining a certain $C_k^{n,l'}$ onto $C_k^{n,l} \vee C_k^{n,4}$ (see Theorem 4) to produce a new digital wedge, we have a natural question, as follows: For $C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,l'}$,

Is $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,l'})$ perfect?

Regarding this query, motivated by Theorem 2 (see the Cases (1)–(6) in the proof of Theorem 2), we have the following: Owing to the cases, (Case 1)–(Case 6), we get some advantages of finding some more features of $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$, $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$, and so on.

In view of Theorem 4, for $l \in \{18, 20, 22\}$, though $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4})$ is not perfect, after joining one $C_k^{n,4}$ onto $C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}$ to produce $C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4}$, using a method similar to Theorem 4, we obtain the following (see Figure 1(8)):

Corollary 1. *If $l \leq 22, l \in \mathbb{N}_0$, then $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect.*

Proof. Motivated by the consideration of (a)–(f) suggested in the proof of Theorem 3, we can consider the following several cases. Though there are many kinds of k-continuous self-maps f of it, regarding $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$, it is sufficient to consider only the maps f such that

(a) $f|_{C_k^{n,4}}(x) = x$; or
(b) $f|_{C_k^{n,l}}(x) = x$; or
(c) $f|_{C_k^{n,l} \vee C_k^{n,4}}(x) = x$; or
(d) $f|_{C_k^{n,4} \vee C_k^{n,4}}(x) = x$; or
(e) $f|_{C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4}}(x) = x$; or
(f) $f|_{C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4}}(x) = x$; or
(g) $f(C_k^{n,l}) \subsetneq C_k^{n,l}$ and $f(C_k^{n,4}) \subsetneq C_k^{n,4}$; or
(h) f does not support any fixed point of it, i.e., there is no $x \in C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4}$ such that $f(x) = x$.

Then, using a method similar to the formulas in (10)–(14), we obtain that if $l \leq 22, l \in \mathbb{N}_0$, then the proof is completed. □

Corollary 2. *If $l \in \{24, 26, 28\}$, while $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$ is not 2-connected, $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect.*

Specifically, Corollary 2 implies that if $l \geq 24$, while $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$ is not 2-connected, if $l \leq 28$, then $F(C_k^{n,l} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4} \vee C_k^{n,4})$ is perfect.

Proof. Using the methods of the proofs of Theorem 3 and Corollary 1, the proof is completed. □

4. Non-Perfectness of $F(C_k^{n,l} \vee C_k^{n,6}), l \geq 6$ and perfectness of $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4}), l \in \{4, 6, \cdots, 18, 20\}$

As mentioned in Sections 1 and 3, owing to the certain significant importance of $C_k^{n,6}$ from the viewpoint of digital topology, this section mainly focuses on investigating some digital topological properties of $F(C_k^{n,l} \vee C_k^{n,6})$, $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$, and so forth. Regarding this study, as proposed in Section 1 with (Q6) and (Q7), we examine if $F(C_k^{n,l} \vee C_k^{n,6})$ is perfect. Moreover, we find some condition that makes $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$ perfect. Indeed, while $C_k^{n,l} \vee C_k^{n,6}, l \geq 6$, is not 2-connected, we find a certain condition that $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$ is perfect. This section, hereafter, mainly investigates some topological properties of $F(C_k^{n,l} \vee C_k^{n,6})$ and perfectness of it depending on the number l. One important consideration is that we need to remind that if $l \geq 6$, then the k-fundamental group of $C_k^{n,l} \vee C_k^{n,6}$ is a free group with two generators of which each of them has an infinite order.

Theorem 5. *(1) If $l \geq 6$, then $F(C_k^{n,l} \vee C_k^{n,6})$ is not perfect.*
(2) If $l \in \{6, 8, 10, 12, 14\}$, then $F(C_k^{n,l} \vee C_k^{n,6})$ has two components.
(3) If ≥ 6, then $l + 4 \notin F(C_k^{n,l} \vee C_k^{n,6})$.

