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On a Relation between the Perfect Roman Domination and Perfect Domination Numbers of a Tree

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Abstract: A *dominating set* in a graph G is a set of vertices $S \subseteq V(G)$ such that any vertex of $V - S$ is adjacent to at least one vertex of S. A *dominating set* S of G is said to be a *perfect dominating set* if each vertex in $V - S$ is adjacent to exactly one vertex in S. The minimum cardinality of a *perfect dominating set* is the perfect domination number $\gamma^p(G)$. A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a perfect Roman dominating function (PRDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a PRDF is the sum of its function values over all vertices, and the minimum weight of a PRDF of G is the perfect Roman domination number $\gamma_R^p(G)$. In this paper, we prove that for any nontrivial tree T, $\gamma_R^p(T) \geq \gamma^p(T) + 1$ and we characterize all trees attaining this bound.

Keywords: Roman domination number; perfect Roman domination number; tree

1. Introduction

In this paper, only simple and undirected graph without isolated vertices will be considered. The set of vertices of the graph G is denoted by $V = V(G)$ and the edge set is $E = E(G)$. The order of a graph G is the number of vertices of the graph G and it is denoted by $n = n(G)$. The size of G is the cardinality of the edge set and it is denoted by $m = m(G)$. For a vertex $v \in V$, the *open neighbourhood* $N(v)$ is the set $\{u \in V(\Gamma) : uv \in E(G)\}$, the *closed neighbourhood* of v is the set $N[v] = N(v) \cup \{v\}$, and the *degree* of v is $\deg_G(u) = |N(v)|$. Any vertex of degree one is called a *leaf*, a *support vertex* is a vertex adjacent to a leaf, a *strong support vertex* is a support vertex adjacent to at least two leaves and an *end support vertex* is a support vertex such that all its neighbors, except possibly one, are leaves. For a graph G, let $L(G) = \{v \in V(G) \mid \deg_G(v) = 1\}$ and $L_v = N(v) \cap L(G)$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum value among distances between all pair of vertices of G. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. Moreover, the depth of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree rooted at* v, denoted by T_v, consists of v and all its descendants. We write P_n for the *path* of order n. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively p and q leaves attached at each support vertex is denoted $DS_{p,q}$. For a real-valued function $f : V \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$.

A *dominating set* (DS) in a graph G is a set of vertices $S \subseteq V(G)$ such that any vertex of $V - S$ is adjacent to at least one vertex of S. A *dominating set* S of G is said to be a *perfect dominating set* (PDS) if each vertex in $V - S$ is adjacent to exactly one vertex in S. The minimum cardinality of a *(perfect) dominating set* of a graph G is the *(perfect) domination number* $\gamma(G)$ $(\gamma^p(G))$. Perfect domination was introduced by Livingston and Stout in [1] and has been studied by several authors [2–6].

A function $f : V(\Gamma) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A *perfect Roman dominating function* (PRDF) on a graph G is an RDF f such that every vertex assigned a 0 is adjacent to exactly one vertex assigned a 2 under f. The minimum weight of a (perfect) RDF on a graph G is the *(perfect) Roman domination number* $\gamma_R(G)$ $(\gamma_R^p(G))$. A (perfect) RDF on G with weight $\gamma_R(G)$ $(\gamma_R^p(G))$ is called a $\gamma_R(G)$-function ($\gamma_R^p(G)$-function). An RDF f on a graph $G = (V, E)$ can be represented by the ordered partition (V_0, V_1, V_2) of V, where $V_i = \{v \in V | f(v) = i\}$ for $i = 0, 1, 2$. The concept of Roman domination was introduced by Cockayne et al. in [7] and was inspired by the manuscript of the authors of [8], and Stewart [9] about the defensive strategy of the Roman Empire decreed by Constantine I The Great, while perfect Roman domination was introduced by Henning, Klostermeyer and MacGillivray in [10] and has been studied in [11–13]. For more on Roman domination, we refer the reader to the book chapters [14,15] and surveys [16–18].

It was shown in [10] that for any tree G of order $n \geq 3$, $\gamma_R^p(G) \leq \frac{4n}{5}$. Moreover, the authors have characterized all trees attaining this upper bound. Note that the previous upper bound have been improved by Henning and Klostermeyer [13] for cubic graphs of order n by showing that $\gamma_R^p(G) \leq \frac{3n}{4}$.

It is worth mentioning that if S is a minimum *(perfect) dominating set* of a graph G, then clearly $(V - S, \emptyset, S)$ is a (perfect) RDF and thus

$$\gamma_R(G) \leq 2\gamma(G) \quad \text{and} \quad \gamma_R^p(G) \leq 2\gamma^p(G). \tag{1}$$

On the other hand, if $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function, then $V_1 \cup V_2$ is a *dominating set* of G yielding

$$\gamma(G) \leq \gamma_R(G). \tag{2}$$

It is natural to ask whether the inequality (2) remains valid between $\gamma^P(G)$ and $\gamma_R^p(G)$ for any graph G. The answer is negative as it can be seen by considering the graph H obtained from a double star $DS_{p,p}$, $(p \geq 3)$ with central vertices u, v by subdividing the edge uv with vertex w, and adding $2k$ $(k \geq 3)$ new vertices, where k vertices are attached to both u and w and the remaining k vertices are attached to both v and w (see Figure 1). Clearly, $\gamma^p(H) = 2k + 3$ while $\gamma_R^p(H) = 5$ and so the difference $\gamma^p(H) - \gamma_R^p(H)$ can be even very large.

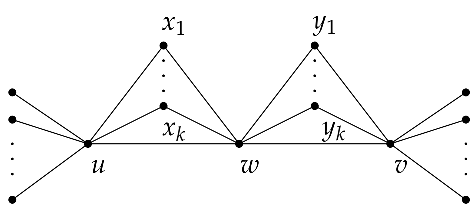

Figure 1. The graph H.

Motivated by the above example, we shall show in this paper that $\gamma_R^p(T) \geq \gamma^p(T) + 1$ for every nontrivial tree T, and we characterize all trees attaining this bound.

2. Preliminaries

We start by providing some useful definitions and observations throughout the paper.

Definition 1. *For any graph G, let*

$$\begin{aligned} W_G^{R,1} &= \{u \in V \mid \text{there exists a } \gamma_R^p(G)\text{-function } f \text{ such that } f(u) = 2\}, \\ W_G^{R,\leq 1} &= \{u \in V \mid f(u) \leq 1 \text{ for some } \gamma_R^p(G)\text{-function } f\}, \\ W_G^{R,\geq 1} &= \{u \in V \mid \text{for each } v \in N_G(u),\ f(v) \leq 1 \text{ for every } \gamma_R^p(G)\text{-function } f\}, \\ W_G^{P,A} &= \{u \in V \mid u \text{ belongs to every } \gamma^p(G)\text{-set}\}. \end{aligned}$$

Definition 2. *Let u be a vertex of a graph G. A set S is said to be an almost perfect dominating set (almost PDS) with respect to u, (i) if each vertex $x \in V \setminus (S \cup \{u\})$ has exactly one neighbor in S, and (ii) if $u \in V \setminus S$, then u has at most one neighbor in S. Let*

$$\gamma^p(G; u) = \min\{|S| : S \text{ is an almost PDS with respect to } u\}.$$

Trivially, every PDS of G is an almost PDS with respect to any vertex of G and thus $\gamma^p(G; u)$ is well defined. Hence $\gamma^p(G; u) \leq \gamma^p(G)$ for each vertex $u \in V$. Let

$$W_G^{APD} = \{u \in V \mid \gamma^p(G; u) = \gamma^p(G)\}.$$

The proof of the following two results are given in [12].

