mathematics

Article

# Pointwise Slant and Pointwise Semi-Slant Submanifolds in Almost Contact Metric Manifolds

**Kwang Soon Park**

Room 4-107, Changgong Hall, Division of General Mathematics, University of Seoul, Seoul 02504, Korea; parkksn1@uos.ac.kr

Received: 27 May 2020; Accepted: 11 June 2020; Published: 16 June 2020

**Abstract:** In almost contact metric manifolds, we consider two kinds of submanifolds: pointwise slant, pointwise semi-slant. On these submanifolds of cosymplectic, Sasakian and Kenmotsu manifolds, we obtain characterizations and study their topological properties and distributions. We also give their examples. In particular, we obtain some inequalities consisting of a second fundamental form, a warping function and a semi-slant function.

**Keywords:** slant; semi-slant; warped product; almost contact metric manifold

## 1. Introduction

Given a Riemannian manifold $(N, g)$ with some additional structures, there are several kinds of submanifolds:

Almost complex submanifolds ([1–4]), totally real submanifolds ([5–8]), CR submanifolds ([9–12]), QR submanifolds ([13–16]), slant submanifolds (([17–22]), pointwise slant submanifolds ([23–25]), semi-slant submanifolds ([26–29]), pointwise semi-slant submanifolds [30], pointwise almost h-slant submanifolds and pointwise almost h-semi-slant submanifolds [31], etc.

As a generalization of almost complex submanifolds and totally real submanifolds of an almost Hermitian manifold, B. Y. Chen [17] introduced a slant submanifold of an almost Hermitian manifold in 1990. After that, many geometers studied slant submanifolds ([18–22,32], etc.).

As a generalization of CR-submanifolds and slant submanifolds of an almost Hermitian manifold, N. Papaghiuc [28] defined the notion of semi-slant submanifolds of an almost Hermitian manifold in 1994. After that, many geometers investigated semi-slant submanifolds ([26,27,29,33,34], etc.).

In 1998, F. Etayo [24] defined pointwise slant submanifolds. In 2012, B. Y. Chen and O. J. Garay [23] investigate pointwise slant submanifolds. In 2013, B. Sahin [30] gives the notion of pointwise semi-slant submanifolds. In 2014, on an almost quaternionic Hermitian manifold the author in [31] obtains some properties of pointwise almost h-slant submanifolds and pointwise almost h-semi-slant submanifolds.

As a generalization of slant submanifolds and semi-slant submanifolds of an almost contact metric manifold, we will define the notions of pointwise slant submanifolds and pointwise semi-slant submanifolds of an almost contact metric manifold. Throughout the paper, we will see the similarity and the difference among cosymplectic manifolds, Sasakian manifolds and Kenmotsu manifolds.

We organize the paper as follows. In Section 2 we deal with some necessary notions. In Section 3 we recall some basic notions in almost contact metric manifolds. In Section 4 we define pointwise slant submanifolds of an almost contact metric manifold and deal with their properties. In Section 5 we investigate their topological properties. In Section 6 we give their examples. In Section 7 we define pointwise semi-slant submanifolds of an almost contact metric manifold. In Section 8 we consider distributions and totally umbilic submanifolds in cosymplectic, Sasakian and Kenmotsu manifolds. In Section 9 we have the non-existence of warped product submanifolds and investigate their properties. In Section 10 we obtain inequalities consisting of a second fundamental form, a warping function and a semi-slant function in cosymplectic, Sasakian and Kenmotsu manifolds. Finally, we give their examples.

## 2. Preliminaries

Let $(N, g)$ be a Riemannian manifold, where $N$ is a $n$-dimensional $C^\infty$-manifold and $g$ is a Riemannian metric on $N$. Let $M$ be a $m$-dimensional submanifold of $(N, g)$.

Denote by $TM^\perp$ the normal bundle of $M$ in $N$.

Denote by $\nabla$ and $\overline{\nabla}$ the Levi–Civita connections of $M$ and $N$, respectively.

Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$
$$\overline{\nabla}_X Z = -A_Z X + D_X Z, \tag{2}$$

respectively, for tangent vector fields $X, Y \in \Gamma(TM)$ and a normal vector field $Z \in \Gamma(TM^\perp)$, where $h$ denotes the second fundamental form, $D$ the normal connection and $A$ the shape operator of $M$ in $N$.

The second fundamental form and the shape operator are related by

$$\langle A_Z X, Y\rangle = \langle h(X, Y), Z\rangle, \tag{3}$$

where $\langle\ ,\ \rangle$ denotes the induced metric on $M$ as well as the Riemannian metric $g$ on $N$.

Choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ of $TN$ such that $e_1, \cdots, e_m$ are tangent to $M$ and $e_{m+1}, \cdots, e_n$ are normal to $M$.

Then the mean curvature vector $H$ is defined by

$$H := \frac{1}{m} trace\ h = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i) \tag{4}$$

and the squared mean curvature is given by $H^2 := \langle H, H\rangle$.

The squared norm of the second fundamental form $h$ is defined by

$$||h||^2 := \sum_{i,j=1}^{m} \langle h(e_i, e_j), h(e_i, e_j)\rangle. \tag{5}$$

Let $(B, g_B)$ and $(\overline{F}, g_{\overline{F}})$ be Riemannian manifolds, where $g_B$ and $g_{\overline{F}}$ are Riemannian metrics on manifolds $B$ and $\overline{F}$, respectively. Let $f$ be a positive $C^\infty$-function on $B$. Consider the product manifold $B \times \overline{F}$ with the natural projections $\pi_1 : B \times \overline{F} \mapsto B$ and $\pi_2 : B \times \overline{F} \mapsto \overline{F}$. The warped product manifold $M = B \times_f \overline{F}$ is the product manifold $B \times \overline{F}$ equipped with the Riemannian metric $g_M$ such that

$$||X||^2 = ||d\pi_1 X||^2 + f^2(\pi_1(x))||d\pi_2 X||^2 \tag{6}$$

for any tangent vector $X \in T_x M$, $x \in M$.

Hence,

$$g_M = g_B + f^2 g_{\overline{F}}.$$

We call the function $f$ the warping function of the warped product manifold $M$ [35].

If the warping function $f$ is constant, then the warped product manifold $M$ is called trivial.

Given vector fields $X \in \Gamma(TB)$ and $Y \in \Gamma(T\overline{F})$, we get their natural horizontal lifts $\widetilde{X}, \widetilde{Y} \in \Gamma(TM)$ such that $d\pi_1 \widetilde{X} = X$ and $d\pi_2 \widetilde{Y} = Y$.

For convenience, we will identify $\widetilde{X}$ and $\widetilde{Y}$ with $X$ and $Y$, respectively.

Choose a local orthonormal frame $\{e_1, \cdots, e_m\}$ of the tangent bundle $TM$ of $M$ such that $e_1, \cdots, e_{m_1} \in \Gamma(TB)$ and $e_{m_1+1}, \cdots, e_m \in \Gamma(T\overline{F})$.

Then we have

$$\triangle f = \sum_{i=1}^{m_1} ((\nabla_{e_i} e_i) f - e_i^2 f). \tag{7}$$

Given unit vector fields $X, Y \in \Gamma(TM)$ such that $X \in \Gamma(TB)$ and $Y \in \Gamma(TF)$, we obtain

$$\nabla_X Y = \nabla_Y X = (X \ln f)Y, \tag{8}$$

where $\nabla$ is the Levi–Civita connection of $(M, g_M)$.

Thus,

$$\begin{aligned} K(X \wedge Y) &= \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X, Y \rangle \\ &= \frac{1}{f}((\nabla_X X)f - X^2 f), \end{aligned} \tag{9}$$

where $K(X \wedge Y)$ denotes the sectional curvature of the plane $< X, Y >$ spanned by $X$ and $Y$ over $\mathbb{R}$.

Hence,

$$\frac{\triangle f}{f} = \sum_{i=1}^{m_1} K(e_i \wedge e_j) \tag{10}$$

for each $j = m_1 + 1, \cdots, m$.

Throughout this paper, we will use the above notations.

## 3. Almost Contact Metric Manifoldsn

In this section, we remind some notions in almost contact metric manifolds and we will use them later.

Let $N$ be a $(2n+1)$-dimensional $C^\infty$-manifold with a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{11}$$

where $I$ denotes the identity endomorphism of $TN$. Then we have [36]

$$\phi\xi = 0, \quad \eta \circ \phi = 0. \tag{12}$$

We call $(\phi, \xi, \eta)$ an almost contact structure and $(N, \phi, \xi, \eta)$ an almost contact manifold. If there is a Riemannian metric $g$ on $N$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{13}$$

for any vector fields $X, Y \in \Gamma(TN)$, then we call $(\phi, \xi, \eta, g)$ an almost contact metric structure and $(N, \phi, \xi, \eta, g)$ an almost contact metric manifold. The metric $g$ is called a compatible metric. By replacing $Y$ by $\xi$ at (13), we obtain

$$\eta(X) = g(X, \xi). \tag{14}$$

Define $\Phi(X, Y) := g(X, \phi Y)$ for vector fields $X, Y \in \Gamma(TN)$. Since $\phi$ is anti-symmetric with respect to $g$, the tensor $\Phi$ is a 2-form on $N$ and is called the fundamental 2-form of the almost contact metric structure $(\phi, \xi, \eta, g)$. We can also choose a local orthonormal frame $\{X_1, \cdots, X_n, \phi X_1, \cdots, \phi X_n, \xi\}$ of $TN$ and we call it a *$\phi$-frame*. An almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is said to be a contact metric manifold (or almost Sasakian manifold) [37] if it satisfies

$$\Phi = d\eta. \tag{15}$$

It is easy to check that given a contact metric manifold $(N, \phi, \xi, \eta, g)$, we get

$$(d\eta)^n \wedge \eta \neq 0. \tag{16}$$

The Nijenhuis tensor of a tensor field $\phi$ is defined by

$$N(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{17}$$

for any vector fields $X, Y \in \Gamma(TN)$. We call the almost contact metric structure $(\phi, \xi, \eta, g)$ normal if

$$N(X,Y) + 2d\eta(X,Y)\xi = 0 \tag{18}$$

for any vector fields $X, Y \in \Gamma(TN)$.

A contact metric manifold $(N, \phi, \xi, \eta, g)$ is said to be a $K$-contact manifold if the characteristic vector field $\xi$ is Killing. It is well-known that for a contact metric manifold $(N, \phi, \xi, \eta, g)$, $\xi$ is Killing if and only if the tensor $\bar{h} := \frac{1}{2} L_\xi \phi$ vanishes, where $L$ denotes the Lie derivative [36].

Given a contact metric manifold $N = (N, \phi, \xi, \eta, g)$, we know that (i) $\bar{h}$ is a symmetric operator, (ii) $\overline{\nabla}_X \xi = -\phi X - \phi \bar{h} X$ for $X \in \Gamma(TN)$, where $\overline{\nabla}$ is the Levi–Civita connection of $N$, (iii) $\bar{h}$ anti-commutes with $\phi$ and $trace(\bar{h}) = 0$ [36]. Using the above three properties, A. Lotta proved Theorem 2 [20].

An almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is called a Sasakian manifold if it is contact and normal. Given an almost contact metric manifold $(N, \phi, \xi, \eta, g)$, we know that it is Sasakian if and only if

$$(\overline{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{19}$$

for $X, Y \in \Gamma(TN)$ [36]. If an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is Sasakian, then we have

$$\overline{\nabla}_X \xi = -\phi X \tag{20}$$

for $X \in \Gamma(TN)$ [36].

Moreover, a Sasakian manifold is a $K$-contact manifold [36].

An almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is said to be a Kenmotsu manifold if it satisfies

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{21}$$

for $X, Y \in \Gamma(TN)$ [36]. From (21), by replacing $Y$ by $\xi$, we easily obtain

$$\overline{\nabla}_X \xi = X - \eta(X)\xi \tag{22}$$

for $X \in \Gamma(TN)$ [36].

An almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is called an almost cosymplectic manifold if $\eta$ and $\Phi$ are closed. An almost cosymplectic manifold $(N, \phi, \xi, \eta, g)$ is said to be a cosymplectic manifold if it is normal [37]. Given an almost contact metric manifold $(N, \phi, \xi, \eta, g)$, we also know that it is cosymplectic if and only if $\phi$ is parallel (i.e., $\overline{\nabla}\phi = 0$) [36].

Given a cosymplectic manifold $(N, \phi, \xi, \eta, g)$, we easily get

$$\overline{\nabla}\phi = 0, \ \overline{\nabla}\eta = 0, \text{ and } \overline{\nabla}\xi = 0. \tag{23}$$

Throughout this paper, we will use the above notations.

## 4. Pointwise Slant Submanifolds

In this section we define the notion of pointwise slant submanifolds of an almost contact metric manifold and study its properties.

**Definition 1.** *Let $N = (N, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold and $M$ a submanifold of $N$. The submanifold $M$ is called a* pointwise slant submanifold *if at each given point $p \in M$ the angle $\theta = \theta(X)$ between $\phi X$ and the space $M_p$ is constant for nonzero $X \in M_p$, where $M_p := \{X \in T_pM \mid g(X, \xi(p)) = 0\}$.*

We call the angle $\theta$ a slant function as a function on $M$.

**Remark 1.**

1. *In other papers ([19,20], etc.), the slant angle $\theta$ of a submanifold $M$ in an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is defined in a little bit different way as follows:*

*Assume that $\xi \in \Gamma(TM)$. Given a point $p \in M$, if the angle $\theta = \theta(X)$ between $\phi X$ and $T_pM$ is constant for nonzero $X \in T_pM - \{\xi(p)\}$, then we call the angle $\theta$ a slant angle.*

*Two definitions for the slant angle of a submanifold in an almost contact metric manifold are essentially same when $\xi \in \Gamma(TM)$. Our definition has some advantages as follows: First of all, our definition does not depend on whether the vector field $\xi$ is tangent to $M$ or the vector field $\xi$ is normal to $M$. Secondly, we have more simple form like this (See Lemma 1): $T^2X = -\cos^2\theta X$ for $X \in M_p$, which is the same form with the case of an almost Hermitian manifold, etc..*

2. *If $\theta : M \mapsto [0, \frac{\pi}{2})$, then by using Theorem 3.3 of [20], we obtain that either $\xi$ is tangent to $M$ or $\xi$ is normal to $M$.*
3. *Like examples of Section 6, we need to deal with our notion both when $\xi$ is tangent to $M$ and when $\xi$ is normal to $M$ so that by (1), our definition is more favorite.*

**Remark 2.**

1. *If the slant function $\theta$ is constant on $M$, then we call $M$ a slant submanifold.*
2. *If $\theta = 0$ on $M$, (which implies $\phi(TM) \subset TM$), then we call $M$ an invariant submanifold.*
3. *If $\theta = \frac{\pi}{2}$ on $M$, (which implies $\phi(TM) \subset TM^{\perp}$), then we call $M$ an anti-invariant submanifold.*

Let $M$ be a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the slant function $\theta$.