Proof. Let us consider any k-continuous self-map f of $C_k^{n,l} \vee C_k^{n,6}$. Indeed, though there are many kind of k-continuous maps f, regarding $F(C_k^{n,l} \vee C_k^{n,6})$, it is sufficient to consider only the maps f such that

(a) $f|_{C_k^{n,6}}(x) = x$; or
(b) $f|_{C_k^{n,l}}(x) = x$; or

(c) $f(C_k^{n,l}) \subsetneq C_k^{n,l}$ and $f(C_k^{n,6}) \subsetneq C_k^{n,6}$; or

(d) f does not support any fixed point of it, i.e., there is no point $x \in C_k^{n,l} \vee C_k^{n,6}$ such that $f(x) = x$,

where $f|_A$ means the restriction function f to the given set A.

First, from (a) above, since $C_k^{n,l} \vee C_k^{n,6}$ has the cardinality $l+5$, we obtain

$$[6, 6 + \frac{l}{2}]_{\mathbb{Z}} \cup \{l+5\} \subset F(C_k^{n,l} \vee C_k^{n,6}). \tag{15}$$

Second, from (b) above, we have

$$[l, l+3]_{\mathbb{Z}} \cup \{l+5\} \subset F(C_k^{n,l} \vee C_k^{n,6}). \tag{16}$$

Third, from (c)–(d) above, we obtain

$$[0, \frac{l}{2} + 4]_{\mathbb{Z}} \subset F(C_k^{n,l} \vee C_k^{n,6}). \tag{17}$$

After comparing the four numbers $6 + \frac{l}{2}$, l, $l+3$, and $\frac{l}{2} + 4$ from (15)–(17), we conclude that

$$F(C_k^{n,l} \vee C_k^{n,6}) = [0, 6 + \frac{l}{2}]_{\mathbb{Z}} \cup [l, l+3]_{\mathbb{Z}} \cup \{l+5\}, \tag{18}$$

Owing to the property (18), for any $C_k^{n,l}$, it is obvious that the element

$$l + 4 \notin F(C_k^{n,l} \vee C_k^{n,6}).$$

Specifically, for any $l, l \geq 6$, of $C_k^{n,l}$, we always obtain the property

$$\{l+4\} \subset [0, l+5]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,6}).$$

Moreover, for the number $l \in \{6, 8, 10, 12, 14\}$, since there is no element between $6 + \frac{l}{2}$ and l in $F(C_k^{n,l} \vee C_k^{n,6})$, let us now check the difference between the two numbers $6 + \frac{l}{2}$ and l of (18). Specifically, if $l \in \{6, 8, 10, 12, 14\}$, then $F(C_k^{n,l} \vee C_k^{n,6})$ has two components such as $[0, l+3]_{\mathbb{Z}}$ and the singleton $\{l+5\}$.

To be specific, with the hypothesis of $l \in \{6, 8, 10, 12, 14\}$, according to equality of (18), let us precisely examine $F(C_k^{n,l} \vee C_k^{n,6})$ (see Figure 3(1–4)).

(Case 1) If $l = 6$, then we see that $F(C_k^{n,6} \vee C_k^{n,6}) = [0, 9]_{\mathbb{Z}} \cup \{11\}$.

(Case 2) If $l = 8$, then it appears that $F(C_k^{n,8} \vee C_k^{n,6}) = [0, 11]_{\mathbb{Z}} \cup \{13\}$.

(Case 3) If $l = 10$, then we observe that $F(C_k^{n,10} \vee C_k^{n,6}) = [0, 13]_{\mathbb{Z}} \cup \{15\}$.

(Case 4) If $l = 12$, then we find that $F(C_k^{n,12} \vee C_k^{n,6}) = [0, 15]_{\mathbb{Z}} \cup \{17\}$.

(Case 5) If $l = 14$, then we see that $F(C_k^{n,14} \vee C_k^{n,6}) = [0, 17]_{\mathbb{Z}} \cup \{19\}$.

At the moment, we need to observe that the set

$$[0, l+5]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,6})$$

has one component up to 2-connectedness.