Observation 1. *Let G be a graph.*

1. *Any strong support vertex belongs to $W_G^{P,A}$.*
2. *Any support vertex adjacent to a strong support vertex, belongs to $W_G^{P,A}$.*
3. *For any leaf u of G, there is a $\gamma_R^p(G)$-function f such that $f(u) \leq 1$.*

Proposition 1. *Let G be a graph. G has a $\gamma_R^p(G)$-function that assigns 2 to every end strong support vertex. Thus every end strong support vertex of a graph G belongs to $W_G^{R,1}$.*

The next result is a consequence of Observation 1 and Proposition 1.

Corollary 1. *Let u be an end strong support vertex of a graph H. If G is the graph obtained from H by adding a vertex x and an edge ux, then $\gamma^p(G) = \gamma^p(H)$ and $\gamma_R^p(G) = \gamma_R^p(H)$.*

Proposition 2. *Let H be a graph and $u \in V(H)$. If G is a graph obtained from H by adding a path $P_2 : x_1x_2$ attached at u by an edge ux_1, then:*

1. *$\gamma^p(G) \leq \gamma^p(H) + 1$ and $\gamma_R^p(G) \geq \gamma_R^p(H) + 1$.*
2. *If $u \in W_H^{R,1} \cup W_H^{R,\geq 1}$, then $\gamma_R^p(G) = \gamma_R^p(H) + 1$.*
3. *If $u \in W_H^{APD}$, then $\gamma^p(G) = \gamma^p(H) + 1$.*

Proof.

1. For a $\gamma^p(H)$-set S, let $S' = S \cup \{x_1\}$ if $u \in S$, and $S' = S \cup \{x_2\}$ if $u \notin S$. Clearly, S' is a PDS of G and thus $\gamma^p(G) \leq \gamma^p(H) + 1$.

   Now let f be a $\gamma_R^p(G)$-function. Obviously, $f(x_1) + f(x_2) \geq 1$. If $f(u) \geq 1$, then the function f restricted to H is a PRDF on H yielding $\gamma_R^p(G) \geq \gamma_R^p(H) + 1$. Thus assume that $f(u) = 0$. Then $f(x_1) + f(x_2) = 2$ and the function $g : V(H) \to \{0,1,2\}$ defined by $g(u) = 1$ and

$g(x) = f(x)$ for $x \in V(H) \setminus \{u\}$ is a PRDF on H of weight $\gamma_R^p(G) - 1$. Hence in any case, $\gamma_R^p(G) \geq \gamma_R^p(H) + 1$.

2. Assume first that $u \in W_H^{R,1}$ and let f be a $\gamma_R^p(H)$-function with $f(u) = 2$. Then f can be extended to a PRDF of G by assigning a 1 to x_2 and a 0 to x_1 and thus $\gamma_R^p(G) \leq \gamma_R^p(H) + 1$. The equality follows by item 1. Assume now that $u \in W_{G'}^{R,\geq 1}$ and let f be a $\gamma_R^p(H)$-function. By the definition of $W_H^{R,\geq 1}$, we must have $f(u) \geq 1$ to Roman dominate u. Now, if $f(u) = 2$, then using the same argument as above we obtain $\gamma_R^p(G) = \gamma_R^p(H) + 1$. Hence assume that $f(u) = 1$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x_1) = 2$, $g(u) = g(x_2) = 0$ and $g(x) = f(x)$ for all $x \in V(H) \setminus \{u\}$ is a PRDF of G of weight $\gamma_R^p(H) + 1$. Therefore $\gamma_R^p(G) \leq \gamma_R^p(H) + 1$, and the equality follows by item 1.
3. Let S be a $\gamma^p(G)$-set. Clearly, $|S \cap \{x_1, x_2\}| \geq 1$ and $S - \{x_1, x_2\}$ is an almost PDS of H with respect to u. Since $u \in W_H^{APD}$, we have $|S - \{x_1, x_2\}| \geq \gamma^p(G'; u) = \gamma^p(H)$. Therefore $\gamma^p(G) = |S| \geq \gamma^p(H) + 1$, and the equality follows from item 1. □

For a graph G and a vertex u of G, we denote by $G^u_{K_{1,3}}$ the graph obtained from G by adding a star $K_{1,3}$ and joining one of its leaf to u.

Proposition 3. *Let G be a graph and u a vertex of G.*

1. $\gamma^p(G^u_{K_{1,3}}) \leq \gamma^p(G) + 2$ *and* $\gamma_R^p(G) + 2 \leq \gamma_R^p(G^u_{K_{1,3}})$.
2. *If* $u \in W_G^{P,A} \cap W_G^{APD}$, *then* $\gamma^p(G^u_{K_{1,3}}) = \gamma^p(G) + 2$.
3. *If* $u \in W_G^{R,\leq 1}$, *then* $\gamma_R^p(G^u_{K_{1,3}}) = \gamma_R^p(G) + 2$.

Proof. Let x be the center of the star $K_{1,3}$ and x_1 a leaf of $K_{1,3}$ attached at u by an edge ux_1.

1. For a $\gamma^p(G)$-set S, let $S' = S \cup \{x, x_1\}$ if $u \in S$, and $S' = S \cup \{x\}$ for otherwise. Clearly, S' is a PDS of $G^u_{K_{1,3}}$ and thus $\gamma^p(G^u_{K_{1,3}}) \leq \gamma^p(G) + 2$.

   Now, let f be a $\gamma_R^p(G^u_{K_{1,3}})$-function. By Proposition 1, we may assume that $f(x) = 2$. If $f(x_1) \leq 1$, then the function f restricted to G is a PRDF on G of weight at most $\gamma_R^p(G^u_{K_{1,3}}) - 2$. Thus, we assume that $f(x_1) = 2$. Then the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(u) = 1$ and $g(x) = f(x)$ for all $x \in V(G) \setminus \{u\}$ is a PRDF on G of weight $\gamma_R^p(G^u_{K_{1,3}}) - 3$. In any case, $\gamma_R^p(G) \leq \gamma_R^p(G^u_{K_{1,3}}) - 2$.
2. Let S be a $\gamma^p(G^u_{K_{1,3}})$-set. By Observation 1-(1), we have $x \in S$. Now, if $u \in S$, then $x_1 \in S$ and clearly $S - \{x, x_1\}$ is a PDS of G, implying that $\gamma^p(G^u_{K_{1,3}}) \geq \gamma^p(G) + 2$. Thus, assume that $u \notin S$. If $x_1 \notin S$, then $S - \{x\}$ is a PDS of G that does not contain u and since $u \in W_G^{P,A}$ we deduce that $|S - \{x\}| \geq \gamma^p(G) + 1$. Hence $\gamma^p(G^u_{K_{1,3}}) \geq \gamma^p(G) + 2$. If $x_1 \in S$, then $S - \{x, x_1\}$ is an almost PDS of G and since $u \in W_G^{APD}$ we conclude that $|S - \{x, x_1\}| \geq \gamma^p(G)$. Hence $\gamma^p(G^u_{K_{1,3}}) \geq \gamma^p(G) + 2$. Whatever the case, the equality follows from item 1.
3. Assume that $u \in W_G^{R,\leq 1}$ and let f be a $\gamma_R^p(G)$-function such that $f(u) \leq 1$. Then f can be extended to a PRDF on $G^u_{K_{1,3}}$ by assigning a 2 to x and a 0 to every neighbor of x and thus $\gamma_R^p(G^u_{K_{1,3}}) \leq \gamma_R^p(G) + 2$. The equality follows from item 1. □