For $X \in \Gamma(TM)$, we write

$$\phi X = TX + FX, \tag{24}$$

where $TX \in \Gamma(TM)$ and $FX \in \Gamma(TM^{\perp})$.

For $Z \in \Gamma(TM^{\perp})$, we obtain

$$\phi Z = tZ + fZ, \tag{25}$$

where $tZ \in \Gamma(TM)$ and $fZ \in \Gamma(TM^{\perp})$.

Let $T^1M := \bigcup_{p \in M} M_p = \bigcup_{p \in M} \{X \in T_pM \mid g(X, \xi(p)) = 0\}$.

Then we get

**Lemma 1.** *Let $M$ be a submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$. Then $M$ is a pointwise slant submanifold of $N$ if and only if $T^2 = -\cos^2\theta \cdot I$ on $T^1M$ for some function $\theta : M \mapsto \mathbb{R}$.*

**Proof.** Suppose that $M$ is a pointwise slant submanifold of $N$ with the slant function $\theta : M \mapsto \mathbb{R}$. Given a point $p \in M$, if $\theta(p) = \frac{\pi}{2}$, then trivial! If $\theta(p) \neq \frac{\pi}{2}$, then for any nonzero $X \in M_p$ we have

$$\cos\theta(p) = \frac{g(\phi X, TX)}{||\phi X||\,||TX||} = \frac{||TX||}{||X||} \tag{26}$$

so that $\cos^2\theta(p) g(X, X) = g(TX, TX) = -g(T^2X, X)$. Replacing $X$ by $X + Y$, $Y \in M_p$, we obtain

$$g((T^2 + \cos^2\theta(p)I)X, Y) + g(X, (T^2 + \cos^2\theta(p)I)Y) = 0.$$

$T^2 + \cos^2\theta(p)I$ is also symmetric so that

$$(T^2 + \cos^2\theta(p)I)X = 0. \tag{27}$$

Conversely, if $T^2 = -\cos^2\theta I$ on $T^1M$ for some function $\theta : M \mapsto \mathbb{R}$, then we have $g(TX, TX) = -g(T^2X, X) = \cos^2\theta(p) g(X, X)$ for any nonzero $X \in M_p$, $p \in M$ so that

$$\cos^2\theta(p) = \frac{g(TX, TX)}{g(X, X)}, \tag{28}$$

which implies that $\arccos(|\cos\theta(p)|)$ is a slant function on $M$.

Hence, $M$ is a pointwise slant submanifold of $N$. □

**Remark 3.** *Let $M$ be a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the slant function $\theta$. By using Lemma 1, we easily get*

$$g(TX, TY) = \cos^2\theta g(X, Y), \tag{29}$$

$$g(FX, FY) = \sin^2\theta g(X, Y), \tag{30}$$

*for $X, Y \in \Gamma(T^1M)$. At each given point $p \in M$ with $0 \leq \theta(p) < \frac{\pi}{2}$, by using (29) we can choose an orthonormal basis $\{X_1, \sec\theta TX_1, \cdots, X_k, \sec\theta TX_k\}$ of $M_p$.*

Using Lemma 1, we obtain

**Corollary 1.** *Let $M$ be a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the nonconstant slant function $\theta : M \mapsto \mathbb{R}$. Then $M$ is even-dimensional.*

In a similar way to Proposition 2.1 of [24], we have

**Proposition 1.** *Let $M$ be a 2-dimensional submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Then $M$ is a pointwise slant submanifold of $N$.*

**Proof.** Given a point $p \in M$, we consider it at two cases.

If $\xi \notin \Gamma(T_pM^\perp)$, then since $\dim M_p = 1$ and $g(\phi X, X) = 0$ for $X \in M_p$, we immediately obtain $\theta(p) = \frac{\pi}{2}$.

If $\xi \in \Gamma(T_pM^\perp)$, then we choose an orthonormal basis $\{X, Y\}$ of $T_pM$. Let $\alpha := g(X, \phi Y)$. Given any nonzero vector $Z = aX + bY \in T_pM$, $a, b \in \mathbb{R}$, we get

$$TZ = g(\phi Z, X)X + g(\phi Z, Y)Y = bg(X, \phi Y)X - ag(X, \phi Y)Y = b\alpha X - a\alpha Y$$

so that

$$\cos\theta(Z) = \frac{g(\phi Z, TZ)}{||\phi Z||\,||TZ||} = \frac{||TZ||}{||Z||} = |\alpha|,$$

which means the result. □

**Remark 4.** *Proposition 1 gives us a kind of examples for pointwise slant submanifolds.*

In a similar way to Theorem 2.4 of [24], we obtain

**Theorem 1.** *Let $M$ be a pointwise slant connected totally geodesic submanifold of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$. Then $M$ is a slant submanifold of $N$.*

**Proof.** Given any two points $p, q \in M$, we choose a $C^\infty$-curve $c : [0, 1] \mapsto M$ such that $c(0) = p$ and $c(1) = q$. For nonzero $X \in M_p$, we take a parallel transport $Z(t)$ along the curve $c$ in $M$ such that $Z(0) = X$ and $Z(1) = Y$. Then since $M$ is totally geodesic,

$$0 = \nabla_{c'}Z(t) = \overline{\nabla}_{c'}Z(t), \tag{31}$$

where $\nabla$ and $\overline{\nabla}$ are the Levi–Civita connections of $M$ and $N$, respectively. By the uniqueness of parallel transports, $Z(t)$ is also a parallel transport in $N$. Since $\xi$ is parallel (see (23)), we have

$$\frac{d}{dt}g(Z(t), \xi) = g(\overline{\nabla}_{c'}Z(t), \xi) + g(Z(t), \overline{\nabla}_{c'}\xi) = 0 \quad \text{and } g(Z(0), \xi) = 0 \tag{32}$$

so that

$$0 = g(Z(1), \xi) = g(Y, \xi),$$

which implies $Y \in M_q$.

By (23),

$$\overline{\nabla}_{c'} \phi Z(t) = (\overline{\nabla}_{c'} \phi) Z(t) + \phi \overline{\nabla}_{c'} Z(t) = 0$$

so that $\phi Z(t)$ becomes a parallel transport along $c$ in $N$ such that $\phi Z(0) = \phi X$ and $\phi Z(1) = \phi Y$.

Define a map $\tau : T_p N \mapsto T_q N$ by $\tau(U) = V$ for $U \in T_p N$ and $V \in T_q N$, where $W(t)$ is the parallel transport along $c$ in $N$ such that $W(0) = U$ and $W(1) = V$. Then $\tau$ is surely isometry. It is easy to check that $\tau(T_p M) = T_p M$ and $\tau(T_p M^\perp) = T_p M^\perp$ so that $\tau(\phi X) = \phi Y$ means $\tau(TX) = TY$.

Hence,

$$\cos \theta(p) = \frac{||TX||}{||X||} = \frac{||TY||}{||Y||} = \cos \theta(q),$$

where $\theta$ is the slant function on $M$.

Therefore, the result follows. □

Using Proposition 1 and Theorem 1, we get

**Corollary 2.** *Let $M$ be a 2-dimensional connected totally geodesic submanifold of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$. Then $M$ is a slant submanifold of $N$.*

**Remark 5.** *Corollary 2 gives us a kind of examples for slant submanifolds.*

Now, we need to mention A. Lotta's result [20], which is the generalization of the well-known result of K. Yano and M. Kon [38].

**Theorem 2** ([20]). *Let $M$ be a submanifold of a contact metric manifold $N = (N, \phi, \xi, \eta, g)$. If $\xi$ is normal to $M$, then $M$ is a anti-invariant submanifold of $N$.*

**Remark 6.**

1. *As we know, Theorem 2 is very strong and it implies that there do not exist submanifolds $M$ with $\xi \in \Gamma(TM^\perp)$ in a contact metric manifold $(N, \phi, \xi, \eta, g)$ such that either $\{X, \phi X\} \subset M_p$ for some nonzero $X \in M_p$, $p \in M$ or $2 \dim M > \dim N + 1$.*
2. *If $N$ is either cosymplectic or Kenmotsu, then Theorem 2 is not true (see Examples 2 and 3) and we easily check that the argument of the proof of Theorem 2 at [20] does not give any information anymore.*
3. *In the view point of (1) and (2), we may think that Sasakian manifolds are somewhat different from cosymplectic manifolds and Kenmotsu manifolds (see Sections 8–10).*

In the same way to Proposition 2.1 of [23], we can obtain

**Proposition 2.** *Let $M$ be a submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Then $M$ is a pointwise slant submanifold of $N$ if and only if*

$$g(TX, TY) = 0 \textit{ whenever } g(X, Y) = 0 \textit{ for } X, Y \in M_p, p \in M. \tag{33}$$

Considering slant functions as conformal invariant, we easily derive

**Proposition 3.** *Let $M$ be a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the slant function $\theta : M \mapsto \mathbb{R}$. Then for any given $C^\infty$-function $f : N \mapsto \mathbb{R}$, $M$ is also a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, e^{-f}\xi, e^f \eta, e^{2f} g)$ with the same slant function $\theta$.*

**Theorem 3.** *Let $M$ be a slant submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ with the slant angle $\theta$. Assume that $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu. Then we have*

$$A_{FX}TX = A_{FTX}X \quad \text{for } X \in \Gamma(T^1M). \tag{34}$$

**Proof.** We will only give its proof when $N$ is Sasakian. For the other cases, we can show them in a similar way. If $\theta = \frac{\pi}{2}$, then done! Assume that $0 \leq \theta < \frac{\pi}{2}$. Given a unit vector field $X \in \Gamma(T^1M)$, we have

$$TX = \cos\theta \cdot X^* \tag{35}$$

for some unit vector field $X^* \in \Gamma(T^1M)$ with $g(X, X^*) = 0$. Then for any $Y \in \Gamma(TM)$, by using (1), (2) and (19), we obtain

$$\begin{aligned} \overline{\nabla}_Y(\phi X) &= \overline{\nabla}_Y(\cos\theta \cdot X^*) + \overline{\nabla}_Y FX \\ &= \cos\theta \cdot \nabla_Y X^* + \cos\theta h(Y, X^*) - A_{FX}Y + D_Y FX \end{aligned} \tag{36}$$

and

$$\begin{aligned} \overline{\nabla}_Y(\phi X) &= (\overline{\nabla}_Y \phi)X + \phi\overline{\nabla}_Y X \\ &= g(Y, X)\xi - \eta(X)Y + T\nabla_Y X + F\nabla_Y X \\ &\quad + th(Y, X) + fh(Y, X) \\ &= g(Y, X)\xi + T\nabla_Y X + F\nabla_Y X + th(Y, X) + fh(Y, X). \end{aligned} \tag{37}$$

Thus, by taking the inner product of right hand sides of (36) and (37) with $X^*$, we derive

$$g(-A_{FX}Y, X^*) = g(th(Y, X), X^*),$$

which gives

$$g(A_{FX}X^*, Y) = g(A_{FX^*}X, Y).$$

Therefore, the result follows. □

## 5. Topological Properties of Pointwise Slant Submanifolds of a Cosymplectic Manifold

In this section we investigate the topological properties of pointwise slant submanifolds of a cosymplectic manifold. A pointwise slant submanifold $M$ of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ is said to be proper if the slant function $\theta$ of $M$ in $N$ is given by $\theta : M \mapsto [0, \frac{\pi}{2})$.

Let $M$ be a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Given $X, Y \in \Gamma(TM)$, we define

$$(\nabla_X T)Y := \nabla_X(TY) - T\nabla_X Y, \tag{38}$$

$$(D_X F)Y := D_X(FY) - F\nabla_X Y. \tag{39}$$

We call the tensors $T$ and $F$ parallel if $\nabla T = 0$ and $\nabla F = 0$, respectively. Then in a similar way to Lemma 3.8 of [31], we easily obtain

**Lemma 2.** *Let $M$ be a pointwise slant submanifold of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$. Then we get*

*1.*

$$(\nabla_X T)Y = A_{FY}X + th(X, Y), \tag{40}$$

$$(D_X F)Y = -h(X, TY) + fh(X, Y) \tag{41}$$

*for* $X, Y \in \Gamma(TM)$.

2.
$$-TA_ZX + tD_XZ = \nabla_X(tZ) - A_{fZ}X, \tag{42}$$
$$-FA_ZX + fD_XZ = h(X, tZ) + D_X(fZ) \tag{43}$$

*for* $X \in \Gamma(TM)$ *and* $Z \in \Gamma(TM^{\perp})$.

Let $M$ be a proper pointwise slant submanifold of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$. Define
$$\Omega(X, Y) := g(X, TY) \quad \text{for } X, Y \in \Gamma(TM). \tag{44}$$

Then $\Omega$ is a 2-form on $M$, which is non-degenerate on $T^1M$ ([23], [31]).

**Theorem 4.** *Let $M$ be a proper pointwise slant submanifold of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$. Then the 2-form $\Omega$ is closed.*

**Proof.** Given $X, Y, Z \in \Gamma(TM)$, we get
$$\begin{aligned} 3d\Omega(X, Y, Z) &= X\Omega(Y, Z) - Y\Omega(X, Z) + Z\Omega(X, Y) \\ &\quad - \Omega([X, Y], Z) + \Omega([X, Z], Y) - \Omega([Y, Z], X) \end{aligned}$$

so that
$$\begin{aligned} 3d\Omega(X, Y, Z) &= g(\nabla_X Y, TZ) + g(Y, \nabla_X TZ) - g(\nabla_Y X, TZ) \\ &\quad - g(X, \nabla_Y TZ) + g(\nabla_Z X, TY) + g(X, \nabla_Z TY) \\ &\quad - g(\nabla_X Y - \nabla_Y X, TZ) + g(\nabla_X Z - \nabla_Z X, TY) - g(\nabla_Y Z - \nabla_Z Y, TX) \\ &= g(Y, (\nabla_X T)Z) - g(X, (\nabla_Y T)Z) + g(X, (\nabla_Z T)Y). \end{aligned}$$

Using Lemma 2 and (3), we obtain
$$\begin{aligned} 3d\Omega(X, Y, Z) &= g(Y, A_{FZ}X + th(X, Z)) - g(X, A_{FZ}Y + th(Y, Z)) \\ &\quad + g(X, A_{FY}Z + th(Z, Y)) \\ &= g(Y, th(X, Z)) - g(Z, th(Y, X)) \\ &\quad - g(X, th(Y, Z)) + g(Z, th(X, Y)) \\ &\quad + g(X, th(Z, Y)) - g(Y, th(X, Z)) \\ &= 0. \end{aligned}$$

Therefore, the result follows. □

Consider the restriction of the 1-form $\eta$ to $M$. We also denote it by $\eta$.