In view of the equality of (Case 1)–(Case 5) above, it appears that at least the element $(C_k^{n,l} \vee C_k^{n,6})^\sharp - 1$ does not belong to $F(C_k^{n,l} \vee C_k^{n,6})$, which implies that $F(C_k^{n,l} \vee C_k^{n,6})$ is not 2-connected so that it is not perfect either, since $F(C_k^{n,l} \vee C_k^{n,6})$ has two components up to 2-connectedness. □

Theorem 6. *If $l \geq 16$, then $F(C_k^{n,l} \vee C_k^{n,6})$ has three components up to 2-connectedness.*

Proof. If $l \geq 16$, then owing to the difference $\frac{l}{2} - 6$ between l and $6 + \frac{l}{2}$ from (18), we have the property that there is at least an element

$$l - 1 \in [0, l+5]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,6})$$

since in $[0, l+5]_{\mathbb{Z}} \setminus F(C_k^{n,l} \vee C_k^{n,6})$ the component containing the element $l-1$ in $F(C_k^{n,l} \vee C_k^{n,6})$ and the component containing the element $l+4$ are disjoint up to 2-connectedness. Hence we concluded that $F(C_k^{n,l} \vee C_k^{n,6})$ has three components. Owing to (18), the volume of the part including the element $l-1$ which does not included in $F(C_k^{n,l} \vee C_k^{n,4})$ depends on the number l of $F(C_k^{n,l} \vee C_k^{n,6})$.

To be specific,

if $l = 16$, then we see that the numbers $15, 20$ do not belong to $F(C_k^{n,16} \vee C_k^{n,6})$.
If $l = 18$, then we see that the numbers $16, 17, 22$ do not belong to $F(C_k^{n,18} \vee C_k^{n,6})$.
If $l = 20$, then we see that the numbers $17, 18, 19, 24$ do not belong to $F(C_k^{n,20} \vee C_k^{n,6})$.
□

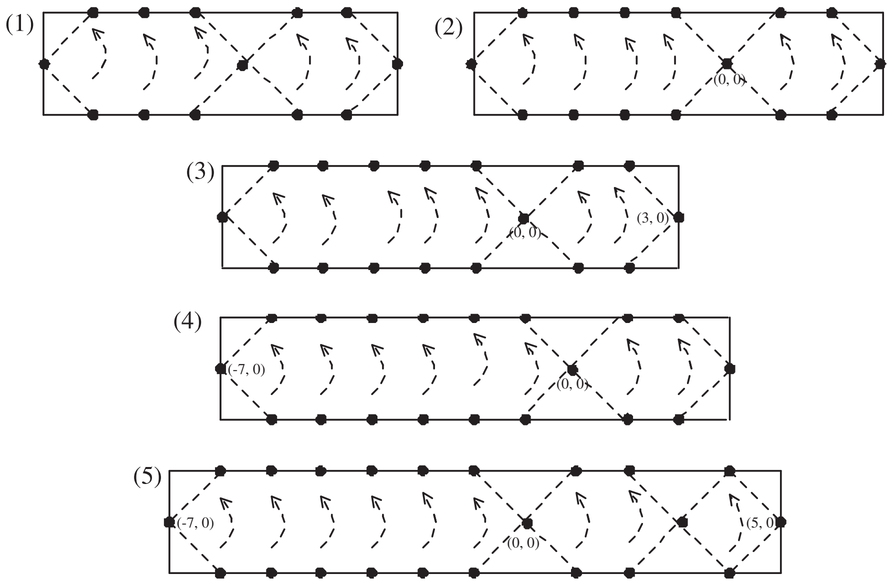

Figure 3. In these figures, the dotted arrows indicate 8-continuous self-mappings. (1)–(4) Explanation of being two components of $F(C_k^{n,l} \vee C_k^{n,6})$, $l \in \{8, 10, 12, 14\}$; (5) Explanation of the perfectness of $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4}) = [0, l+8]_{\mathbb{Z}}$, $l \in \{4, 6, 8, 10, 12, 14, 16, 18, 20\}$ (see Theorem 7). In particular, $l = 14$.