Proposition 4. *Let G' be a graph and let u be an end support vertex of G' which is adjacent to a strong support vertex v. If G is a graph obtained from G' by adding a vertex x and an edge ux, then $\gamma^p(G) = \gamma^p(G')$ and $\gamma_R^p(G) \geq \gamma_R^p(G')$. Moreover, if $u \in W_{G'}^{R,1}$, then $\gamma_R^p(G) = \gamma_R^p(G')$.*

Proof. Let S be a $\gamma^p(G')$-set. By Observation 1, $v \in S$. Thus $u \in S$ for otherwise u would have two neighbors in S. Hence S is a PDS of G and so $\gamma^p(G) \leq \gamma^p(G')$. On the other hand, by Observation 1, any $\gamma^p(G)$-set contains both u and v, and thus remains a PDS of G'. It follows that $\gamma^p(G) \geq \gamma^p(G')$, and the desired equality is obtained.

Since u is an end strong support vertex in G, $u \in W_G^{R,1}$. By Proposition 1, there is a $\gamma_R^p(G)$-function f such that $f(u) = 2$, and clearly f restricted to G' is a PRDF on G' yielding $\gamma_R^p(G) \geq \gamma_R^p(G')$.

Now, assume that $u \in W_{G'}^{R,1}$ and let g be a $\gamma_R^p(G')$-function with $g(u) = 2$. Then g can be extended to a PRDF on G by assigning a 0 to x. Thus $\gamma_R^p(G) \leq \gamma_R^p(G')$, and the desired equality follows. □

Proposition 5. *Let G' be a graph and u a vertex of G'. If G is a graph obtained from G' by adding a double star $DS_{2,2}$ attached at u by one of its leaves, then:*

1. $\gamma^p(G) \leq \gamma^p(G') + 3$ *and* $\gamma_R^p(G) \geq \gamma_R^p(G') + 3$.
2. *If* $u \in W_{G'}^{R,1}$, *then* $\gamma_R^p(G) = \gamma_R^p(G') + 3$.
3. *If* $u \in W_{G'}^{P,A} \cap W_{G'}^{APD}$, *then* $\gamma^p(G) = \gamma^p(G') + 3$.

Proof. Let x, y be the non-leaf vertices of the double star $DS_{2,2}$, and let $L_x = \{x_1, x_2\}$ and $L_y = \{y_1, y_2\}$. We assume that $x_1u \in E(G)$.

1. For a $\gamma^p(G')$-set S, let $S' = S \cup \{x, y\}$ if $u \notin S$, and $S' = S \cup \{x_1, x, y\}$ if $u \in S$. Clearly, S' is a PDS of G and thus $\gamma^p(G) \leq \gamma^p(G') + 3$.

   Consider now a $\gamma_R^p(G)$-function f such that $f(y) = 2$ (according to Proposition 1). Clearly, $f(x) + f(x_2) \geq 1$. If $f(x_1) \leq 1$, then f restricted to G' is a PRDF on G' of weight at most $\gamma_R^p(G) - 3$ and thus $\gamma_R^p(G) \geq \gamma_R^p(G') + 3$. If $f(x_1) = 2$, then $f(u) = 0$ and the function $g : V(G') \to \{0, 1, 2\}$ defined by $g(u) = 1$ and $g(w) = f(w)$ otherwise, is a PRDF onG' of weight at most $\gamma_R^p(G) - 4$ yielding $\gamma_R^p(G) \geq \gamma_R^p(G') + 4$. In any case we have $\gamma_R^p(G) \geq \gamma_R^p(G') + 3$.
2. Assume that $u \in W_{G'}^{R,1}$ and let f be a $\gamma_R^p(G')$-function such that $f(u) = 2$. Then f can be extended to a PRDF on G by assigning a 2 to y, a 1 to x_2 and a 0 to x, x_1, y_1, y_2. Hence $\gamma_R^p(G) \leq \gamma_R^p(G') + 3$, and the desired equality follows from item 1.
3. Assume that $u \in W_{G'}^{P,A} \cap W_{G'}^{APD}$, and let S be a $\gamma^p(G)$-set. By items 1 and 2 of Observation 1, $x, y \in S$. If $u \in S$, then $x_1 \in S$ and thus $S - \{x, y, x_1\}$ is a PDS of G', implying that $\gamma^p(G) \geq \gamma^p(G') + 3$. Hence, assume that $u \notin S$. If $x_1 \notin S$, then $S - \{x, y\}$ is a PDS of G' that does not contain u. But since $u \in W_{G'}^3$, we deduce that $|S - \{x, y\}| \geq \gamma^p(G') + 1$ which yields $\gamma^p(G) \geq \gamma^p(G') + 3$. Thus suppose that $x_1 \in S$. Then $S - \{x, y, x_1\}$ is an almost PDS of G', and since $u \in W_{G'}^{APD}$ we conclude that $|S - \{x, y, x_1\}| \geq \gamma^p(G'; u) = \gamma^p(G')$. Hence $\gamma^p(G) \geq \gamma^p(G') + 3$, and the desired equality is obtained by item 1.

□

Proposition 6. *Let G' be a graph and let u be an end strong support vertex of degree 3 whose non-leaf neighbor is a support vertex, say v, of degree 3, where $|L_v| = 1$. Let G be a graph obtained from G' by adding four vertices, where two are attached to a leaf of u and the other two are attached to the leaf of v. Then $\gamma^p(G) = \gamma^p(G') + 2$ and $\gamma_R^p(G) = \gamma_R^p(G') + 2$.*

Proof. Let $L_u = \{x, x'\}$ and $L_v = \{y\}$. Let x_1, x_2, y_1 and y_2 be the four added vertices, where $xx_1, xx_2, yy_1, yy_2 \in E(G)$. By items 1 and 2 of Observation 1, any $\gamma^p(G')$-set contains u and v. Clearly such a set can be extended to a PDS of G by adding x, y which yields $\gamma^p(G) \leq \gamma^p(G') + 2$. On the other hand, let D be a $\gamma^p(G)$-set. Then by items 1 and 2 of Observation 1, we have $x, u, y, v \in D$, and thus $D \setminus \{x, y\}$ is a PDS of G', implying that $\gamma^p(G) \geq \gamma^p(G') + 2$. Therefore $\gamma^p(G) = \gamma^p(G') + 2$.