Denote by $[\Omega]$ and $[\eta]$ the de Rham cohomology classes of 2-form $\Omega$ and 1-form $\eta$ on $M$, respectively. As we know, a cosymplectic manifold is locally a Riemannian product of a Kähler manifold and an interval and the cosymplectic condition (i.e., $\overline{\nabla}\phi = 0$) naturally corresponds to the Kähler condition ($\overline{\nabla}J = 0$) (See [37]).

Hence, in a similar way to Theorem 5.1 of [23] and to Theorem 5.2 of [31], by using Theorem 4, we obtain

**Theorem 5.** *Let $M$ be a $2m$-dimensional compact proper pointwise slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold $(N, \phi, \xi, \eta, g)$ such that $\xi$ is normal to $M$.*

*Then $[\Omega] \in H^2(M, \mathbb{R})$ is non-vanishing.*

**Proof.** Since $TM = T^1M$, by the definition of $\Omega$, $\Omega$ is non-degenerate on $M$.

Therefore, the result follows. $\square$

**Remark 7.** *By the proof of Theorem 5, we have*

$$\dim H^{2i}(M, \mathbb{R}) \geq 1 \quad \text{for } 0 \leq i \leq m. \tag{45}$$

**Theorem 6.** *Let $M$ be a $(2m+1)$-dimensional compact proper pointwise slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold $(N, \phi, \xi, \eta, g)$ such that $\xi$ is tangent to $M$.*

*Then both $[\eta] \in H^1(M, \mathbb{R})$ and $[\Omega] \in H^2(M, \mathbb{R})$ are non-vanishing.*

**Proof.** Using (29), we can choose a local orthonormal frame $\{\xi, X_1, \sec\theta TX_1, \cdots, X_m, \sec\theta TX_m\}$ of $TM$.

Thus,

$$\eta \wedge \Omega^m = \eta \wedge g(\ ,T)^m \neq 0 \text{ at each point of } M \tag{46}$$

so that it gives a volume form on $M$.

Hence, both $[\eta]$ and $[\Omega]$ are never vanishing. $\square$

**Remark 8.** *By the proof of Theorem 6, we get*

$$\dim H^{i}(M, \mathbb{R}) \geq 1 \quad \text{for } 0 \leq i \leq 2m+1. \tag{47}$$

By using (45) and (47), we obtain

**Corollary 3.** *Every m-sphere $S^m$, $m \geq 3$, cannot be immersed in a cosymplectic manifold as a proper pointwise slant submanifold.*

**Corollary 4.** *Any m-dimensional real projective space $\mathbb{RP}^m$, $m \geq 3$, cannot be immersed in a cosymplectic manifold as a proper pointwise slant submanifold.*

**Remark 9.** *For 2-sphere $S^2$ and 2-torus $T^2$, they satisfy the condition (45). By Proposition 1, they are pointwise slant submanifolds of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$ if they are just submanifolds of $N$.*

## 6. Examples

In this section we give some examples of pointwise slant submanifolds.

**Example 1.** *Define a map $i : \mathbb{R}^3 \mapsto \mathbb{R}^5$ by*

$$i(x_1, x_2, x_3) = (y_1, y_2, y_3, y_4, t) = (x_1, \sin x_2, 0, \cos x_2, x_3).$$

*Let $M := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 < x_2 < \frac{\pi}{2}\}$.*

*We define $(\phi, \xi, \eta, g)$ on $\mathbb{R}^5$ as follows:*

$$\begin{aligned}
&\phi(a_1 \tfrac{\partial}{\partial y_1} + \cdots + a_4 \tfrac{\partial}{\partial y_4} + a_5 \tfrac{\partial}{\partial t}) = -a_2 \tfrac{\partial}{\partial y_1} + a_1 \tfrac{\partial}{\partial y_2} - a_4 \tfrac{\partial}{\partial y_3} + a_3 \tfrac{\partial}{\partial y_4},\\
&\xi := \tfrac{\partial}{\partial t}, \quad \eta := dt,\ a_i \in \mathbb{R}, 1 \leq i \leq 5,
\end{aligned}$$

*$g$ is the Euclidean metric on $\mathbb{R}^5$. It is easy to check that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $\mathbb{R}^5$.*

*Then $M$ is a pointwise slant submanifold of an almost contact metric manifold $(\mathbb{R}^5, \phi, \xi, \eta, g)$ with the slant function $k(x_1, x_2, x_3) = x_2$ such that $\xi$ is tangent to $M$.*

**Example 2.** *Define a map $i : \mathbb{R}^2 \mapsto \mathbb{R}^5$ by*

$$i(x_1, x_2) = (y_1, y_2, y_3, y_4, t) = (0, \cos x_1, x_2, \sin x_1, 0).$$

*Let $M := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \frac{\pi}{2}\}$.*

*We define $(\phi, \xi, \eta, g)$ on $\mathbb{R}^5$ as follows:*

$$\phi(a_1 \frac{\partial}{\partial y_1} + \cdots + a_4 \frac{\partial}{\partial y_4} + a_5 \frac{\partial}{\partial t}) = -a_2 \frac{\partial}{\partial y_1} + a_1 \frac{\partial}{\partial y_2} - a_4 \frac{\partial}{\partial y_3} + a_3 \frac{\partial}{\partial y_4},$$
$$\xi := \frac{\partial}{\partial t}, \quad \eta := dt, \ a_i \in \mathbb{R}, 1 \le i \le 5,$$

*$g$ is the Euclidean metric on $\mathbb{R}^5$. It is easy to check that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $\mathbb{R}^5$.*

*We also know that $(\mathbb{R}^5, \phi, \xi, \eta, g)$ is a cosymplectic manifold.*

*Then $M$ is a pointwise slant submanifold of a cosymplectic manifold $(\mathbb{R}^5, \phi, \xi, \eta, g)$ with the slant function $k(x_1, x_2) = x_1$ such that $\xi$ is normal to $M$.*

**Example 3.** *Let $t$ be a coordinate of $\mathbb{R}$ and $(y_1, y_2, y_3, y_4)$ coordinates of $\mathbb{R}^4$. Let $N := \mathbb{R} \times_f \mathbb{R}^4$ be a warped product manifold of the Euclidean space $\mathbb{R}$ and the Euclidean space $\mathbb{R}^4$ with the natural projections $\pi_1 : N \mapsto \mathbb{R}$ and $\pi_2 : N \mapsto \mathbb{R}^4$ such that the warping function $f(t) = e^t$.*

*Let $\mathbb{R}^4 = (\mathbb{R}^4, \bar{g}, J)$, where $\bar{g}$ is the Euclidean metric on $\mathbb{R}^4$ and $J$ is a complex structure on $\mathbb{R}^4$ defined by*

$$J(a_1 \frac{\partial}{\partial y_1} + \cdots + a_4 \frac{\partial}{\partial y_4}) = -a_2 \frac{\partial}{\partial y_1} + a_1 \frac{\partial}{\partial y_2} - a_4 \frac{\partial}{\partial y_3} + a_3 \frac{\partial}{\partial y_4}.$$

*Then $\mathbb{R}^4$ is obviously Kähler.*

*We define $(\phi, \xi, \eta, g)$ on $N$ as follows:*

$$\phi(a_1 \frac{\partial}{\partial y_1} + \cdots + a_4 \frac{\partial}{\partial y_4} + a_5 \frac{d}{dt}) := J(a_1 \frac{\partial}{\partial y_1} + \cdots + a_4 \frac{\partial}{\partial y_4}),$$
$$\xi := \frac{d}{dt}, \quad \eta := dt,$$
$$g(Z, W) := \eta(Z)\eta(W) + f(t)^2 \bar{g}(d\pi_2(Z), d\pi_2(W))$$

*for $Z, W \in \Gamma(TN)$, $a_i \in \mathbb{R}$, $1 \le i \le 5$.*

*We easily check that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $N$.*

*Furthermore, by Proposition 3 of [39], $(N, \phi, \xi, \eta, g)$ is a Kenmotsu manifold.*

*Let $M := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \frac{\pi}{2}\}$.*

*Define a map $i : \mathbb{R}^2 \mapsto \mathbb{R}^4 \subset N$ by*

$$i(x_1, x_2) = (y_1, y_2, y_3, y_4) = (x_2, \sin x_1, 1972, \cos x_1).$$

*Then $M$ is a pointwise slant submanifold of a Kenmotsu manifold $(N, \phi, \xi, \eta, g)$ with the slant function $k(x_1, x_2) = x_1$ such that $\xi$ is normal to $M$.*

**Example 4.** *Let $M$ be a submanifold of a hyperkähler manifold $(\overline{M}, J_1, J_2, J_3, \bar{g})$ such that $M$ is complex with respect to the complex structure $J_1$ (i.e., $J_1(TM) = TM$) and totally real with respect to the complex structure $J_2$ (i.e., $J_2(TM) \subset TM^\perp$) [40]. Let $f : \overline{M} \mapsto [0, \frac{\pi}{2}]$ be a $C^\infty$-function and $N := \overline{M} \times \mathbb{R}$ with the natural projections $\pi_1 : N \mapsto \overline{M}$ and $\pi_2 : N \mapsto \mathbb{R}$.*

*We define $(\phi, \xi, \eta, g)$ on $N$ as follows:*

$$\phi(X + h\frac{d}{dt}) := \cos(f \circ \pi_1) J_1 X - \sin(f \circ \pi_1) J_2 X,$$
$$\xi := \frac{d}{dt}, \quad \eta := dt,$$
$$g(Z, W) := \bar{g}(d\pi_1 Z, d\pi_1 W) + \eta(Z) \cdot \eta(W)$$

*for $X \in \Gamma(T\overline{M})$, $h \in C^\infty(N)$, $Z, W \in \Gamma(TN)$ and $t$ is a coordinate of $\mathbb{R}$.*

*It is easy to show that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $N$.*

*Then $M$ is a pointwise slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the slant function $f \circ \pi_1$ such that $\xi$ is normal to $M$.*

**Example 5.** *Given a Euclidean space $\mathbb{R}^5 = \mathbb{R}^4 \times \mathbb{R}$ with coordinates $(y_1, \cdots, y_4, t)$, we consider complex structures $J_1$ and $J_2$ on $\mathbb{R}^4$ as follows:*

$$J_1(\tfrac{\partial}{\partial y_1}) = \tfrac{\partial}{\partial y_2}, J_1(\tfrac{\partial}{\partial y_2}) = -\tfrac{\partial}{\partial y_1}, J_1(\tfrac{\partial}{\partial y_3}) = \tfrac{\partial}{\partial y_4}, J_1(\tfrac{\partial}{\partial y_4}) = -\tfrac{\partial}{\partial y_3},$$
$$J_2(\tfrac{\partial}{\partial y_1}) = \tfrac{\partial}{\partial y_3}, J_2(\tfrac{\partial}{\partial y_2}) = -\tfrac{\partial}{\partial y_4}, J_2(\tfrac{\partial}{\partial y_3}) = -\tfrac{\partial}{\partial y_1}, J_2(\tfrac{\partial}{\partial y_4}) = \tfrac{\partial}{\partial y_2},$$

*Let $f : \mathbb{R}^5 \mapsto [0, \frac{\pi}{2}]$ be a $C^\infty$-function.*

*We define $(\phi, \xi, \eta, g)$ on $\mathbb{R}^5$ as follows:*

$$\phi(X + h\tfrac{d}{dt}) := \cos f \cdot J_1 X - \sin f \cdot J_2 X,$$
$$\xi := \tfrac{d}{dt}, \quad \eta := dt,$$

*$g$ is the Euclidean metric on $\mathbb{R}^5$, $X \in \Gamma(T\mathbb{R}^4)$ and $h \in C^\infty(\mathbb{R}^5)$.*

*We can easily check that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $\mathbb{R}^5$.*

*Define a map $i : \mathbb{R}^2 \mapsto \mathbb{R}^5$ by*

$$i(x_1, x_2) = (y_1, y_2, y_3, y_4, t) = (e, -\pi, x_2, x_1, \sqrt{2}).$$

*Then $\mathbb{R}^2$ is a pointwise slant submanifold of an almost contact metric manifold $(\mathbb{R}^5, \phi, \xi, \eta, g)$ with the slant function $f$ such that $\xi$ is normal to $\mathbb{R}^2$.*

**Example 6.** *With all the conditions of Example 5, define a function $f : \mathbb{R}^5 \mapsto [0, \frac{\pi}{2}]$ by $f(y_1, \cdots, y_4, t) = \arctan(|y_1 + y_2 + y_3 + y_4|)$.*

*Then $\mathbb{R}^2$ is a pointwise slant submanifold of an almost contact metric manifold $(\mathbb{R}^5, \phi, \xi, \eta, g)$ with the slant function $(f \circ i)(x_1, x_2) = \arctan(|e - \pi + x_1 + x_2|)$ such that $\xi$ is normal to $\mathbb{R}^2$.*

## 7. Pointwise Semi-Slant Submanifolds

In this section we introduce the notion of pointwise semi-slant submanifolds of an almost contact metric manifold and obtain a characterization of pointwise semi-slant submanifolds.

**Definition 2.** *Let $(N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M$ a submanifold of $N$. The submanifold $M$ is called a pointwise semi-slant submanifold if there is a distribution $\mathcal{D}_1 \subset TM$ on $M$ such that*

$$TM = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad \phi(\mathcal{D}_1) \subset \mathcal{D}_1,$$

*and at each given point $p \in M$ the angle $\theta = \theta(X)$ between $\phi X$ and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $TM$.*

We call the angle $\theta$ a *semi-slant function* as a function on $M$.