If $l \geq 6$, while $F(C_k^{n,l} \vee C_k^{n,6})$ is not perfect, we obtain the following:

Theorem 7. *If $l \leq 20$, then $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$ is perfect.*

Proof. Let us consider any k-continuous self-map f of $C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4}$ (see Figure 3(5)). Indeed, though there are many kinds of k-continuous maps f, regarding $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$, it is sufficient to consider only the maps f such that

(a) $f|_{C_k^{n,4}}(x) = x$;
(b) $f|_{C_k^{n,6}}(x) = x$;
(c) $f|_{C_k^{n,l}}(x) = x$;
(d) $f|_{C_k^{n,6} \vee C_k^{n,4}}(x) = x$:
(e) $f|_{C_k^{n,l} \vee C_k^{n,6}}(x) = x$;

(f) $f|_{C_k^{n,l} \vee C_k^{n,4}}(x) = x$;

(g) $f(C_k^{n,l}) \subsetneq C_k^{n,l}$, $f(C_k^{n,6}) \subsetneq C_k^{n,6}$, and $f(C_k^{n,4}) \subsetneq C_k^{n,4}$ (in particular, see Figure 3(5)); and

(h) f does not support any fixed point of it, i.e., there is no point $x \in C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4}$ such that $f(x) = x$,

where $f|_A$ means the restriction function f to the given set A.

As already mentioned above, though there are indeed so many cases such as the cases (a)–(h) above for calculating $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$, it is sufficient to consider the above cases from (a)–(h) exhausting $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,4})$.

Using a method similar to the proofs of Theorems 5 and 6, the proof is completed. $\square$

Remark 4. *Using a method similar to the proof of Theorems 5, 6, and 7, if* $6 \leq l \leq 12$, *then* $F(C_k^{n,l} \vee C_k^{n,6} \vee C_k^{n,6})$ *is not perfect.*

Example 2. *In view of Theorem 7, while* $F(C_8^{2,20} \vee C_8^{2,6} \vee C_8^{2,4})$ *is perfect, let us show that* $F(C_8^{2,22} \vee C_8^{2,6} \vee C_8^{2,4})$ *is not perfect. According to the property (a) of the proof of Theorem 7, we obtain* $[4, 18]_{\mathbb{Z}} \cup \{20\} \cup \{30\}$. *Owing to (b), we have* $[6, 20]_{\mathbb{Z}} \cup \{30\}$. *By (c), we have* $[22, 28]_{\mathbb{Z}} \cup \{30\}$. *By (d), we have* $[9, 20]_{\mathbb{Z}}$. *Owing to (e), we have* $[26, 27]_{\mathbb{Z}}$. *By (f), we have* $[27, 30]_{\mathbb{Z}}$. *Finally, by (g) and (h), we have* $[0, 17]_{\mathbb{Z}}$. *In view of this situation, we obtain* $F(C_k^{n,22} \vee C_k^{n,6} \vee C_k^{n,4}) = [0, 30]_{\mathbb{Z}} \setminus \{21\}$.

The more generalized cases of Corollaries 1 and 2, and Theorems 5–7 will be dealt in a consecutive paper shortly.

5. Conclusions and Further Work

We have addressed several issues raised in Section 1, which can facilitate the studies of both digital topology and fixed point theory. Since there are at least countably many categories of digital spaces [25,35–37], using many kinds of continuous maps that are involved in the corresponding digital spaces, we can further study an alignment of homotopy fixed point sets in the categories of $KDTC(k)$-, $CTC(k)$-, and $DTC(k)$, and so on [5]. As a further work, we need to investigate the various properties of alignments of fixed point sets in the $KDTC(k)$- or the $CTC(k)$-setting, and further, to explore some features of $KDTC(k)$- or $CTC(k)$-rigidity. In addition, using an alignment of homotopy fixed point sets in a $KDTC(k)$- or a $CTC(k)$-setting, we can further investigate their digital topological properties.

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