Next we shall show that $\gamma_R^p(G) = \gamma_R^p(G') + 2$. First we show that $\gamma_R^p(G) \leq \gamma_R^p(G') + 2$. Since u is an end strong support vertex of G', let f be a $\gamma_R^p(G')$-function with $f(u) = 2$ (by Proposition 1) such that $f(v)$ is as small as possible. If $f(v) \leq 1$, then $f(y) = 1$, and thus the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(x) = g(y) = 2$, $g(x') = 1$, $g(u) = g(x_1) = g(x_2) = g(y_1) = g(y_2) = 0$ and $g(w) = f(w)$ otherwise, is a PRDF on G. Hence $\gamma_R^p(G) \leq \gamma_R^p(G') + 2$. If $f(v) = 2$, then by our choice of f, we have $f(z) = 0$ for any $z \in N(v) \setminus \{u\}$ and thus the function $h : V(G) \to \{0, 1, 2\}$ defined by $h(z) = 1$ for $z \in N(v) \setminus \{u, y\}$ and $h(x') = 1$, $h(x) = h(y) = 2$, $h(u) = h(v) = h(x_1) = h(x_2) = h(y_1) = h(y_2) = 0$

and $h(w) = f(w)$ otherwise, is a PRDF on G yielding $\gamma_R^p(G) \le \gamma_R^p(G') + 2$. Hence $\gamma_R^p(G) \le \gamma_R^p(G') + 2$. Now we show that $\gamma_R^p(G) \ge \gamma_R^p(G') + 2$. By Proposition 1, let g be a $\gamma_R^p(G)$-function such that $g(x) = g(y) = 2$. It can be seen that $g(x') = 1$. If $f(v) = 0$, then the function $h : V(G') \to \{0,1,2\}$ defined by $h(u) = 2, h(y) = 1, h(x) = h(x') = 0$ and $h(w) = g(w)$ otherwise, is a PRDF on G' of weight at most $\gamma_R^p(G) - 2$. If $f(v) \ge 1$, then the function $h : V(G') \to \{0,1,2\}$ defined by $h(u) = h(v) = 2$, $h(x) = h(x') = h(y) = 0$ and $h(w) = g(w)$ otherwise, is a PRDF on G' of weight at most $\gamma_R^p(G) - 2$. In any case, $\gamma_R^p(G) \ge \gamma_R^p(G') + 2$, and the equality follows. □

3. The Family $\mathcal{T}$

In this section, we define the family $\mathcal{T}$ of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \ge 1)$ of trees such that $T_1 \in \{P_2, P_3\}$ and $T = T_k$. If $k \ge 2$, then T_{i+1} is obtained recursively from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $u \in V(T_i)$ is an end strong support vertex, then $\mathcal{O}_1$ adds a vertex x attached at u by an edge ux to obtain T_{i+1}.

Operation $\mathcal{O}_2$: If $u \in (W_{T_i}^{R,1} \cup W_{T_i}^{R,\ge 1}) \cap W_{T_i}^{APD}$, then $\mathcal{O}_2$ adds a path $P_2 = x_1x_2$ attached at u by an edge ux_1 to obtain T_{i+1}.

Operation $\mathcal{O}_3$: If $u \in W_{T_i}^{R,\le 1} \cap W_{T_i}^{P,A} \cap W_{T_i}^{APD}$, then $\mathcal{O}_3$ adds a star $K_{1,3}$ centered at x by attaching one of its leaves, say x_1, to u to obtain T_{i+1}.

Operation $\mathcal{O}_4$: If $u \in W_{T_i}^{R,1}$ is an end support vertex which is adjacent to a strong support vertex, then $\mathcal{O}_4$ adds a vertex x attached at u by an edge ux to obtain T_{i+1}.

Operation $\mathcal{O}_5$: If $u \in W_{T_i}^{R,1} \cap W_{T_i}^{P,A} \cap W_{T_i}^{APD}$, then $\mathcal{O}_5$ adds a double star $DS_{2,2}$ by attaching one of its leaves, say x_1, to u to obtain T_{i+1}.

Operation $\mathcal{O}_6$: If $u \in V(T_i)$ is an end strong support vertex of degree 3 with $x \in L_u$ such that u is adjacent to a support vertex v of degree 3 with $L_v = \{y\}$, then $\mathcal{O}_6$ adds four vertices x_1, x_2, y_1, y_2 attached at x and y by edges xx_1, xx_2, yy_1, yy_2 to obtain T_{i+1}.

Lemma 1. *If T_i is a tree with $\gamma_R^p(T_i) = \gamma^p(T_i) + 1$ and T_{i+1} is a tree obtained from T_i by one of the Operations $\mathcal{O}_1, \ldots, \mathcal{O}_6$, then $\gamma_R^p(T_{i+1}) = \gamma^p(T_{i+1}) + 1$.*

Proof. If T_{i+1} is obtained from T_i by Operation $\mathcal{O}_1$, then by Corollary 1 and the assumption $\gamma_R^p(T_i) = \gamma^p(T_i) + 1$, we have $\gamma_R^p(T_{i+1}) = \gamma_R^p(T_i) = \gamma^p(T_i) + 1 = \gamma^p(T_{i+1}) + 1$. If T_{i+1} is obtained from T_i by Operation $\mathcal{O}_2$, then as above the result follows from Proposition 2 (items 2, 3 and 4). If T_{i+1} is obtained from T_i by Operation $\mathcal{O}_3$, then the result follows from Proposition 3 (items 2 and 3). If T_{i+1} is obtained from T_i by Operation $\mathcal{O}_4$, then the result follows from Proposition 4. If T_{i+1} is obtained from T_i by Operation $\mathcal{O}_5$, then the result follows from Proposition 5. Finally, if T_{i+1} is obtained from T_i by Operation $\mathcal{O}_6$, then the result follows from Proposition 6. □

In the rest of the paper, we shall prove our main result:

Theorem 1. *For any tree T of order $n \ge 2$,*

$$\gamma_R^p(T) \ge \gamma^p(T) + 1,$$

with equality if and only if $T \in \mathcal{T}$.

4. Proof of Theorem 1

Lemma 2. *If $T \in \mathcal{T}$, then $\gamma_R^p(T) = \gamma^p(T) + 1$.*

Proof. Let T be a tree of $\mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \ge 1)$ such that $T_1 \in \{P_2, P_3\}$, and if $k \ge 2$, then T_{i+1} can be obtained from T_i by one of the aforementioned operations.

We proceed by induction on the number of operations used to construct T. If $k = 1$, then $T \in \{P_2, P_3\}$ and clearly $\gamma_R^p(T) = \gamma^p(T) + 1$. This establishes our basis case. Let $k \geq 2$ and assume that the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_R^p(T') = \gamma^p(T') + 1$. Since $T = T_k$ is obtained from T' by one of the Operations $\mathcal{O}_i$ ($i \in \{1, 2, \ldots, 6\}$) , we conclude from Lemma 1 that $\gamma_R^p(T) = \gamma^p(T) + 1$. □

Theorem 2. *For any tree T of order $n \geq 2$,*

$$\gamma_R^p(T) \geq \gamma^p(T) + 1,$$

with equality only if $T \in \mathcal{T}$.

Proof. We use an induction on n. If $n \in \{2, 3\}$, then $T \in \{P_2, P_3\}$, where $\gamma_R^p(T) = 2 = \gamma^p(T) + 1$ and $T \in \mathcal{T}$. If $n = 4$ and $\mathrm{diam}(T) = 2$, then $T = K_{1,3}$, where $\gamma_R^p(T) = 2 = \gamma^p(T) + 1$ and $T \in \mathcal{T}$ because it can be obtained from P_3 by applying Operation $\mathcal{O}_1$. If $n = 4$ and $\mathrm{diam}(T) = 3$, then $T = P_4$, where $\gamma_R^p(T) = 3 = \gamma^p(T) + 1$ and clearly $T \in \mathcal{T}$ since it can be obtained from P_2 by Operation $\mathcal{O}_2$. Let $n \geq 5$ and assume that every tree T' of order n' with $2 \leq n' < n$ satisfies $\gamma_R^p(T') \geq \gamma^p(T') + 1$ with equality only if $T' \in \mathcal{T}$.