**Remark 10.** *Let $M$ be a pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$.*

1. *Given a point $p \in M$, if $\xi(p) \in T_pM$, then $\xi(p)$ should belong to $(\mathcal{D}_1)_p$ (i.e., $\xi(p) \in (\mathcal{D}_1)_p$).*

   *If not, we can induce contradiction as follows:*

   *Assume that $\xi(p) = X + Y$ for some $X \in (\mathcal{D}_1)_p$ and some nonzero $Y \in (\mathcal{D}_2)_p$. Then $0 = \phi\xi(p) = \phi X + \phi Y$ with $\phi X \in (\mathcal{D}_1)_p$ and $\phi Y \in (\mathcal{D}_2)_p \oplus T_pM^\perp$ so that $\phi X = 0$ and $\phi Y = 0$. Since $g(X, Y) = 0$ and $\ker \phi =< \xi >$, we must have $X = 0$ and $Y = \xi(p)$. $\theta(Y) = \theta(\xi(p))$ is not defined, contradiction.*
2. *Let $(\overline{\mathcal{D}}_1)_p := \{X \in (\mathcal{D}_1)_p \mid g(X, \xi(p)) = 0\}$ for $p \in M$.*

   *Then we have either $(\mathcal{D}_1)_p = (\overline{\mathcal{D}}_1)_p$ or $(\mathcal{D}_1)_p =< \xi(p) > \oplus (\overline{\mathcal{D}}_1)_p$.*

   *We can check this as follows:*

*Since $\phi(\mathcal{D}_1) \subset \mathcal{D}_1$, we get*

$$\phi((\overline{\mathcal{D}}_1)_p) \subset (\mathcal{D}_1)_p \text{ and } g(\phi((\overline{\mathcal{D}}_1)_p), \xi(p)) = 0$$

*so that $\phi((\overline{\mathcal{D}}_1)_p) \subset (\overline{\mathcal{D}}_1)_p$ implies $\phi((\overline{\mathcal{D}}_1)_p) = (\overline{\mathcal{D}}_1)_p$. Thus, we can choose an orthonormal basis $\{Z_1, \phi Z_1, \cdots, Z_k, \phi Z_k\}$ of $(\overline{\mathcal{D}}_1)_p$. Assume that $(\mathcal{D}_1)_p \neq (\overline{\mathcal{D}}_1)_p$. Then there is a vector $Z = a\xi(p) + X \in (\mathcal{D}_1)_p$ with $a \neq 0$ and $g(\xi(p), X) = 0$. We know $\phi Z = \phi X \in (\mathcal{D}_1)_p$ and $g(\phi X, \xi(p)) = 0$ so that $\phi X \in (\overline{\mathcal{D}}_1)_p$ implies $\phi X = \sum_{i=1}^{k}(a_i Z_i + a_{k+i}\phi Z_i)$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq 2k$.*

*Hence, $-X = \phi^2 X = \sum_{i=1}^{k}(-a_{k+i}Z_i + a_i \phi Z_i) \in (\overline{\mathcal{D}}_1)_p \subset (\mathcal{D}_1)_p$, which implies $\frac{1}{a}(Z - X) = \xi(p) \in (\mathcal{D}_1)_p$. Therefore, the result follows.*

3. *From (2), we have either $\mathcal{D}_1 = \overline{\mathcal{D}}_1$ or $\mathcal{D}_1 =< \xi > \oplus \overline{\mathcal{D}}_1$, where $\overline{\mathcal{D}}_1 := \bigcup_{p \in M} (\overline{\mathcal{D}}_1)_p$.*

   *If not, then we can choose a $C^\infty$-curve $c : (-\epsilon, \epsilon) \mapsto M$ for sufficiently small $\epsilon > 0$ such that either $(\mathcal{D}_1)_{c(0)} = (\overline{\mathcal{D}}_1)_{c(0)}$ and $(\mathcal{D}_1)_{c(t)} =< \xi(c(t)) > \oplus (\overline{\mathcal{D}}_1)_{c(t)}$ for $t \in (-\epsilon, \epsilon) - \{0\}$ or $(\mathcal{D}_1)_{c(0)} =< \xi(c(0)) > \oplus (\overline{\mathcal{D}}_1)_{c(0)}$ and $(\mathcal{D}_1)_{c(t)} = (\overline{\mathcal{D}}_1)_{c(t)}$ for $t \in (-\epsilon, \epsilon) - \{0\}$.*

   *Take an orthonormal frame $\{X_1(t), X_2(t), \cdots, X_l(t)\}$ of $\mathcal{D}_1$ along $c$. At the first case, we obtain*

$$\xi(c(t)) = \sum_{i=1}^{l} a_i(t) X_i(t) \tag{48}$$

   *for some $a_i(t) \in \mathbb{R}$, $1 \leq i \leq l$, $t \in (-\epsilon, \epsilon) - \{0\}$. Since $\xi$ is a $C^\infty$-vector field on $N$, we can obtain the $C^\infty$-extension of right hand side of (48) along $c$. $\xi(c(0)) \notin (\mathcal{D}_1)_{c(0)}$ and $\sum_{i=1}^{l} a_i(0) X_i(0) \in (\mathcal{D}_1)_{c(0)}$ with $a_i(0) := \lim_{t \to 0} a_i(t)$, $1 \leq i \leq l$, contradiction. In a similar way, we can also induce contradiction at the second case.*
4. *From (1), we get $\xi(p) \notin (\mathcal{D}_2)_p$ for any $p \in M$.*
5. *If $\theta : M \mapsto (0, \frac{\pi}{2})$, then $M$ is said to be proper.*

Let $M$ be a pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Then there is a distribution $\mathcal{D}_1 \subset TM$ on $M$ such that

$$TM = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad \phi(\mathcal{D}_1) \subset \mathcal{D}_1,$$

and at each given point $p \in M$ the angle $\theta = \theta(X)$ between $\phi X$ and the space $(\mathcal{D}_2)_p$ is constant for nonzero $X \in (\mathcal{D}_2)_p$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $TM$.

For $X \in \Gamma(TM)$, we write

$$X = PX + QX, \tag{49}$$

where $PX \in \Gamma(\mathcal{D}_1)$ and $QX \in \Gamma(\mathcal{D}_2)$.

For $X \in \Gamma(TM)$, we have

$$\phi X = TX + FX, \tag{50}$$

where $TX \in \Gamma(TM)$ and $FX \in \Gamma(TM^\perp)$.

For $Z \in \Gamma(TM^\perp)$, we get

$$\phi Z = tZ + fZ, \tag{51}$$

where $tZ \in \Gamma(TM)$ and $fZ \in \Gamma(TM^\perp)$.

Denote by $(TN)|_M$ the restriction of $TN$ to $M$ (i.e., $(TN)|_M = TM \oplus TM^\perp$).

For $U \in \Gamma((TN)|_M)$, we write

$$U = \mathcal{H}U + \mathcal{V}U, \tag{52}$$

where $\mathcal{H}U \in \Gamma(TM)$ and $\mathcal{V}U \in \Gamma(TM^\perp)$.

Hence,

$$T(\mathcal{D}_1) \subset \mathcal{D}_1, F(\mathcal{D}_1) = 0, T(\mathcal{D}_2) \subset \mathcal{D}_2, t(TM^\perp) \subset \mathcal{D}_2, \tag{53}$$

$$T^2 + tF = -I + \eta \otimes \mathcal{H}(\xi) \text{ and } FT + fF = \eta \otimes \mathcal{V}(\xi) \text{ on } TM \tag{54}$$

$$Tt + tf = \eta \otimes \mathcal{H}(\xi) \text{ and } Ft + f^2 = -I + \eta \otimes \mathcal{V}(\xi) \text{ on } TM^\perp \tag{55}$$

Then we obtain

$$TM^\perp = F\mathcal{D}_2 \oplus \mu, \tag{56}$$

where $\mu$ is the orthogonal complement of $F\mathcal{D}_2$ in $TM^\perp$.

For $X, Y \in \Gamma(TM)$, we define

$$(\nabla_X T)Y := \nabla_X(TY) - T\nabla_X Y, \tag{57}$$

$$(D_X F)Y := D_X(FY) - F\nabla_X Y. \tag{58}$$

The tensors $T$ and $F$ are called *parallel* if $\nabla T = 0$ and $\nabla F = 0$, respectively.

In the same way to Lemma 2, we have

**Lemma 3.** *Let $M$ be a pointwise semi-slant submanifold of a cosymplectic manifold $(N, \phi, \xi, \eta, g)$. Then we obtain*

*1.*

$$(\nabla_X T)Y = A_{FY}X + th(X, Y), \tag{59}$$

$$(D_X F)Y = -h(X, TY) + fh(X, Y) \tag{60}$$

*for $X, Y \in \Gamma(TM)$.*

*2.*

$$-TA_Z X + tD_X Z = \nabla_X(tZ) - A_{fZ}X, \tag{61}$$

$$-FA_Z X + fD_X Z = h(X, tZ) + D_X(fZ) \tag{62}$$

*for $X \in \Gamma(TM)$ and $Z \in \Gamma(TM^\perp)$.*

**Proposition 4.** *Let $M$ be a pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that either $\mathcal{D}_2 \subset \ker \eta$ or $\mu \subset \ker \eta$.*

*Then $\mu$ is $\phi$-invariant (i.e., $\phi\mu \subset \mu$).*

**Proof.** Given $Y \in \Gamma(\mu)$ and $X \in \Gamma(TM)$ with $X = X_1 + X_2$, $X_1 \in \Gamma(\mathcal{D}_1)$, $X_2 \in \Gamma(\mathcal{D}_2)$, we have

$$g(X, \phi Y) = -g(\phi X, Y) = -g(\phi X_1 + \phi X_2, Y) = 0$$

so that

$$\phi\mu \subset TM^\perp. \tag{63}$$

Given $Y \in \Gamma(\mu)$ and $X \in \Gamma(F\mathcal{D}_2)$ with $X = FX'$ for some $X' \in \Gamma(\mathcal{D}_2)$, by using (54) and the hypothesis, we get

$$\begin{aligned} g(X, \phi Y) &= -g(\phi X, Y) = -g(fFX', Y) \\ &= g(FTX' - \eta(X')\mathcal{V}(\xi), Y) \\ &= -\eta(X') \cdot \eta(Y) = 0. \end{aligned}$$

with (63), it implies $\phi\mu \subset \mu$. □

In a similar way to Proposition 3.9 of [31], we have

**Lemma 4.** *Let $M$ be a pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$.*

*Then*

$$g((T^2 + \cos^2\theta(I - \eta \otimes \xi))(X), Y) = 0 \quad \text{for } X, Y \in \Gamma(\mathcal{D}_2). \tag{64}$$

**Proof.** We will prove this at each point of $M$.

Gven a point $p \in M$, if $X \in (\mathcal{D}_2)_p$ is vanishing, then done! Given a nonzero $X \in (\mathcal{D}_2)_p$, we obtain

$$\cos\theta(p) = \frac{g(\phi X, TX)}{||\phi X||\,||TX||} = \frac{||TX||}{||\phi X||} \tag{65}$$

so that $\cos^2\theta(p) g(\phi X, \phi X) = g(TX, TX) = -g(T^2 X, X)$. Substituting $X$ by $X + Y$, $Y \in (\mathcal{D}_2)_p$, at the above equation, we induce

$$g((T^2 + \cos^2\theta(I - \eta \otimes \xi))(X), Y) + g(X, (T^2 + \cos^2\theta(I - \eta \otimes \xi))(Y)) = 0. \tag{66}$$

$T^2 + \cos^2\theta(I - \eta \otimes \xi)$ is also symmetric so that

$$g((T^2 + \cos^2\theta(I - \eta \otimes \xi))(X), Y) = 0.$$

□

**Remark 11.** *Let $M$ be a pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$. Assume that either $\xi$ is tangent to $M$ or $\xi$ is normal to $M$.*

1. *By using (64) and Remark 10 (1), we get*

$$T^2 X = -\cos^2\theta \cdot X \quad \text{for } X \in \Gamma(\mathcal{D}_2). \tag{67}$$

2. *By (67), we obtain*

$$g(TX, TY) = \cos^2\theta g(X, Y), \tag{68}$$

$$g(FX, FY) = \sin^2\theta g(X, Y), \tag{69}$$

*for $X, Y \in \Gamma(\mathcal{D}_2)$.*

3. *At each given point $p \in M$ with $0 \leq \theta(p) < \frac{\pi}{2}$, by using (68), we can choose an orthonormal basis $\{X_1, \sec\theta T X_1, \cdots, X_k, \sec\theta T X_k\}$ of $(\mathcal{D}_2)_p$.*

## 8. Distributions

In this section we consider distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ and deal with the notion of totally umbilic submanifolds.

Notice that if $N = (N, \phi, \xi, \eta, g)$ is Sasakian, then from Theorem 2, there does not exist a proper pointwise semi-slant submanifold $M$ of $N$ such that $\xi$ is normal to $M$.

**Lemma 5.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is tangent to $M$ and $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*Then the distribution $\mathcal{D}_1$ is integrable if and only if*

$$g(h(X, \phi Y) - h(Y, \phi X), FZ) = 0 \tag{70}$$

*for $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$.*

**Proof.** We will only give its proof when $N$ is Sasakian. For the other cases, we can show them in the same way.

Given $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$, by using Remark 10 and (19), we obtain

$$\begin{aligned}
&g([X,Y],Z) \\
&= g(\phi[X,Y],\phi Z) + \eta([X,Y])\eta(Z) \\
&= g(\phi(\overline{\nabla}_X Y - \overline{\nabla}_Y X), TZ + FZ) \\
&= -g(\overline{\nabla}_X Y - \overline{\nabla}_Y X, T^2 Z + FTZ) \\
&+ g(h(X,\phi Y) - h(Y,\phi X) - (g(X,Y)\xi - \eta(Y)X - g(Y,X)\xi + \eta(X)Y), FZ) \\
&= \cos^2\theta g([X,Y],Z) + g(h(X,\phi Y) - h(Y,\phi X), FZ)
\end{aligned}$$

so that

$$\sin^2\theta g([X,Y],Z) = g(h(X,\phi Y) - h(Y,\phi X), FZ).$$

Therefore, we get the result. □

In the same way to Lemma 5, we obtain

**Lemma 6.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N,\phi,\xi,\eta,g)$. Assume that $\xi$ is normal to $M$ and $N$ is one of the following two manifolds: cosymplectic, Kenmotsu.*

*Then the distribution $\mathcal{D}_1$ is integrable if and only if*

$$g(h(X,\phi Y) - h(Y,\phi X), FZ) = 0 \tag{71}$$

*for $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$.*

**Lemma 7.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N,\phi,\xi,\eta,g)$. Assume that $\xi$ is normal to $M$ and $N$ is one of the following two manifolds: cosymplectic, Kenmotsu.*

*Then the distribution $\mathcal{D}_2$ is integrable if and only if*

$$g(A_{FTW}Z - A_{FTZ}W, X) = g(A_{FW}Z - A_{FZ}W, \phi X) \tag{72}$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$.*

**Proof.** We only give its proof when $N$ is Kenmotsu.

Given $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$, by using (21) and Remark 11, we get

$$\begin{aligned}
&g([Z,W],X) \\
&= g(\phi[Z,W],\phi X) + \eta([Z,W])\eta(X) \\
&= g(\phi(\overline{\nabla}_Z W - \overline{\nabla}_W Z), \phi X) \\
&= g(\overline{\nabla}_Z(TW + FW) - \overline{\nabla}_W(TZ + FZ), \phi X) \\
&- g(g(\phi Z,W)\xi - \eta(W)\phi Z - g(\phi W,Z)\xi + \eta(Z)\phi W, \phi X) \\
&= -g(\overline{\nabla}_Z(T^2 W + FTW) - \overline{\nabla}_W(T^2 Z + FTZ), X) \\
&+ g(g(\phi Z,TW)\xi - \eta(TW)\phi Z - g(\phi W,TZ)\xi + \eta(TZ)\phi W, X) \\
&+ g(A_{FZ}W - A_{FW}Z, \phi X) \\
&= \cos^2\theta g([Z,W],X) + g(A_{FTW}Z - A_{FTZ}W, X) + g(A_{FZ}W - A_{FW}Z, \phi X)
\end{aligned}$$

so that

$$\sin^2\theta g([Z,W],X) = g(A_{FTW}Z - A_{FTZ}W, X) + g(A_{FZ}W - A_{FW}Z, \phi X).$$

Therefore, the result follows. □

**Lemma 8.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N,\phi,\xi,\eta,g)$. Assume that $\xi$ is tangent to $M$ and $N$ is one of the following two manifolds: cosymplectic, Kenmotsu.*

*Then the distribution $\mathcal{D}_2$ is integrable if and only if*

$$g(A_{FTW}Z - A_{FTZ}W, X) = g(A_{FW}Z - A_{FZ}W, \phi X) \tag{73}$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$.*

**Proof.** We will show it when $N$ is Kenmotsu.

Given $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$, from the proof of Lemma 7, we have

$$\begin{aligned} \sin^2\theta g([Z,W],X) \quad &= \quad \eta([Z,W])\eta(X) + g(A_{FTW}Z - A_{FTZ}W, X) \\ &\quad + g(A_{FZ}W - A_{FW}Z, \phi X). \end{aligned} \tag{74}$$

Replacing $X$ by $\xi$ at (74), by using (3) and (22), we get

$$\begin{aligned} -\cos^2\theta\eta([Z,W]) &= g(h(Z,\xi), FTW) - g(h(W,\xi), FTZ) \\ &= g(\overline{\nabla}_Z\xi, FTW) - g(\overline{\nabla}_W\xi, FTZ) \\ &= g(Z - \eta(Z)\xi, FTW) - g(W - \eta(W)\xi, FTZ) \\ &= 0 \end{aligned}$$

so that $\eta([Z,W]) = 0$.

Hence, the result follows. □

**Remark 12.** *For the case when both N is Sasakian and $\xi$ is tangent to M, confer Proposition 5.4 of [26].*

**Theorem 7.** *Let M be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is tangent to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*Then the distribution $\mathcal{D}_1$ defines a totally geodesic foliation if and only if*

$$g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0 \tag{75}$$

*for $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$.*

**Proof.** We will give its proof when $N$ is Kenmotsu.

Given $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$, by using Remark 10, (21) and Remark 11, we obtain

$$\begin{aligned} & g(\nabla_Y X, Z) \\ & = g(\phi\overline{\nabla}_Y X, \phi Z) + \eta(\overline{\nabla}_Y X)\eta(Z) \\ & = g(\phi\overline{\nabla}_Y X, TZ + FZ) \\ & = -g(\overline{\nabla}_Y X, T^2Z + FTZ) \\ & \ + g(\overline{\nabla}_Y \phi X - (g(\phi Y, X)\xi - \eta(X)\phi Y), FZ) \\ & = \cos^2\theta g(\nabla_Y X, Z) - g(A_{FTZ}X, Y) + g(A_{FZ}\phi X, Y) \end{aligned}$$

so that

$$\sin^2\theta g(\nabla_Y X, Z) = g(A_{FZ}\phi X - A_{FTZ}X, Y).$$

Therefore, we obtain the result. □

In the same way to Theorem 7, we get

**Theorem 8.** *Let M be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is normal to M and N is one of the following two manifolds: cosymplectic, Kenmotsu.*

*Then the distribution $\mathcal{D}_1$ defines a totally geodesic foliation if and only if*

$$g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0 \tag{76}$$

*for $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$.*

**Theorem 9.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is normal to $M$ and $N$ is one of the following two manifolds: cosymplectic, Kenmotsu.*

*Then the distribution $\mathcal{D}_2$ defines a totally geodesic foliation if and only if*

$$g(A_{FZ}\phi X - A_{FTZ}X, W) = 0 \tag{77}$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$.*

**Proof.** We give its proof when $N$ is Kenmotsu.

Given $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$, by using (21) and Remark 11, we get

$$\begin{aligned}
& g(\nabla_W Z, X) \\
& = g(\phi\overline{\nabla}_W Z, \phi X) + \eta(\overline{\nabla}_W Z)\eta(X) \\
& = g(\overline{\nabla}_W (TZ + FZ) - (g(\phi W, Z)\xi - \eta(Z)\phi W), \phi X) \\
& = -g(\overline{\nabla}_W (T^2 Z + FTZ) - (g(\phi W, TZ)\xi - \eta(TZ)\phi W), X) - g(A_{FZ}W, \phi X) \\
& = \cos^2\theta g(\nabla_W Z, X) + g(A_{FTZ}W, X) - g(A_{FZ}W, \phi X)
\end{aligned}$$

so that

$$\sin^2\theta g(\nabla_W Z, X) = g(A_{FTZ}X - A_{FZ}\phi X, W).$$

Therefore, the result follows. □

In a similar way, we have

**Theorem 10.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is tangent to $M$*

1. *If $N$ is one of the following two manifolds: cosymplectic, Sasakian, then $\mathcal{D}_2$ defines a totally geodesic foliation if and only if*

$$g(A_{FZ}\phi X - A_{FTZ}X, W) = 0 \tag{78}$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$.*

2. *If $N$ is Kenmotsu, then $\mathcal{D}_2$ defines a totally geodesic foliation if and only if*

$$g(A_{FZ}\phi X - A_{FTZ}X, W) + \sin^2\theta\eta(X)g(W, Z) = 0 \tag{79}$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$.*

**Proof.** We only give its proof when $N$ is Sasakian. For the other cases, we can show them in the same way.

Given $X \in \Gamma(\mathcal{D}_1)$ and $Z, W \in \Gamma(\mathcal{D}_2)$, by using (19) and Remark 11, we obtain

$$
\begin{aligned}
& g(\nabla_W Z, X) \\
&= g(\phi\overline{\nabla}_W Z, \phi X) + \eta(\overline{\nabla}_W Z)\eta(X) \\
&= g(\overline{\nabla}_W (TZ + FZ) - (g(W, Z)\xi - \eta(Z)W), \phi X) + \eta(\nabla_W Z)\eta(X) \\
&= -g(\overline{\nabla}_W (T^2 Z + FTZ) - (g(W, TZ)\xi - \eta(TZ)W), X) \\
&\quad - g(A_{FZ}W, \phi X) + \eta(\nabla_W Z)\eta(X) \\
&= \cos^2\theta g(\nabla_W Z, X) + g(A_{FTZ}W, X) + g(W, TZ)\eta(X) \\
&\quad - g(A_{FZ}W, \phi X) + \eta(\nabla_W Z)\eta(X)
\end{aligned}
$$

so that

$$
\begin{aligned}
\sin^2\theta g(\nabla_W Z, X) \quad = \quad & g(A_{FTZ}X - A_{FZ}\phi X, W) \\
& + g(W, TZ)\eta(X) + \eta(\nabla_W Z)\eta(X).
\end{aligned} \tag{80}
$$

Replacing $X$ by $\xi$ at (80), we get

$$
\sin^2\theta\eta(\nabla_W Z) = g(h(W, \xi), FTZ) + g(W, TZ) + \eta(\nabla_W Z)
$$

so that by using (20) and Remark 11,

$$
\begin{aligned}
-\cos^2\theta\eta(\nabla_W Z) &= g(\overline{\nabla}_W \xi, FTZ) + g(W, TZ) \\
&= g(-\phi W, FTZ) + g(W, TZ) \\
&= -\sin^2\theta g(W, TZ) + g(W, TZ) \\
&= \cos^2\theta g(W, TZ),
\end{aligned}
$$

which implies $\eta(\nabla_W Z) = -g(W, TZ)$.

Hence, from (79),

$$
\sin^2\theta g(\nabla_W Z, X) = g(A_{FTZ}X - A_{FZ}\phi X, W).
$$

Therefore, the result follows. □

Using Theorem 7 and Theorem 10, we obtain

**Corollary 5.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is tangent to $M$ and $N$ is one of the following two manifolds: cosymplectic, Sasakian.*

*Then $M$ is locally a Riemannian product manifold of $M_1$ and $M_2$ if and only if*

$$
A_{FZ}\phi X = A_{FTZ}X \tag{81}
$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$, where $M_1$ and $M_2$ are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

Using Theorems 8 and 9, we also obtain

**Corollary 6.** *Let $M$ be a proper pointwise semi-slant submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is normal to $M$ and $N$ is one of the following two manifolds: cosymplectic, Kenmotsu.*

*Then $M$ is locally a Riemannian product manifold of $M_1$ and $M_2$ if and only if*

$$
A_{FZ}\phi X = A_{FTZ}X \tag{82}
$$

*for $X \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mathcal{D}_2)$, where $M_1$ and $M_2$ are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

Let $M$ be a submanifold of a Riemannian manifold $(N, g)$. We call $M$ a *totally umbilic submanifold* of $(N, g)$ if

$$h(X, Y) = g(X, Y)H \quad \text{for } X, Y \in \Gamma(TM), \tag{83}$$

where $H$ is the mean curvature vector field of $M$ in $N$.

**Lemma 9.** *Let $M$ be a pointwise semi-slant totally umbilic submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$. Assume that $\xi$ is tangent to $M$ and $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*Then*

$$H \in \Gamma(F\mathcal{D}_2). \tag{84}$$

**Proof.** We give its proof when $N$ is Kenmotsu.

Since $\xi$ is tangent to $M$, by Proposition 4, $\mu$ is $\phi$-invariant (i.e., $\phi(\mu) = \mu$). Given $X, Y \in \Gamma(\mathcal{D}_1)$ and $Z \in \Gamma(\mu)$, we have

$$\begin{aligned}
&\nabla_X \phi Y + h(X, \phi Y) \\
&= \overline{\nabla}_X \phi Y \\
&= g(\phi X, Y)\xi - \eta(Y)\phi X + \phi \overline{\nabla}_X Y \\
&= g(\phi X, Y)\xi - \eta(Y)\phi X + T\nabla_X Y + F\nabla_X Y + th(X, Y) + fh(X, Y)
\end{aligned}$$

so that by taking the inner product of both sides with $Z$,

$$g(h(X, \phi Y), Z) = g(fh(X, Y), Z). \tag{85}$$

From (85), by (83) we obtain

$$g(X, \phi Y)g(H, Z) = -g(X, Y)g(H, \phi Z). \tag{86}$$

Interchanging the role of $X$ and $Y$,

$$g(Y, \phi X)g(H, Z) = -g(Y, X)g(H, \phi Z). \tag{87}$$

Comparing (86) with (87), we have

$$g(X, Y)g(H, \phi Z) = 0,$$

which means $H \in \Gamma(F\mathcal{D}_2)$. □

Using Lemma 9, we immediately obtain

**Corollary 7.** *Let $M$ be a pointwise semi-slant totally umbilic submanifold of an almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$. Assume that $\xi$ is tangent to $M$ and $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*If $\theta = 0$ on $M$, then $M$ is a totally geodesic submanifold of $N$.*

## 9. Warped Product Submanifolds

In this section we consider the non-existence of some type of warped product pointwise semi-slant submanifolds and investigate the properties of some warped product pointwise semi-slant submanifolds.

**Theorem 11.** *Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M = B \times_f \overline{F}$ a nontrivial warped product submanifold of $N$. Assume that $\xi$ is normal to $M$ and $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*Then there does not exist a proper pointwise semi-slant submanifold $M$ of $N$ such that $\mathcal{D}_1 = T\overline{F}$ and $\mathcal{D}_2 = TB$.*

**Proof.** If $N$ is Sasakian, then by Theorem 2, it is obviously true.

We will prove it when $N$ is Kenmotsu. For the case of $N$ to be cosymplectic, we can prove it in the same way.

Suppose that there exists a proper pointwise semi-slant submanifold $M = B \times_f \overline{F}$ of $N$ such that $\mathcal{D}_1 = T\overline{F}$ and $\mathcal{D}_2 = TB$. We will induce contradiction.

Given $X, Y \in \Gamma(T\overline{F})$ and $Z \in \Gamma(TB)$, by using (8), (21) and Remark 11, we get

$$\begin{aligned}
& Z(\ln f)g(X,Y) \\
&= g(\overline{\nabla}_X Z, Y) \\
&= g(\phi\overline{\nabla}_X Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y) \\
&= g(\overline{\nabla}_X (TZ + FZ) - (g(\phi X, Z)\xi - \eta(Z)\phi X), \phi Y) \\
&= g(\overline{\nabla}_X (TZ + FZ), \phi Y) \\
&= -g(\overline{\nabla}_X (T^2 Z + FTZ) - (g(\phi X, TZ)\xi - \eta(TZ)\phi X), Y) + g(\overline{\nabla}_X FZ, \phi Y) \\
&= \cos^2\theta g(\nabla_X Z, Y) + g(h(X,Y), FTZ) - g(h(X,\phi Y), FZ)
\end{aligned}$$

so that

$$\sin^2\theta Z(\ln f)g(X,Y) = g(h(X,Y), FTZ) - g(h(X,\phi Y), FZ). \tag{88}$$

Interchanging the role of $X$ and $Y$, we have

$$\sin^2\theta Z(\ln f)g(Y,X) = g(h(Y,X), FTZ) - g(h(Y,\phi X), FZ). \tag{89}$$

Comparing (88) with (89), we obtain

$$g(h(X,\phi Y), FZ) = g(h(Y,\phi X), FZ). \tag{90}$$

On the other hand,

$$\begin{aligned}
& g(h(X,\phi Y), FZ) \\
&= g(A_{FZ} X, \phi Y) \\
&= g(-\overline{\nabla}_X FZ, \phi Y) \\
&= g(-\overline{\nabla}_X (\phi Z - TZ), \phi Y) \\
&= -g(g(\phi X, Z)\xi - \eta(Z)\phi X + \phi\overline{\nabla}_X Z, \phi Y) + g(\overline{\nabla}_X TZ, \phi Y) \\
&= -g(\nabla_X Z, Y) + \eta(\overline{\nabla}_X Z)\eta(Y) + g(\nabla_X TZ, \phi Y) \\
&= -Z(\ln f)g(X,Y) + TZ(\ln f)g(X,\phi Y).
\end{aligned}$$

From (90), by using the above result, we obtain

$$TZ(\ln f)g(X,\phi Y) = 0. \tag{91}$$

Replacing $Z$ by $\phi Z$ and $X$ by $\phi X$ at (91), by Remark 11 we get

$$\cos^2\theta Z(\ln f)g(X,Y) = 0,$$

which implies $Z(\ln f) = 0$ so that $f$ is constant, contradiction. □

**Theorem 12.** *Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M = B \times_f \overline{F}$ a nontrivial warped product submanifold of $N$. Assume that $\xi$ is tangent to $M$ and $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*Then there does not exist a proper pointwise semi-slant submanifold $M$ of $N$ such that $\mathcal{D}_1 = T\overline{F}$ and $\mathcal{D}_2 = TB$.*

**Proof.** We will only give its proof when $N$ is Sasakian. For the other cases, we can show them in the same way.

Suppose that there exists a proper pointwise semi-slant submanifold $M = B \times_f \overline{F}$ of $N$ such that $\mathcal{D}_1 = T\overline{F}$ and $\mathcal{D}_2 = TB$. We will also induce contradiction.