Let T be a tree of order n. If $\mathrm{diam}(T) = 2$, then T is a star, where $\gamma_R^p(T) = 2 = \gamma^p(T) + 1$ and $T \in \mathcal{T}$ because T it can be obtained from P_3 by frequently use of Operation $\mathcal{O}_1$. Hence assume that $\mathrm{diam}(T) = 3$, and thus T is a double star $DS_{p,q}$, $(q \geq p \geq 1)$. If $T = DS_{1,q}$ $(q \geq 2)$, then $\gamma_R^p(T) = 3 = \gamma^p(T) + 1$ and $T \in \mathcal{T}$ since it is obtained from P_3 by applying Operation $\mathcal{O}_2$. If $T = DS_{p,q}$, $(q \geq p \geq 2)$, then $\gamma^p(T) = 2$, $\gamma_R^p(T) = 4$ and so $\gamma_R^p(T) > \gamma^p(T) + 1$. Henceforth, we assume that $\mathrm{diam}(T) \geq 4$. Let $v_1 v_2 \ldots v_k$ $(k \geq 5)$ be a diametrical path in T such that $\deg_T(v_2)$ is as large as possible. Root T at v_k and consider the following cases.

Case 1. $\deg_T(v_2) \geq 4$.
Let $T' = T - v_1$. By Corollary 1 and the induction hypothesis on T', we obtain

$$\gamma_R^p(T) = \gamma_R^p(T') \geq \gamma^p(T') + 1 = \gamma^p(T) + 1.$$

Further if $\gamma_R^p(T) = \gamma^p(T) + 1$, then we have equality throughout this inequality chain. In particular, $\gamma_R^p(T') = \gamma^p(T') + 1$. By induction on T', we have $T' \in \mathcal{T}$. It follows that $T \in \mathcal{T}$ since it can be obtained from T' by applying operation $\mathcal{O}_1$.

Case 2. $\deg_T(v_2) = \deg_T(v_3) = 2$.
Let $T' = T - T_{v_3}$. For a $\gamma^p(T')$-set S, let $S' = S \cup \{v_1\}$ if $v_4 \in S$ and $S' = S \cup \{v_2\}$ for otherwise. Clearly S' is a PDS of T and thus $\gamma^p(T) \leq \gamma^p(T') + 1$. Consider now a $\gamma_R^p(T)$-function f. If $f(v_3) \in \{0, 1\}$, then $f(v_1) + f(v_2) = 2$ and the function f, restricted to T' is a PRDF on T' of weight at most $\gamma_R^p(T) - 2$. If $f(v_3) = 2$, then $f(v_4) = 0$ and the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_4) = 1$ and $g(z) = f(z)$ otherwise, is a PRDF on T'. In any case, $\gamma_R^p(T) \geq \gamma_R^p(T') + 2$. By the induction hypothesis on T', we obtain

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 2 \geq \gamma^p(T') + 1 + 2 \geq \gamma^p(T) - 1 + 3 > \gamma^p(T) + 1.$$

Case 3. $\deg_T(v_2) = 2$ and $\deg_T(v_3) \geq 3$.
Let $T' = T - T_{v_2}$. By Proposition 2, we have $\gamma^p(T) \leq \gamma^p(T') + 1$ and $\gamma_R^p(T) \geq \gamma_R^p(T') + 1$. It follows from the induction hypothesis that

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 1 \geq \gamma^p(T') + 1 + 1 \geq \gamma^p(T) + 1.$$

Further if $\gamma_R^p(T) = \gamma^p(T) + 1$, then we have equality throughout this inequality chain. In particular, $\gamma^p(T) = \gamma^p(T') + 1$, $\gamma_R^p(T) = \gamma_R^p(T') + 1$ and $\gamma_R^p(T') = \gamma^p(T') + 1$. By induction on T', we deduce that $T' \in \mathcal{T}$. Next, we shall show that $v_3 \in (W^{R,1}_{T'} \cup W^{R,\geq 1}_{T'}) \cap W^{APD}_{T'}$. Let f be a $\gamma^p_R(T)$-function. If $f(v_3) = 2$, then $f(v_1) = 1$ and $f(v_2) = 0$ and the function $f|_{V(T')}$ is a $\gamma^p_R(T')$-function with $f(v_3) = 2$ and hence $v_3 \in W^{R,1}_{T'}$. Hence, assume that $f(v_3) \leq 1$. Then $f(v_1) + f(v_2) = 2$. If $f(v_2) \leq 1$ or $f(v_2) = 2$ and $f(v_3) = 1$, then the function f restricted to T' is a PRDF on T' of weight $\gamma^p_R(T) - 2$, contradicting the fact $\gamma^p_R(T) = \gamma^p_R(T') + 1$. Hence we assume $f(v_2) = 2$ and $f(v_3) = 0$. Then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_3) = 1$ and $g(x) = f(x)$ otherwise, is a $\gamma^p_R(T')$-function and so $v_3 \in W^{R,\geq 1}_{T'}$. Hence $v_3 \in W^{R,1}_{T'} \cup W^{R,\geq 1}_{T'}$. It remains to show that $v_3 \in W^{APD}_{T'}$. Suppose that $v_3 \notin W^5_{T'}$ and let S be an almost PDS of T' of size less that $\gamma^p(T')$. Clearly, $v_3 \notin S$ and v_3 has no neighbor in S. Therefore, $S \cup \{v_2\}$ is a PDS of T of size at most $\gamma^p(T') = \gamma^p(T) - 1$, a contradiction. Hence $v_3 \in W^{APD}_{T'}$. It follows that $T \in \mathcal{T}$ since it can be obtained from T' by Operation $\mathcal{O}_2$.

Case 4. $\deg_T(v_2) = 3$.

Let $L_{v_2} = \{v_1, w\}$. According to Cases 1, 2 and 3, we may assume that any end support vertex on a diametrical path has degree 3. Consider the following subcases.

Subcase 4.1. $\deg_T(v_3) = 2$.

Let $T' = T - T_{v_3}$. By Proposition 3-(1) and the induction hypothesis we have:

$$\gamma^p_R(T) \geq \gamma^p_R(T') + 2 \geq \gamma^p(T') + 1 + 2 \geq \gamma^p(T) + 1.$$

Further if $\gamma^p_R(T) = \gamma^p(T) + 1$, then we have equality throughout this inequality chain. In particular, $\gamma^p_R(T) = \gamma^p_R(T') + 2$, $\gamma^p(T) = \gamma^p(T') + 2$ and $\gamma^p_R(T') = \gamma^p(T') + 1$. It follows from the induction hypothesis that $T' \in \mathcal{T}$. In the next, we shall show that $v_4 \in W^{R,\leq 1}_{T'} \cap W^{P,A}_{T'} \cap W^{APD}_{T'}$.