Given $X, Y \in \Gamma(T\overline{F})$ and $Z \in \Gamma(TB)$, by using (8), (19), Remarks 10 and 11, we have

$$\begin{aligned}
&Z(\ln f)g(X,Y)\\
&= g(\overline{\nabla}_X Z, Y)\\
&= g(\phi\overline{\nabla}_X Z, \phi Y) + \eta(\overline{\nabla}_X Z)\eta(Y)\\
&= g(\overline{\nabla}_X (TZ + FZ) - (g(X,Z)\xi - \eta(Z)X), \phi Y) + Z(\ln f)\eta(X)\eta(Y)\\
&= g(\overline{\nabla}_X (TZ + FZ), \phi Y) + Z(\ln f)\eta(X)\eta(Y)\\
&= -g(\overline{\nabla}_X (T^2 Z + FTZ) - (g(X,TZ)\xi - \eta(TZ)X), Y)\\
&\ + g(\overline{\nabla}_X FZ, \phi Y) + Z(\ln f)\eta(X)\eta(Y)\\
&= \cos^2\theta g(\nabla_X Z, Y) + g(h(X,Y), FTZ) - g(h(X,\phi Y), FZ) + Z(\ln f)\eta(X)\eta(Y)
\end{aligned}$$

so that

$$\begin{aligned}
\sin^2\theta Z(\ln f)g(X,Y) \quad &= \quad g(h(X,Y), FTZ) - g(h(X,\phi Y), FZ)\\
&\quad\ +Z(\ln f)\eta(X)\eta(Y).
\end{aligned} \tag{92}$$

Replacing $X$ and $Y$ by $\xi$ at (92), by using (20) we obtain

$$\begin{aligned}
\cos^2\theta Z(\ln f) &= -g(h(\xi,\xi), FTZ)\\
&= -g(\overline{\nabla}_\xi \xi, FTZ)\\
&= -g(-\phi\xi, FTZ)\\
&= 0,
\end{aligned}$$

which implies $Z(\ln f) = 0$ so that $f$ is constant, contradiction. □

Now, we will study nontrivial warped product pointwise semi-slant submanifold $M = B \times_f \overline{F}$ of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ such that $\mathcal{D}_1 = TB$ and $\mathcal{D}_2 = T\overline{F}$.

**Lemma 10.** *Let $M = B \times_f \overline{F}$ be a nontrivial warped product proper pointwise semi-slant submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ such that $\mathcal{D}_1 = TB$ and $\mathcal{D}_2 = T\overline{F}$. Assume that $N$ is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

*Then we get*

$$g(A_{FZ}W, X) = g(A_{FW}Z, X) \tag{93}$$

*for $X \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*

**Proof.** We give its proof when $N$ is Kenmotsu.

Given $X \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$, by using (21), (53) and (8), we obtain

$$\begin{aligned}
& g(A_{FZ}W, X) \\
& = g(A_{FZ}X, W) \\
& = -g(\overline{\nabla}_X(\phi Z - TZ), W) \\
& = -g(g(\phi X, Z)\xi - \eta(Z)\phi X + \phi\overline{\nabla}_X Z, W) + g(\overline{\nabla}_X TZ, W) \\
& = g(\overline{\nabla}_X Z, TW + FW) + g(\nabla_X TZ, W) \\
& = g(X(\ln f)Z, TW) + g(A_{FW}Z, X) + g(X(\ln f)TZ, W) \\
& = g(A_{FW}Z, X).
\end{aligned}$$

□

**Lemma 11.** *Let $M = B \times_f \overline{F}$ be a nontrivial warped product proper pointwise semi-slant submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ such that $\mathcal{D}_1 = TB$ and $\mathcal{D}_2 = T\overline{F}$.*

1. *If $N$ is cosymplectic, then*

$$g(A_{FTZ}W, X) = -\phi X(\ln f)g(W, TZ) - \cos^2\theta X(\ln f)g(W, Z) \tag{94}$$

*and*

$$g(A_{FZ}W, \phi X) = (X - \eta(X)\xi)(\ln f)g(W, Z) - \phi X(\ln f)g(TW, Z) \tag{95}$$

*for $X \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*

2. *If $N$ is Sasakian, then*

$$\begin{aligned} g(A_{FTZ}W, X) \quad = \quad & -\eta(X)g(TZ, W) - \phi X(\ln f)g(W, TZ) \\ & -\cos^2\theta X(\ln f)g(W, Z) \end{aligned} \tag{96}$$

*and*

$$g(A_{FZ}W, \phi X) = (X - \eta(X)\xi)(\ln f)g(W, Z) - \phi X(\ln f)g(TW, Z) \tag{97}$$

*for $X \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*

3. *If $N$ is Kenmotsu, then*

$$\begin{aligned} g(A_{FTZ}W, X) \quad = \quad & \cos^2\theta\eta(X)(g(Z, W) - \eta(Z)\eta(W)) \\ & -\phi X(\ln f)g(W, TZ) - \cos^2\theta X(\ln f)g(W, Z) \end{aligned} \tag{98}$$

*and*

$$g(A_{FZ}W, \phi X) = (X - \eta(X)\xi)(\ln f)g(W, Z) - \phi X(\ln f)g(TW, Z) \tag{99}$$

*for $X \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*

**Proof.** We only give its proof when $N$ is Kenmotsu.

Given $X \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$, by using Lemma 10, (21), Lemma 4 and (8), we have

$$\begin{aligned}
& g(A_{FTZ}W, X) \\
& = g(A_{FW}TZ, X) \\
& = -g(\overline{\nabla}_{TZ}(\phi W - TW), X) \\
& = -g(g(\phi TZ, W)\xi - \eta(W)\phi TZ + \phi\overline{\nabla}_{TZ}W, X) + g(\overline{\nabla}_{TZ}TW, X) \\
& = \cos^2\theta\eta(X)g(Z - \eta(Z)\xi, W) + g(\overline{\nabla}_{TZ}W, \phi X) - g(TW, \overline{\nabla}_{TZ}X) \\
& = \cos^2\theta\eta(X)(g(Z, W) - \eta(Z)\eta(W)) - \phi X(\ln f)g(W, TZ) - \cos^2\theta X(\ln f)g(W, Z).
\end{aligned}$$

Replacing $TZ$ and $X$ by $Z$ and $\phi X$, respectively,

$$g(A_{FZ}W, \phi X) = (X - \eta(X)\xi)(\ln f)g(W, Z) - \phi X(\ln f)g(TW, Z).$$

□

To obtain some inequalities on nontrivial warped product proper pointwise semi-slant submanifolds of cosymplectic, Sasakian, Kenmotsu manifolds in the next section, we need to have

**Lemma 12.** *Let $M = B \times_f \overline{F}$ be a nontrivial warped product proper pointwise semi-slant submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ such that $\mathcal{D}_1 = TB$ and $\mathcal{D}_2 = T\overline{F}$.*

1. *If N is cosymplectic, then*
$$g(h(X, Y), FZ) = 0 \tag{100}$$
*and*
$$g(h(X, W), FZ) = -\phi X(\ln f)g(W, Z) + (X - \eta(X)\xi)(\ln f)g(W, TZ) \tag{101}$$
*for $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*
2. *If N is Sasakian, then*
$$g(h(X, Y), FZ) = \eta(Z)g(X, Y) \tag{102}$$
*and*
$$\begin{aligned} g(h(X, W), FZ) \;\; = \;\; & -\eta(X)g(FW, FZ) - \phi X(\ln f)g(W, Z) \\ & +(X - \eta(X)\xi)(\ln f)g(W, TZ) \end{aligned} \tag{103}$$
*for $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*
3. *If N is Kenmotsu, then*
$$g(h(X, Y), FZ) = \eta(Z)g(\phi X, Y) \tag{104}$$
*and*
$$\begin{aligned} g(h(X, W), FZ) \;\; = \;\; & -\eta(X)\eta(W)\eta(FZ) - \phi X(\ln f)g(W, Z) \\ & +(X - \eta(X)\xi)(\ln f)g(W, TZ) \end{aligned} \tag{105}$$
*for $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$.*

**Proof.** We will give its proof when $N$ is Sasakian.

Given $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(T\overline{F})$, by using (19) and (8), we get

$$\begin{aligned} & g(h(X, Y), FZ) \\ & = g(\overline{\nabla}_X Y, \phi Z - TZ) \\ & = -g(\phi\overline{\nabla}_X Y, Z) - g(\overline{\nabla}_X Y, TZ) \\ & = -g(\overline{\nabla}_X \phi Y - (g(X, Y)\xi - \eta(Y)X), Z) + g(Y, \overline{\nabla}_X TZ) \\ & = g(\phi Y, X(\ln f)Z) + \eta(Z)g(X, Y) + g(Y, X(\ln f)TZ) \\ & = \eta(Z)g(X, Y), \end{aligned}$$

which gives (102).

Replacing $X$ by $\phi X$ at (97), we obtain

$$g(h(X, W), FZ) = \eta(X)\eta(A_{FZ}W) - \phi X(\ln f)g(W, Z) + (X - \eta(X)\xi)(\ln f)g(W, TZ).$$

By using (3) and (20),

$$\begin{aligned}\eta(A_{FZ}W) &= g(A_{FZ}W,\xi)\\ &= g(h(W,\xi),FZ)\\ &= g(\overline{\nabla}_W\xi,FZ)\\ &= g(-\phi W,FZ)\\ &= -g(FW,FZ),\end{aligned}$$

which gives (103). □

## 10. Inequalities

We will consider inequalities for the squared norm of the second fundamental form in terms of a warping function and a semi-slant function for a warped product submanifold in cosymplectic manifolds, Sasakian manifolds and Kenmotsu manifolds.

Let $M = B\times_f \overline{F}$ be a $m$-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional almost contact metric manifold $(N,\phi,\xi,\eta,g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is tangent to $M$.

Then by using Remark 11 we can choose a local orthonormal frame $\{e_1, e_2,\cdots, e_{2m_1+1}, v_1, \cdots, v_{2m_2}, w_1,\cdots, w_{2m_2}, u_1,\cdots,u_{2r}\}$ of $TN$ such that $\{e_1,\cdots,e_{2m_1+1}\}\subset\Gamma(\mathcal{D}_1)$, $\{v_1,\cdots,v_{2m_2}\}\subset\Gamma(\mathcal{D}_2)$, $\{w_1,\cdots,w_{2m_2}\}\subset\Gamma(F\mathcal{D}_2)$, $\{u_1,\cdots,u_{2r}\}\subset\Gamma(\mu)$ with the following conditions:

1. $e_{m_1+i} = \phi e_i$, $1\le i\le m_1$, $e_{2m_1+1} = \xi$,
2. $v_{m_2+i} = \sec\theta T v_i$, $1\le i\le m_2$,
3. $w_i = \csc\theta F v_i$, $1\le i\le 2m_2$,
4. $u_{r+i} = \phi u_i$, $1\le i\le r$.

We have $m = 2m_1 + 2m_2 + 1$ and $n = m_1 + 2m_2 + r$.

Using the above notations, we obtain

**Theorem 13.** *Let $M = B\times_f\overline{F}$ be a $m$-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Sasakian manifold $(N,\phi,\xi,\eta,g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is tangent to $M$.*

*Assume that $n = m_1 + 2m_2$.*

*Then we have*

$$||h||^2 \ge 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 + 4m_2\sin^2\theta \tag{106}$$

*with equality holding if and only if $g(h(Z,W),V) = 0$ for $Z,W\in\Gamma(T\overline{F})$ and $V\in\Gamma(TM^{\perp})$.*

**Proof.** Since $\mu = 0$, we get

$$\begin{aligned}||h||^2 &= \sum_{i,j=1}^{2m_1+1} g(h(e_i,e_j),h(e_i,e_j)) + \sum_{i,j=1}^{2m_2} g(h(v_i,v_j),h(v_i,v_j))\\ &\quad + 2\sum_{i=1}^{2m_1+1}\sum_{j=1}^{2m_2} g(h(e_i,v_j),h(e_i,v_j))\\ &= \sum_{i,j=1}^{2m_1+1}\sum_{k=1}^{2m_2} g(h(e_i,e_j),w_k)^2 + \sum_{i,j=1}^{2m_2}\sum_{k=1}^{2m_2} g(h(v_i,v_j),w_k)^2\\ &\quad + 2\sum_{i=1}^{2m_1+1}\sum_{j,k=1}^{2m_2} g(h(e_i,v_j),w_k)^2.\end{aligned}$$

By using Lemma 12 and Remark 10, we obtain

$$
\begin{aligned}
||h||^2 &= \sum_{i,j,k=1}^{2m_2} g(h(v_i,v_j),w_k)^2 \\
&\quad +2\csc^2\theta \sum_{i=1}^{2m_1+1}\sum_{j,k=1}^{2m_2}(-\eta(e_i)g(Fv_j,Fv_k) \\
&\quad -\phi e_i(\ln f)g(v_j,v_k)+(e_i-\eta(e_i)\xi)(\ln f)g(v_j,Tv_k))^2 \\
&= \sum_{i,j,k=1}^{2m_2} g(h(v_i,v_j),w_k)^2 \\
&\quad +2\csc^2\theta \sum_{i=1}^{2m_1}\sum_{j,k=1}^{2m_2}(-\phi e_i(\ln f)\delta_{jk}+e_i(\ln f)g(v_j,Tv_k))^2 \\
&\quad +2\csc^2\theta \sum_{j,k=1}^{2m_2}(-\sin^2\theta\delta_{jk})^2 \\
&= \sum_{i,j,k=1}^{2m_2} g(h(v_i,v_j),w_k)^2 \\
&\quad +2\csc^2\theta \sum_{i=1}^{2m_1}\sum_{j,k=1}^{2m_2}((\phi e_i(\ln f))^2\delta_{jk}+(e_i(\ln f)g(v_j,Tv_k))^2 \\
&\quad -2\phi e_i(\ln f)\delta_{jk}\cdot e_i(\ln f)g(v_j,Tv_k))+4m_2\sin^2\theta,
\end{aligned} \tag{107}
$$

where $\delta_{jk}$ is the Kronecker delta for $1 \le j,k \le 2m_2$.