Suppose that $v_4 \notin W^{P,A}_{T'}$ and let S be a $\gamma^p(T')$-set that does not contain v_4. Then $S \cup \{v_2\}$ is a PDS of T, contradicting the fact $\gamma^p(T) = \gamma^p(T') + 2$. Hence $v_4 \in W^{P,A}_{T'}$. Suppose now that $v_4 \notin W^{APD}_{T'}$ and let D be an almost PDS of T' with respect to v_4 such that $|D| < \gamma^p(T')$. Then $v_4 \notin D$ and v_4 has no neighbor in D, and thus $D \cup \{v_2, v_3\}$ is a PDS of T of cardinality less $\gamma^p(T') + 2$, a contradiction. Hence $v_4 \in W^{APD}_{T'}$. It remains to show that $v_4 \in W^{R,\leq 1}_{T'}$. By Proposition 1, let f be a $\gamma^p_R(T)$-function such that $f(v_2) = 2$. If $f(v_4) = 2$, then we must have $f(v_3) \geq 1$. But f restricted to T' is a PRDF on T' of weight at most $\gamma^p_R(T) - 3$, contradicting $\gamma^p_R(T) = \gamma^p_R(T') + 2$. Hence $f(v_4) \leq 1$. If $f(v_4) = 0$ and $f(v_3) = 2$, then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_4) = 1$ and $g(x) = f(x)$ otherwise, is a PRDF of T' of weight at most $\gamma^p_R(T) - 3$, a contradiction as above. Thus $f(v_4) = 1$ or $f(v_4) = 0$ and $f(v_3) \leq 1$. Then f restricted to T' is a $\gamma^p_R(T')$-function showing that $v_4 \in W^{R,\leq 1}_{T'}$. Hence $v_4 \in W^{R,\leq 1}_{T'} \cap W^{P,A}_{T'} \cap W^{APD}_{T'}$. Therefore, $T \in \mathcal{T}$ because it can be obtained from T' by Operation $\mathcal{O}_3$.

Subcase 4.2. $\deg_T(v_3) \geq 3$.

We distinguish between some situations.

(a) v_3 is a strong support vertex.

Let $T' = T - v_1$. By Proposition 4 and the induction hypothesis we have:

$$\gamma^p_R(T) \geq \gamma^p_R(T') \geq \gamma^p(T') + 1 = \gamma^p(T) + 1.$$

Further if $\gamma^p_R(T) = \gamma^p(T) + 1$, then we have equality throughout this inequality chain. In particular, $\gamma^p_R(T) = \gamma^p_R(T')$, $\gamma^p(T) = \gamma^p(T')$ and $\gamma^p_R(T') = \gamma^p(T') + 1$. By the induction hypothesis, $T' \in \mathcal{T}$. To show $v_2 \in W^{R,1}_{T'}$, let f be a $\gamma^p_R(T)$-function such that $f(v_2) = 2$ (by Proposition 1). Since $\gamma^p_R(T) = \gamma^p_R(T')$, f is also a $\gamma^p_R(T')$-function with $f(v_2) = 2$, implying that $v_2 \in W^{R,1}_{T'}$. Therefore $T \in \mathcal{T}$ because it can be obtained from T' by Operation $\mathcal{O}_4$.

(b) v_3 has two children x, y with depth one, different from v_2.

Then u and w are both strong support vertices of degree 3. Let $T' = T - T_{v_2}$. By Observation 1, any $\gamma^p(T')$-set S contains x and y and thus $v_3 \in S$. Hence $S \cup \{v_2\}$ is a PDS of T yielding $\gamma^p(T) \leq \gamma^p(T') + 1$. Now, let f be a $\gamma^p_R(T)$ function such that $f(v_2) = 2$ and $f(x) = 2$

(by Proposition 1). Then $f(v_3) \geq 1$. It follows that the function f restricted to T' is a PRDF on T' of weight $\gamma_R^p(T) - 2$, and hence $\gamma_R^p(T) \geq \gamma_R^p(T') + 2$. By the induction hypothesis we have

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 2 \geq \gamma^p(T') + 3 \geq \gamma^p(T) + 2 > \gamma^p(T) + 1.$$

(c) v_3 is a support vertex and has a child u with depth one different from v_2.
Let w_1 be the unique leaf adjacent to v_3. Note that u is a strong support vertices of degree 3. Let $T' = T - T_{v_2}$. If S is a $\gamma^p(T')$-set, then by Observation 1-(2), $v_3 \in S$ and thus $S \cup \{v_2\}$ is a PDS of T yielding $\gamma^p(T) \leq \gamma^p(T') + 1$. By Proposition 1, let f be a $\gamma_R^p(T)$-function such that $f(v_2) = 2$ and $f(u) = 2$. By the definition of perfect Roman dominating functions, we have $f(v_3) \geq 1$. Then, the function f restricted to T' is a PRDF on T' of weight $\gamma_R^p(T) - 2$ and thus $\gamma_R^p(T) \geq \gamma_R^p(T') + 2$. It follows from the induction hypothesis that

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 2 \geq \gamma^p(T') + 3 \geq \gamma^p(T) + 2 > \gamma^p(T) + 1.$$

According to (a), (b) and (c), we can assume for the next that $\deg_T(v_3) = 3$.

(d) $\deg_T(v_3) = 3$ and v_3 has a child x with depth one different from v_2.
Note that x is a strong support vertices of degree 3. Let $L_x = \{x_1, x_2\}$ and let T' be the tree obtained from T by removing the set of vertices $\{v_1, v_2, w, x, x_1, x_2\}$. For a $\gamma^p(T')$-set S, let $S' = S \cup \{v_2, x\}$ if $v_3 \in S$ and $S' = S \cup \{v_2, v_3, x\}$ when $v_3 \notin S$. Clearly, S' is a PDS of T and so $\gamma^p(T) \leq \gamma^p(T') + 3$. Now let f be a $\gamma_R^p(T)$-function such that $f(v_2) = f(x) = 2$. Then $f(v_3) \geq 1$ and the function f restricted to T' is a PRDF on T' of weight at most $\gamma_R^p(T) - 4$. By the induction hypothesis we have:

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 4 \geq \gamma^p(T') + 1 + 4 \geq \gamma^p(T) - 3 + 5 > \gamma^p(T) + 1.$$

(e) $\deg_T(v_3) = 3$ and v_3 is adjacent to exactly one leaf w'.
If v_4 has a child s with depth one and degree two, then let T' be the tree obtained from T by removing s and its unique leaf. This case can be treated in the same way as in Case 3. Moreover, if v_4 has a child s with depth one and degree at least four, then let T' be the tree obtained from T by removing a leaf neighbor of s. This case can be treated in the same way as in Case 1. Hence, we may assume that each child of v_4 is a leaf or a vertex with depth one and degree 3 or a vertex with depth two whose maximal subtree is isomorphic to T_{v_3}. First assume that $\deg_T(v_4) \geq 4$, and let $T = T - T_{v_3}$. Clearly, any $\gamma^p(T')$-set contains v_4 and such a set can be extended to a PDS of T by adding v_2, v_3. Hence $\gamma^p(T) \leq \gamma^p(T') + 2$. Now let f be a $\gamma_R^p(T)$-function such that $f(v_2) = 2$. Clearly, $f(v_3) + f(w') \geq 1$. If $f(v_3) \leq 1$ or $f(v_3) = 2$ and $f(v_4) \geq 1$, then the function f restricted to T' is a PRDF on T' and thus $\gamma_R^p(T) \geq \gamma_R^p(T') + 3$. Hence assume that $f(v_3) = 2$ and $f(v_4) = 0$. Then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_4) = 1$ and $g(u) = f(u)$ otherwise, is a PRDF of T' of weight $\gamma_R^p(T) - 3$ and thus $\gamma_R^p(T) \geq \gamma_R^p(T') + 3$. By the induction hypothesis we have:

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 3 \geq \gamma^p(T') + 1 + 3 \geq \gamma^p(T) - 2 + 4 > \gamma^p(T) + 1.$$

From now on, we can assume that $\deg_T(v_4) \leq 3$. We examine different cases.