But

$$
\begin{aligned}
\sum_{i=1}^{2m_1}(\phi e_i(\ln f))^2 &= \sum_{i=1}^{2m_1} g(\phi e_i,\nabla(\ln f))^2 \\
&= \sum_{i=1}^{2m_1} g(e_i,\phi\nabla(\ln f))^2 \\
&= g(\phi\nabla(\ln f),\phi\nabla(\ln f)) \\
&= ||\phi\nabla(\ln f)||^2,
\end{aligned} \tag{108}
$$

$$
\begin{aligned}
\sum_{i=1}^{2m_1}(e_i(\ln f))^2 &= \sum_{i=1}^{2m_1} g(e_i,\nabla(\ln f))^2 \\
&= g(\nabla(\ln f),\nabla(\ln f))-(\eta(\nabla(\ln f)))^2 \\
&= g(\phi\nabla(\ln f),\phi\nabla(\ln f)) \\
&= ||\phi\nabla(\ln f)||^2,
\end{aligned} \tag{109}
$$

$$\delta_{jk}g(v_j,Tv_k)=0, \tag{110}$$

By Remark 11,

$$\begin{aligned}
&\sum_{j,k=1}^{2m_2} g(v_j, Tv_k)^2 &(111)\\
=& \sum_{k=1}^{m_2}\sum_{j=1}^{2m_2} g(v_j, Tv_k)^2 + \sum_{k=1}^{m_2}\sum_{j=1}^{2m_2} g(v_j, Tv_{m_2+k})^2\\
=& \sum_{k=1}^{m_2} g(\sec\theta Tv_k, Tv_k)^2 + \sum_{k=1}^{m_2} g(v_k, \sec\theta(-\cos^2\theta)v_k)^2\\
=& \sum_{k=1}^{m_2} \sec^2\theta\cdot\cos^4\theta + \sum_{k=1}^{m_2}\cos^2\theta\\
=& 2m_2\cos^2\theta.
\end{aligned}$$

Applying (108), (109), (110), (111) to (107), we have

$$\begin{aligned}
||h||^2 = &\sum_{i,j,k=1}^{2m_2} g(h(v_i, v_j), w_k)^2 + 2\csc^2\theta(2m_2||\phi\nabla(\ln f)||^2\\
&+ 2m_2\cos^2\theta||\phi\nabla(\ln f)||^2) + 4m_2\sin^2\theta
\end{aligned}$$

so that

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 + 4m_2\sin^2\theta$$

with equality holding if and only if $g(h(v_i, v_j), w_k) = 0$ for $1 \leq i, j, k \leq 2m_2$.

Therefore, the result follows. □

In the same way, we get

**Theorem 14.** *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is tangent to $M$.*

*Assume that $n = m_1 + 2m_2$.*

*Then we have*

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 \tag{112}$$

*with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.*

**Theorem 15.** *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is tangent to $M$.*

*Assume that $n = m_1 + 2m_2$.*

*Then we have*

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 \tag{113}$$

*with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.*

Let $M = B \times_f \overline{F}$ be a $m$-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional almost contact metric manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is normal to $M$ with $\xi \in \Gamma(\mu)$.

Then by Propositin 4, $\mu$ is $\phi$-invariant.

Using Remark 11, we can choose a local orthonormal frame $\{e_1, e_2, \cdots, e_{2m_1}, v_1, \cdots, v_{2m_2}, w_1, \cdots, w_{2m_2}, u_1, \cdots, u_{2r+1}\}$ of $TN$ such that $\{e_1, \cdots, e_{2m_1}\} \subset \Gamma(\mathcal{D}_1)$, $\{v_1, \cdots, v_{2m_2}\} \subset \Gamma(\mathcal{D}_2)$, $\{w_1, \cdots, w_{2m_2}\} \subset \Gamma(F\mathcal{D}_2)$, $\{u_1, \cdots, u_{2r+1}\} \subset \Gamma(\mu)$ with the following conditions:

1. $e_{m_1+i} = \phi e_i, 1 \leq i \leq m_1$,
2. $v_{m_2+i} = \sec\theta T v_i, 1 \leq i \leq m_2$,
3. $w_i = \csc\theta F v_i, 1 \leq i \leq 2m_2$,
4. $u_{r+i} = \phi u_i, 1 \leq i \leq r, u_{2r+1} = \xi$.

We have $m = 2m_1 + 2m_2$ and $n = m_1 + 2m_2 + r$.

Notice that if $N$ is Sasakian, then from Theorem 2, there does not exist such a proper pointwise semi-slant submanifold $M$ of $N$.

Using these notations, in a similar way, we obtain

**Theorem 16.** *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is normal to $M$ with $\xi \in \Gamma(\mu)$.*

*Assume that $n = m_1 + 2m_2$.*

*Then we have*

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2 + 2m_1 \tag{114}$$

*with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.*

**Proof.** Since $\mu =< \xi >$, we obtain

$$\begin{aligned}
||h||^2 &= \sum_{i,j=1}^{2m_1} g(h(e_i, e_j), h(e_i, e_j)) + \sum_{i,j=1}^{2m_2} g(h(v_i, v_j), h(v_i, v_j)) \\
&+ 2\sum_{i=1}^{2m_1}\sum_{j=1}^{2m_2} g(h(e_i, v_j), h(e_i, v_j)) \\
&= \sum_{i,j=1}^{2m_1}(\sum_{k=1}^{2m_2} g(h(e_i, e_j), w_k)^2 + (\eta(h(e_i, e_j)))^2) \\
&+ \sum_{i,j=1}^{2m_2}(\sum_{k=1}^{2m_2} g(h(v_i, v_j), w_k)^2 + (\eta(h(v_i, v_j)))^2) \\
&+ 2\sum_{i=1}^{2m_1}(\sum_{j,k=1}^{2m_2} g(h(e_i, v_j), w_k)^2 + (\eta(h(e_i, v_j)))^2).
\end{aligned}$$

Using (22), we can easily check that $\eta(h(e_i, e_j)) = -\delta_{ij}$ and $\eta(h(e_i, v_k)) = 0$ for $1 \leq i, j \leq 2m_1$ and $1 \leq k \leq 2m_2$ so that by using Lemma 12,

$$\begin{aligned}
||h||^2 &= 2m_1 + \sum_{i,j=1}^{2m_2}(\sum_{k=1}^{2m_2} g(h(v_i, v_j), w_k)^2 + (\eta(h(v_i, v_j)))^2) \\
&+ 2\csc^2\theta \sum_{i=1}^{2m_1}\sum_{j,k=1}^{2m_2}(-\eta(e_i)\eta(v_j)\eta(Fv_k) \\
&- \phi e_i(\ln f)g(v_j, v_k) + (e_i - \eta(e_i)\xi)(\ln f)g(v_j, Tv_k))^2 \\
&= 2m_1 + \sum_{i,j=1}^{2m_2}(\sum_{k=1}^{2m_2} g(h(v_i, v_j), w_k)^2 + (\eta(h(v_i, v_j)))^2) \\
&+ 2\csc^2\theta \sum_{i=1}^{2m_1}\sum_{j,k=1}^{2m_2}((\phi e_i(\ln f))^2\delta_{jk} + (e_i(\ln f)g(v_j, Tv_k))^2 \\
&- 2\phi e_i(\ln f)\delta_{jk} \cdot e_i(\ln f)g(v_j, Tv_k)).
\end{aligned}$$

In a similar way to the proof of Theorem 13, we also derive the following:

$$\sum_{i=1}^{2m_1}(\phi e_i(\ln f))^2 = ||\nabla(\ln f)||^2,$$
$$\sum_{i=1}^{2m_1}(e_i(\ln f))^2 = ||\nabla(\ln f)||^2,$$
$$\sum_{j,k=1}^{2m_2}(g(v_j, Tv_k))^2 = 2m_2\cos^2\theta,$$
$$\delta_{jk}g(v_j, Tv_k) = 0$$

so that

$$||h||^2 \geq 2m_1 + 4m_2(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2$$

with equality holding if and only if $g(h(v_i, v_j), w_k) = 0$ and $g(h(v_i, v_j), \xi) = 0$ for $1 \leq i, j, k \leq 2m_2$.

Therefore, the result follows. □

In the same way, we get

**Theorem 17.** *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold $(N, \phi, \xi, \eta, g)$ with the semi-slant function $\theta$ such that $\mathcal{D}_1 = TB$, $\mathcal{D}_2 = T\overline{F}$ and $\xi$ is normal to M with $\xi \in \Gamma(\mu)$.*

*Assume that $n = m_1 + 2m_2$.*

*Then we have*

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2 \tag{115}$$

*with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.*

## 11. Examples

**Example 7.** *Define a map $i : \mathbb{R}^4 \mapsto \mathbb{R}^{11}$ by*

$$i(x_1, x_2, x_3, x_4) = (y_1, y_2, \cdots, y_{10}, t) = (x_2\sin x_3, x_1\sin x_3,$$
$$x_2\sin x_4, x_1\sin x_4, x_2\cos x_3, x_1\cos x_3, x_2\cos x_4, x_1\cos x_4, x_3, x_4, 0)$$

*Let $M := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 0 < x_1, x_2 < 1,\ 0 < x_3, x_4 < \frac{\pi}{2}\}$.*

*We define $(\phi, \xi, \eta, g)$ on $\mathbb{R}^{11}$ as follows:*

$$\phi(a_1\tfrac{\partial}{\partial y_1} + \cdots + a_{10}\tfrac{\partial}{\partial y_{10}} + a_{11}\tfrac{\partial}{\partial t}) := \sum_{i=1}^{5}(-a_{2i}\tfrac{\partial}{\partial y_{2i-1}} + a_{2i-1}\tfrac{\partial}{\partial y_{2i}}),$$
$$\xi := \tfrac{\partial}{\partial t},\ \eta := dt,\ a_i \in \mathbb{R},\ 1 \leq i \leq 11,$$

*g is the Euclidean metric on $\mathbb{R}^{11}$.*

*We easily check that $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $\mathbb{R}^{11}$. Then M is a pointwise semi-slant submanifold of $\mathbb{R}^{11}$ with the semi-slant function $k(x_1, x_2, x_3, x_4) = \arccos(\dfrac{1}{x_1^2 + x_2^2 + 1})$ such that $\xi$ is normal to M and*

$$\mathcal{D}_1 = <\sin x_3\tfrac{\partial}{\partial y_2} + \cos x_3\tfrac{\partial}{\partial y_6} + \sin x_4\tfrac{\partial}{\partial y_8} + \cos x_4\tfrac{\partial}{\partial y_{10}},$$
$$\sin x_3\tfrac{\partial}{\partial y_1} + \cos x_3\tfrac{\partial}{\partial y_5} + \sin x_4\tfrac{\partial}{\partial y_7} + \cos x_4\tfrac{\partial}{\partial y_9}>,$$

$$\mathcal{D}_2 = < x_2 \cos x_3 \frac{\partial}{\partial y_1} + x_1 \cos x_3 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3} - x_2 \sin x_3 \frac{\partial}{\partial y_5} - x_1 \sin x_3 \frac{\partial}{\partial y_6},$$
$$\frac{\partial}{\partial y_4} + x_2 \cos x_4 \frac{\partial}{\partial y_7} + x_1 \cos x_4 \frac{\partial}{\partial y_8} - x_2 \sin x_4 \frac{\partial}{\partial y_9} - x_1 \sin x_4 \frac{\partial}{\partial y_{10}} > .$$

*Notice that* $(\mathbb{R}^{11}, \phi, \xi, \eta, g)$ *is cosymplectic.*

**Example 8.** *Define a map* $i : \mathbb{R}^5 \mapsto \mathbb{R}^7$ *by*

$$\begin{aligned} i(x_1, x_2, \cdots, x_5) &= (y_1, y_2, \cdots, y_6, t) \\ &= (x_3, x_1, x_5, \sin x_4, 0, \cos x_4, x_2). \end{aligned}$$

*Let* $M := \{(x_1, x_2, \cdots, x_5) \in \mathbb{R}^5 \mid 0 < x_4 < \frac{\pi}{2}\}$.

*We define* $(\phi, \xi, \eta, g)$ *on* $\mathbb{R}^7$ *as follows:*

$$\phi(a_1 \frac{\partial}{\partial y_1} + \cdots + a_6 \frac{\partial}{\partial y_6} + a_7 \frac{\partial}{\partial t}) := \sum_{i=1}^{3} (-a_{2i} \frac{\partial}{\partial y_{2i-1}} + a_{2i-1} \frac{\partial}{\partial y_{2i}}),$$
$$\xi := \frac{\partial}{\partial t},\ \eta := dt,\ a_i \in \mathbb{R},\ 1 \leq i \leq 7,$$

*g is the Euclidean metric on* $\mathbb{R}^7$. *It is easy to check that* $(\phi, \xi, \eta, g)$ *is an almost contact metric structure on* $\mathbb{R}^7$.

*Then M is a pointwise semi-slant submanifold of* $\mathbb{R}^7$ *with the semi-slant function* $k(x_1, \cdots, x_5) = x_4$ *such that* $\xi$ *is tangent to M and*

$$\mathcal{D}_1 =< \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \xi >$$
$$\mathcal{D}_2 =< \frac{\partial}{\partial y_3}, \cos x_4 \frac{\partial}{\partial y_4} - \sin x_4 \frac{\partial}{\partial y_6} > .$$

**Example 9.** *Let* $(N, \phi, \xi, \eta, g_N)$ *be an almost contact metric manifold. Let M be a submanifold of a hyperkähler manifold* $(\overline{M}, J_1, J_2, J_3, g_{\overline{M}})$ *such that M is complex with respect to the complex structure* $J_1$ *(i.e.,* $J_1(TM) = TM$*) and totally real with respect to the complex structure* $J_2$ *(i.e.,* $J_2(TM) \subset TM^{\perp}$*) [40]. Let* $f : \overline{M} \mapsto [0, \frac{\pi}{2}]$ *be a* $C^\infty$*-function. Let* $\overline{N} := \overline{M} \times N$ *with the natural projections* $\pi_1 : \overline{N} \mapsto \overline{M}$ *and* $\pi_2 : \overline{N} \mapsto N$.

*We define* $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *on* $\overline{N}$ *as follows:*

$$\begin{aligned} &\overline{\phi}(X+Y) := \cos(f \circ \pi_1) J_1 X - \sin(f \circ \pi_1) J_2 X + \phi Y, \\ &\overline{\xi} := \xi, \quad \overline{\eta} := \eta, \\ &\overline{g}(Z, W) := g_{\overline{M}}(d\pi_1(Z), d\pi_1(W)) + g_N(d\pi_2(Z), d\pi_2(W)) \end{aligned}$$

*for* $X \in \Gamma(T\overline{M})$, $Y \in \Gamma(TN)$, $Z, W \in \Gamma(T\overline{N})$.

*Here,* $\overline{\xi}$ *is exactly the horizontal lift of* $\xi$ *along* $\pi_2$ *and* $\overline{\eta}(Z) := \eta(d\pi_2(Z))$. *Conveniently, we identify a vector field on* $\overline{M}$ *(or on N) with its horizontal lift.*

*We can easily check that* $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *is an almost contact metric structure on* $\overline{N}$.

*Then* $M \times N$ *is a pointwise semi-slant submanifold of an almost contact metric manifold* $(\overline{N}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *with the semi-slant function* $f \circ \pi_1$ *such that* $\overline{\xi}$ *is tangent to* $M \times N$ *and* $\mathcal{D}_1 = TN$, $\mathcal{D}_2 = TM$.

**Example 10.** *Define* $(\phi, \xi, \eta, g)$ *on* $\mathbb{R}^{11}$ *as follows:*

$$\phi(a_1 \frac{\partial}{\partial y_1} + \cdots + a_{10} \frac{\partial}{\partial y_{10}} + a_{11} \frac{\partial}{\partial t}) = \sum_{i=1}^{5} (-a_{2i} \frac{\partial}{\partial y_{2i-1}} + a_{2i-1} \frac{\partial}{\partial y_{2i}}),$$
$$\xi = \frac{\partial}{\partial t},\ \eta = dt,\ a_i \in \mathbb{R},\ 1 \leq i \leq 11,$$

*g is the Euclidean metric on* $\mathbb{R}^{11}$. *Then we know that* $\mathbb{R}^{11} = (\mathbb{R}^{11}, \phi, \xi, \eta, g)$ *is a cosymplectic manifold.*

*Let*

$$M = \{(x_1, x_2, u, v) \mid 0 < x_i < 1, i = 1, 2, \quad 0 < u, v < \frac{\pi}{2}\}.$$

*Take two points* $P_1, P_2$ *in the unit sphere* $S^1$ *such that*

$$
\begin{aligned}
&P_i = (a_{1i}, a_{2i}), \quad i = 1, 2, \\
&a_{11}a_{12} + a_{21}a_{22} = 0, \\
&-a_{11}a_{22} + a_{21}a_{12} \neq 0.
\end{aligned}
$$

*We define a map* $i : M \subset \mathbb{R}^4 \mapsto \mathbb{R}^{11}$ *by*

$$
\begin{aligned}
&i(x_1, x_2, u, v) = (y_1, y_2, \cdots, y_{10}, t) \\
&= (x_1 \cos u, x_2 \cos u, x_1 \cos v, x_2 \cos v, \\
&x_1 \sin u, x_2 \sin u, x_1 \sin v, x_2 \sin v, \\
&a_{11}u + a_{12}v, a_{21}u + a_{22}v, 2020).
\end{aligned}
$$

*Then the tangent bundle* $TM$ *is spanned by* $X_1, X_2, Y_1, Y_2$*, where*

$$
\begin{aligned}
&X_1 = \cos u \tfrac{\partial}{\partial y_1} + \cos v \tfrac{\partial}{\partial y_3} + \sin u \tfrac{\partial}{\partial y_5} + \sin v \tfrac{\partial}{\partial y_7}, \\
&X_2 = \cos u \tfrac{\partial}{\partial y_2} + \cos v \tfrac{\partial}{\partial y_4} + \sin u \tfrac{\partial}{\partial y_6} + \sin v \tfrac{\partial}{\partial y_8}, \\
&Y_1 = -x_1 \sin u \tfrac{\partial}{\partial y_1} - x_2 \sin u \tfrac{\partial}{\partial y_2} + x_1 \cos u \tfrac{\partial}{\partial y_5} \\
&+ x_2 \cos u \tfrac{\partial}{\partial y_6} + a_{11} \tfrac{\partial}{\partial y_9} + a_{21} \tfrac{\partial}{\partial y_{10}}, \\
&Y_2 = -x_1 \sin v \tfrac{\partial}{\partial y_3} - x_2 \sin v \tfrac{\partial}{\partial y_4} + x_1 \cos v \tfrac{\partial}{\partial y_7} \\
&+ x_2 \cos v \tfrac{\partial}{\partial y_8} + a_{12} \tfrac{\partial}{\partial y_9} + a_{22} \tfrac{\partial}{\partial y_{10}}.
\end{aligned}
$$

*We can easily check that* $M$ *is a proper pointwise semi-slant submanifold of a* 11*-dimensional cosymplectic manifold* $\mathbb{R}^{11} = (\mathbb{R}^{11}, \phi, \xi, \eta, g)$ *such that* $\mathcal{D}_1 =< X_1, X_2 >$, $\mathcal{D}_2 =< Y_1, Y_2 >$*, the semi-slant functions* $\theta$ *with*

$$
\cos\theta = \frac{|-a_{11}a_{22} + a_{21}a_{12}|}{1 + x_1^2 + x_2^2},
$$

$\xi$ *is normal to* $M$ *with* $\xi \in \Gamma(\mu)$.

*We see that the distributions* $\mathcal{D}_1$, $\mathcal{D}_2$ *are integrable. Denote by* $B$, $F$ *the integral manifolds of* $\mathcal{D}_1$, $\mathcal{D}_2$*, respectively.*

*Then we see that* $M = (M, g)$ *is a non-trivial warped product Riemannian submanifold of* $\mathbb{R}^{11}$ *such that*

$$
\begin{aligned}
&M = B \times_f F, \\
&g = 2(dx_1^2 + dx_2^2) + (1 + x_1^2 + x_2^2)(du^2 + dv^2), \\
&\text{the warping function } f = \sqrt{1 + x_1^2 + x_2^2}.
\end{aligned}
$$

*Hence,* $M$ *is a non-trivial warped product proper pointwise semi-slant submanifold of* $(\mathbb{R}^{11}, \phi, \xi, \eta, g)$.

*By Theorem 17, we obtain*

$$
||h||^2 \geq 4\left(\frac{(1 + x_1^2 + x_2^2)^2 + (-a_{11}a_{22} + a_{21}a_{12})^2}{(1 + x_1^2 + x_2^2)^2 - (-a_{11}a_{22} + a_{21}a_{12})^2}\right)||\nabla(\frac{1}{2}\ln(1 + x_1^2 + x_2^2))||^2. \tag{116}
$$

**Funding:** This research received no external funding.

**Conflicts of Interest:** The author declares no conflict of interest.

## References


1. Gray, A. Almost Complex Submanifolds of the Six Sphere. *Proc. Am. Math. Soc.* **1969**, *20*, 277–279. [CrossRef]
2. Kon, M. On some complex submanifolds in Kaehler manifolds. *Can. J. Math.* **1974**, *26*, 1442–1449. [CrossRef]
3. Kimura, M. Real hypersurfaces and complex submanifolds in complex projective space. *Trans. Am. Math. Soc.* **1986**, *296*, 137–149. [CrossRef]
4. Namba, M. On maximal families of compact complex submanifolds of complex manifolds. *Tohoku Math. J.* **1972**, *24*, 581–609. [CrossRef]
5. Chen, B.Y.; Dillen, F.; Verstraelen, L.; Vrancken, L. Totally real submanifolds of $\mathbb{CP}^n$ satisfying a basic equality. *Arch. Math. (Basel)* **1994**, *63*, 553–564. [CrossRef]
6. Chen, B.Y.; Ogiue, K. On totally real submanifolds. *Trans. Am. Math. Soc.* **1974**, *193*, 257–266. [CrossRef]
7. Chen, B.Y.; Nagano, T. Totally geodesic submanifolds of symmetric spaces, I. *Duke Math. J.* **1977**, *44*, 745–755. [CrossRef]
8. Ejiri, N. Totally real submanifolds in a 6-sphere. *Proc. Am. Math. Soc.* **1981**, *83*, 759–763.
9. Bejancu, A. *Geometry of CR-Submanifolds*; Mathematics and its Applications (East European Series); 23. D. Reidel Publishing Co.: Dordrecht, The Netherland, 1986.
10. Chen, B.Y. CR-submanifolds of a Kaehler manifold. I. *J. Differ. Geom.* **1981**, *16*, 305–322. [CrossRef]
11. Sekigawa, K. Some CR-submanifolds in a 6-dimensional sphere. *Tensor (N.S.)* **1984**, *41*, 13–20.
12. Yano, K.; Kon, M. *CR Submanifolds of Kaehlerian and Sasakian Manifolds*; Progress in Mathematics, 30; Birkhauser: Boston, MA, USA, 1983.
13. Bejancu, A. QR-submanifolds of quaternion Kaehler manifolds. *Chin. J. Math.* **1986**, *14*, 81–94.
14. Bejancu, A.; Farran, H.R. On totally umbilical QR-submanifolds of quaternion Kaehlerian manifolds. *Bull. Aust. Math. Soc.* **2000**, *62*, 95–103. [CrossRef]
15. Güneş, R.; Şahin, B.; Keleş, S. QR-submanifolds and almost contact 3-structure. *Turk. J. Math.* **2000**, *24*, 239–250.
16. Kwon, J.H.; Pak, J.S. QR-submanifolds of $(p-1)$ QR-dimension in a quaternionic projective space $\mathbb{QP}^{(n+p)/4}$. *Acta Math. Hungar.* **2000**, *86*, 89–116. [CrossRef]
17. Chen, B.Y. Slant immersions. *Bull. Aust. Math. Soc.* **1990**, *41*, 135–147. [CrossRef]
18. Chen, B.Y. *Geometry of Slant Submaniflods*; Katholieke Universiteit Leuven: Leuven, Belgium, 1990.
19. Cabrerizo, J.L.; Carriazo, A.; Fernandez, L.M.; Fernandez, M. Slant submanifolds in Sasakian manifolds. *Glasg. Math. J.* **2000**, *42*, 125–138. [CrossRef]
20. Lotta, A. Slant submanifolds in contact geometry. *Bull. Math. Soc. Roumanie* **1996**, *39*, 183–198.
21. Matsumoto, K.; Mihai, I.; Tazawa, Y. Ricci tensor of slant submanifolds in complex space forms. *Kodai Math. J.* **2003**, *26*, 85–94. [CrossRef]
22. Şahin, B. Slant submanifolds of quaternion Kaehler manifolds. *Commum. Korean Math. Soc.* **2007**, *22*, 123–135. [CrossRef]
23. Chen, B.Y.; Garay, O.J. Pointwise slant submanifolds in almost Hermitian manifolds. *Turk. J. Math.* **2012**, *36*, 630–640.
24. Etayo, F. On quasi-slant submanifolds of an almost Hermitian manifold. *Publ. Math. Debrecen* **1998**, *53*, 217–223.
25. Park, K.S. On the Pointwise Slant Submanifolds. In *Hermitian-Grassmannian Submanifolds*; Suh, Y., Ohnita, Y., Zhou, J., Kim, B., Lee, H., Eds.; Springer Proceedings in Mathematics & Statistics, 203; Springer: Singapore, 2017.
26. Cabrerizo, J.L.; Carriazo, A.; Fernandez, L.M.; Fernandez, M. Semi-slant submanifolds of a Sasakian manifold. *Geom. Dedicata* **1999**, *78*, 183–199. [CrossRef]
27. Khan, V.A.; Khan, M.A.; Khan, K.A. Slant and semi-slant submanifolds of a Kenmotsu manifold. *Math. Slovaca* **2007**, *57*, 483–494. [CrossRef]
28. Papaghiuc, N. Semi-slant submanifolds of a Kaehlerian manifold. *An. Ştiinţ. Al. I. Univ. Iaşi.* **1994**, *40*, 55–61.
29. Şahin, B. Non-existence of warped product semi-slant submanifolds of Kaehler manifolds. *Geom. Dedicata* **2006**, *117*, 195–202. [CrossRef]
30. Şahin, B. Warped product pointwise semi-slant submanifolds of Kaehler manifolds. *Port. Math.* **2013**, *70*, 252–268. [CrossRef]
31. Park, K.S. Pointwise almost h-semi-slant submanifolds. Internat. *J. Math.* **2015**, *26*, 1550099.
32. Arslan, K.; Carriazo, A.; Chen, B.Y.; Murathan, C. On slant submanifolds of neutral Kaehler manifolds. *Taiwan. J. Math.* **2010**, *14*, 561–584. [CrossRef]

33. Khan, V.A.; Khan, M.A. Semi-Slant Submanifolds of a Nearly Kaehler Manifold. *Turkish J. Math.* **2007**, *31*, 341–353.
34. Li, H.; Liu, X. Semi-Slant Submanifolds of a Locally Product Manifold. *Georgian Math. J.* **2005**, *12*, 273–282.
35. Chen, B.Y. A survey on geometry of warped product submanifolds. *arXiv* **2013**, arXiv:1307.0236v1.
36. Blair, D.E. *Riemannian Geometry of Contact and Symplectic Manifolds*; Springer:Heidelberg, Germany, 2010.
37. Falcitelli, M.; Ianus, S.; Pastore, A.M. *Riemannian Submersions and Related Topics*; World Scientific Publishing Co.: Singapore, 2004.
38. Yano, K.; Kon, M. *Anti-Invariants Submanifolds*; Lecture Notes in Pure and Applied Mathematics, 21; Marcel Dekker: New York, NY, USA, 1976.
39. Kenmotsu, K. A class of almost contact Riemannian manifolds. *Tohoku Math. J.* **1972**, *24*, 93–103. [CrossRef]
40. Besse, A.L. *Einstein Manifolds*; Springer: Berlin, Germany, 1987.