(e.1.) v_4 has a child x of degree 3 and depth 1.

Let $L_x = \{z_1, z_2\}$ and let T' be the tree obtained from T by removing the set $\{v_1, w, z_1, z_2\}$. By Proposition 6, we have $\gamma^p(T) = \gamma^p(T') + 2$ and $\gamma_R^p(T) = \gamma_R^p(T') + 2$. We deduce from the induction hypothesis that

$$\gamma_R^p(T) = \gamma_R^p(T') + 2 \geq \gamma^p(T') + 1 + 2 = \gamma^p(T) - 2 + 3 = \gamma^p(T) + 1.$$

Further if $\gamma_R^p(T) = \gamma^p(T) + 1$, then we have equality throughout this inequality chain. In particular, $\gamma_R^p(T') = \gamma^p(T') + 1$. By induction on T', we have $T' \in \mathcal{T}$. Therefore $T \in \mathcal{T}$ since it can be obtained from T' by Operation $\mathcal{O}_6$.

(e.2.) v_4 has a child v_3' with depth two.
Note that $T_{v_3'}$ and T_{v_3} are isomorphic. Let $T' = T - (T_{v_3} \cup T_{v_3'})$, and observe that v_4 is a leaf in T'. Since any $\gamma^p(T')$-set can be extended to a PDS of T by adding v_4 and the support vertices of $T_{v_3} \cup T_{v_3'}$ we obtain $\gamma^p(T) \leq \gamma^p(T') + 5$. Moreover, as above we can see that $\gamma_R^p(T) \geq \gamma_R^p(T') + 6$. Now, by induction hypothesis we obtain:

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 6 \geq \gamma^p(T') + 1 + 6 \geq \gamma^p(T) - 5 + 7 > \gamma^p(T) + 1.$$

(e.3.) $\deg_T(v_4) = 2$.
Let $T' = T - T_{v_4}$. If $V(T') = \{v_5\}$, then it can be seen that T is tree with $\gamma_R^p(T) = 5$ and $\gamma^p(T) = 3$, implying that $\gamma_R^p(T) > \gamma^p(T) + 1$. Hence we assume that T' is nontrivial. By Proposition 5 and by the inductive hypothesis we have:

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 3 \geq \gamma^p(T') + 1 + 3 \geq \gamma^p(T) - 3 + 4 = \gamma^p(T) + 1.$$

Further if $\gamma_R^p(T) = \gamma^p(T) + 1$, then we have equality throughout this inequality chain. In particular, $\gamma_R^p(T) = \gamma_R^p(T') + 3$, $\gamma^p(T) = \gamma^p(T') + 3$ and $\gamma_R^p(T') = \gamma^p(T') + 1$. By induction on T', we have $T' \in \mathcal{T}$. Next, we shall show that $v_5 \in W_{T'}^{R,1} \cap W_{T'}^{P,A} \cap W_{T'}^{APD}$. Suppose that $v_5 \notin W_{T'}^{P,A}$ and let S be a $\gamma^p(T')$-set that does not contain v_5. Then $S \cup \{v_2, v_3\}$ is a PDS of T contradicting $\gamma^p(T') = \gamma^p(T') + 3$. Hence $v_5 \in W_{T'}^{P,A}$. Suppose that $v_5 \notin W_{T'}^{APD}$ and let S be an almost PDS of T' such that $|S| \leq \gamma^p(T') - 1$. Clearly, $v_5 \notin S$ and v_5 has no neighbor in S. It follows that $S \cup \{v_4, v_3, v_2\}$ is a PDS of T of size $|S| + 3 \leq \gamma^p(T) - 1$, a contradiction. Thus $v_5 \in W_{T'}^{APD}$. Next we show that $v_5 \in W_{T'}^{R,1}$. Let f be a $\gamma_R^p(T)$-function such that $f(v_2) = 2$. To Roman dominate w', we must have either $f(w') = 1$ or $f(v_3) = 2$. We claim that $f(v_4) \leq 1$. Suppose, to the contrary, that $f(v_4) = 2$. By definition of perfect Roman dominating functions, we may assume that $f(v_3) = 2$. But then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = 1$ and $g(x) = f(x)$ otherwise, is a PRDF of T' of weight $\gamma_R^p(T') - 5$ contradicting $\gamma_R^p(T) = \gamma_R^p(T') + 3$. Hence $f(v_4) \leq 1$. It follows that the function f restricted to T' is a PRDF of T' of weight at most $\gamma_R^p(T) - 3$ for which we conclude from $\gamma_R^p(T) = \gamma_R^p(T') + 3$ that $f(v_3) = f(v_4) = 0$ and $f(w') = 1$. Hence to Roman dominate v_4, we must have $f(v_5) = 2$ and thus function f restricted to T' is a $\gamma_R^p(T')$-function that assigns a a 2 to v_5. Hence $v_5 \in W_{T'}^{R,1}$, and thus $v_5 \in W_{T'}^{R,1} \cap W_{T'}^{P,A} \cap W_{T'}^{APD}$. Therefore, $T \in \mathcal{T}$ since it can be obtained from T' by Operation $\mathcal{O}_5$.

(e.4.) $\deg_T(v_4) = 3$ and v_4 has a child z with depth 0.
Seeing the above Cases and Subcases as we did in the beginning of Case (e), we may assume that any child of v_5 is a leaf, or an end strong support vertex of degree 3, or a vertex with depth 2 whose maximal subtree is isomorphic to T_{v_3}, or a vertex with depth 3 whose maximal subtree is isomorphic to T_{v_4}. Assume first that $\deg_T(v_5) \geq 4$, and let $T' = T - T_{v_4}$. Clearly, v_5 belongs to any $\gamma^p(T')$-set and such a set $\gamma^p(T')$-set can be extended to a PDS of T by adding v_2, v_3, v_4, implying that $\gamma^p(T) \leq \gamma^p(T') + 3$. Next we show that $\gamma_R^p(T) \geq \gamma_R^p(T') + 4$. Let f be a $\gamma_R^p(T)$-function such that $f(v_2) = 2$. Clearly $f(v_3) + f(w') \geq 1$ and $f(v_4) + f(z) \geq 1$. If $f(v_4) \leq 1$ or $f(v_5) \geq 1$, then the function f restricted to T' is a PRDF on T' yielding $\gamma_R^p(T) \geq \gamma_R^p(T') + 4$. Hence assume that $f(v_4) = 2$ and $f(v_5) = 0$. Then the function

$g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = 1$ and $g(u) = f(u)$ otherwise, is a PRDF of T' yielding $\gamma_R^p(T) \geq \gamma_R^p(T') + 4$. By induction on T', it follows that

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 4 \geq \gamma^p(T') + 5 \geq \gamma^p(T) + 2 > \gamma^p(T) + 1.$$

For the next, we assume that $\deg_T(v_5) \leq 3$. If $\deg_T(v_5) = 1$, then it can be seen that T is a tree with $\gamma_R^p(T) = 6$, $\gamma^p(T)$ and so $\gamma_R^p(T) > \gamma^p(T) + 1$. Hence we assume that $\deg_T(v_5) \in \{2, 3\}$. Consider the following situations.

(e.4.1.) $\deg_T(v_5) = 2$.
Let $T' = T - T_{v_5}$. If $V(T') = \{v_6\}$, then T is a tree with $\gamma_R^p(T) = 6$ and $\gamma^p(T) = 4$, yielding $\gamma_R^p(T) > \gamma^p(T) + 1$. Hence, assume that T' is nontrivial. For a $\gamma^p(T')$-set S, let $S' = S \cup \{v_2, v_3, v_4, v_5\}$ if $v_6 \in S$, and $S' = S \cup \{v_2, v_3, v_4\}$ if $v_6 \notin S$. Then S' is a PDS of T, implying that $\gamma^p(T) \leq \gamma^p(T') + 4$. Moreover, it is easy to see that $\gamma_R^p(T) \geq \gamma_R^p(T') + 5$. By induction on T', we obtain $\gamma_R^p(T) > \gamma^p(T) + 1$.

(e.4.3.) $\deg_T(v_5) = 3$ and v_5 has a child v_4' with depth 3.
Then T_{v_4} and $T_{v_4'}$ are isomorphic. If u is a vertex in T_{v_4}, then let u' be the vertex of $T_{v_4'}$ corresponding to u in T_{v_4}. Let $T = T - (T_{v_4} \cup T_{v_4'})$. Clearly, any $\gamma^p(T')$-set can be extended to a PDS of T by adding $v_5, v_2, v_3, v_4, v_2', v_3', v_4'$ and thus $\gamma^p(T) \leq \gamma^p(T') + 7$. Moreover, it is not hard to see that $\gamma_R^p(T) \geq \gamma_R^p(T') + 8$. By induction on T', we obtain $\gamma_R^p(T) > \gamma^p(T) + 1$.

(e.4.4.) $\deg_T(v_5) = 3$ and v_5 has a children y with depth 1 and degree 3.
Let $T' = T - (T_{v_4} \cup T_y)$. Clearly, any $\gamma^p(T')$-set can be extended to a PDS of T by adding v_5, v_2, v_3, v_4, y and thus $\gamma^p(T) \leq \gamma^p(T') + 5$. Next, we show that $\gamma_R^p(T) \geq \gamma_R^p(T') + 6$. Let f be a $\gamma_R^p(T)$-function such that $f(v_2) = 2$ and $f(y) = 2$ (by Proposition 1). Clearly $f(v_3) + f(w') \geq 1$ and $f(v_4) + f(z) \geq 1$. If $f(v_5) \geq 1$, then the function f restricted to T' is a PRDF on T' yielding $\gamma_R^p(T) \geq \gamma_R^p(T') + 6$. Thus, let $f(v_5) = 0$. Then to Roman dominate z, v_4, w', we must have $f(z) + f(v_4) + f(v_3) + f(w') \geq 4$. Then the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = 1$ and $g(u) = f(u)$ otherwise, is a PRDF on T' yielding $\gamma_R^p(T) \geq \gamma_R^p(T') + 6$. It follows from the induction hypothesis that

$$\gamma_R^p(T) \geq \gamma_R^p(T') + 6 \geq \gamma^p(T') + 7 \geq \gamma^p(T) + 2 > \gamma^p(T) + 1.$$

(e.4.5.) $\deg_T(v_5) = 3$ and v_5 has a child v_3' with depth 2 such that $T_{v_3'} \cong T_{v_3}$.
If u is a vertex in T_{v_3}, then let u' be the vertex of $T_{v_3'}$ corresponding to u in T_{v_3}. Let $T = T - (T_{v_4} \cup T_{v_3'})$. Clearly, any $\gamma^p(T')$-set can be extended to a PDS of T by adding $v_5, v_2, v_3, v_4, v_2', v_3'$ and so $\gamma^p(T) \leq \gamma^p(T') + 6$. Moreover, it is not hard to see that $\gamma_R^p(T) \geq \gamma_R^p(T') + 7$. By the induction hypothesis we obtain $\gamma_R^p(T) > \gamma^p(T) + 1$.

(e.4.6.) $\deg_T(v_5) = 3$ and v_5 has a children z' with depth 0.
If $V(T') = \{v_6\}$, then T is a tree with $\gamma_R^p(T) = 6$ and $\gamma^p(T) = 4$, yielding $\gamma_R^p(T) > \gamma^p(T) + 1$. Hence we assume that $\deg_T(v_6) \geq 2$. Suppose first that $\deg_T(v_6) = 2$ and let $T' = T - T_{v_6}$. If $V(T') = \{v_7\}$, then T is a tree with $\gamma_R^p(T) \geq 7$ and $\gamma^p(T) = 5$, yielding $\gamma_R^p(T) > \gamma^p(T) + 1$. Hence assume that T' is nontrivial. Clearly, any $\gamma^p(T')$-set can be extended to a PDS of T by adding v_2, v_3, v_4, v_5, v_6 and thus $\gamma^p(T) \leq \gamma^p(T') + 5$. On the other hand, it is not hard to see that $\gamma_R^p(T) \geq \gamma_R^p(T') + 7$. By the induction hypothesis we obtain $\gamma_R^p(T) > \gamma^p(T) + 1$.

Assume now that $\deg_T(v_6) \geq 3$. By above Cases and Subcases, we may assume that any child of v_6 is a leaf, or a vertex with depth j whose maximal subtree is isomorphic to $T_{v_{j+1}}$ for $j = 2, 3, 4$. Let T' be a tree obtained from T by removing v_3, w', v_4, z, v_5, z' and joining v_2 to v_6. Clearly, any $\gamma^p(T')$-set contains v_2, v_6 and such a set can be extended to a PDS of T by v_3, v_4, v_5 yielding $\gamma^p(T) \le \gamma^p(T') + 3$. Now, let f be a $\gamma_R^p(T)$-function, and let $r = f(v_3) + f(w') + f(v_4) + f(z) + f(v_5) + f(z')$. To Roman dominate the vertices v_3, w', v_4, z, v_5, z', we must have $r \ge 5$ when $f(v_5) \le 1$ or $r = 4$ when $f(v_5) = 2$. If $r = 4$ or $r \ge 5$ and $f(v_6) \ge 1$, then the function f restricted to T' is a PRDF on T' implying that $\gamma_R^p(T) \ge \gamma_R^p(T') + 4$. Hence assume that $r \ge 5$ and $f(v_6) = 0$. Then the function $h : V(T') \to \{0, 1, 2\}$ defined by $h(v_6) = 1$ and $h(x) = f(x)$ otherwise, is a PRDF on T' yielding $\gamma_R^p(T) \ge \gamma_R^p(T') + 4$. By the induction hypothesis we obtain

$$\gamma_R^p(T) \ge \gamma_R^p(T') + 4 \ge \gamma^p(T') + 1 + 4 \ge \gamma^p(T) - 3 + 5 > \gamma^p(T) + 1,$$

and the proof is complete.

□

According to Lemma 2 and Theorem 2, we have proven Theorem 1.

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