Sum of Soft Topological Spaces

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Abstract: In this paper, we introduce the concept of sum of soft topological spaces using pairwise disjoint soft topological spaces and study its basic properties. Then, we define additive and finitely additive properties which are considered a link between soft topological spaces and their sum. In this regard, we show that the properties of being p-soft T_i, soft paracompactness, soft extremally disconnectedness, and soft continuity are additive. We provide some examples to elucidate that soft compactness and soft separability are finitely additive; however, soft hyperconnected, soft indiscrete, and door soft spaces are not finitely additive. In addition, we prove that soft interior, soft closure, soft limit, and soft boundary points are interchangeable between soft topological spaces and their sum. This helps to obtain some results related to some important generalized soft open sets. Finally, we observe under which conditions a soft topological space represents the sum of some soft topological spaces.

Keywords: sum of soft topological spaces; additive property; p-soft T_i; soft compactness; soft paracompactness; soft extremally disconnectedness; soft separable; soft continuity

1. Introduction

In 1999, Molodtsov [1] introduced the concept of soft sets as an innovative approach to deal with uncertainties. He demonstrated that soft sets are a beneficial mathematical method to handle with uncertainty in a parametric manner. Then, Maji et al. have completed two significant works, the first one has showed an application of soft set theory on decision-making problems [2], and the second one has presented the basic concepts between two soft sets such as soft union, intersection, and equality relations [3]. Ali et al. [4] have made some amendments for some results obtained by [3] and have defined new types of soft union and intersection between two soft sets. They have presented these types along with a relative complement of a soft set to keep the De Morgan's laws on a soft setting.

In 2011, Shabir and Naz [5] initiated the concept of soft topological spaces using soft sets that are defined over an initial universe set with a fixed set of parameters. After the inception of soft topology, many authors have investigated soft topological concepts analogously with their counterparts on classical topology. The different types of belong and non-belong relations on soft setting leads to introducing several types of soft axioms in terms of ordinary points [5–8] and soft points [9,10]. The authors of [11–14] have corrected some alleged findings on soft separation axioms. The author in [15] has presented and studied soft compactness. Then, the authors in [16] have defined other types of soft compactness depending on the natural belonging of the ordinary points to the covers. The authors in [17] have explored weak types of soft compact spaces, namely almost soft compact and mildly soft compact spaces. Investigation of soft compactness using soft pre-open and soft semi-open sets have been done by [18] and [19], respectively. The authors in [20] have discussed the concepts of

soft connectedness and soft paracompactness. The authors in [21] have introduced a concept of soft hyperconnected spaces, and the authors [22] have studied a concept of soft extremally disconnected spaces. Kharal and Ahmad [23] were the first one who introduced the concept of soft mappings and proved main properties. Then, the authors of [24] have defined and discussed the concepts of soft continuous and homeomorphism maps. The study of generalized soft open sets began by Chen [25]. He defined soft semi-open sets and discussed main properties. Following Chen, the concepts of soft α-open, soft b-open, soft pre-open, soft β-open, and soft somewhere dense sets have been given in [26–29] and [30], respectively. In [31], the authors have presented the concept of the pointwise topology of soft topological spaces and investigated the properties of soft mapping spaces. Recently, the authors [32] have proved the equivalence between two types of soft topology and have investigated the links between soft topology and its parameterized topologies.

We aim through this work to introduce and study the concept of sum of soft topological spaces using pairwise disjoint soft topological spaces. Our results mainly investigate invariant properties between soft topological spaces and their sum. We define additive and finitely additive properties. With the help of illustrative examples, many properties such as p-soft T_i, soft paracompactness, soft continuity, soft compactness, soft separability, and soft discrete are investigated regarding whether they are additive, finitely additive, or not. In addition, we made use of interchangeability of soft interior and soft closure operators between soft topological spaces and their sum to obtain some results related to some important generalized soft open sets. Ultimately, we study under what conditions a soft topological space represents the sum of some soft topological spaces.

2. Preliminaries

In what follows, we first recall the main definitions and results which will be used through this work.

2.1. Soft Sets

Throughout the paper, Y will be a nonempty set, called an *initial universal set*, 2^Y its power set, E a nonempty set, called the *set of parameters*, $A, B, C, ...$ subsets of E. Let us mention that almost all definitions are given for soft sets having a common set of parameters A.

Definition 1. *[1] A soft set over Y is an ordered pair (ξ, A) such that $A \subset E$ is a set of parameters and ξ is a mapping of A into 2^Y.*

We often write a soft set (ξ, A) as

$$(\xi, A) = \{(a, \xi(a)) : a \in A \text{ and } \xi(a) \in 2^Y\}.$$

Through this work, the collection of all soft sets over Y under a set of parameters A is denoted by $SS(Y_A)$. In addition, we use the different notations $(\omega, B), (\delta, C), (\eta, D)$ for soft sets.

Definition 2. *[3] A soft set (ξ, A) over Y is said to be the null soft set if $\xi(a) = \varnothing$ for all $a \in A$; and it is said to be the absolute soft set if $\xi(a) = Y$ for all $a \in A$.*
The null and absolute soft sets are denoted by $\widetilde{\varnothing_Y}$ and $\widetilde{Y}$, respectively.
A soft set (P, A) over Y is called a soft point if there exist $a \in A$ and $y \in Y$ such that $P(a) = \{y\}$ and $P(b) = \varnothing$, for each $b \in A \setminus \{a\}$. A soft point will be shortly denoted by P_a^y and we say that $P_a^y \in (\xi, A)$, if $y \in \xi(a)$ [33].

Definition 3. *[34] (ξ, A) is a soft subset of (ω, B), denoted by $(\xi, A) \widetilde{\subseteq} (\omega, B)$, if A is a subset of B, and $\xi(a)$ is a subset of $\omega(a)$ for all $a \in A$. The two soft sets are soft equal if each of them is a soft subset of the other.*

Definition 4. *[4] The relative complement of (ξ, A) is a soft set $(\xi, A)^c = (\xi^c, A)$ such that the map $\xi^c : A \rightarrow 2^Y$ is defined by $\xi^c(a) = Y \setminus \xi(a)$ for each $a \in A$.*

Definition 5. *[3] The union of soft sets (ξ, A) and (ω, B) over Y, denoted by $(\xi, A) \widetilde{\bigcup} (\omega, B)$, is the soft set (δ, C), where $C = A \bigcup B$ and $\delta : C \to 2^Y$ is a mapping defined by*

$$
\delta(c) = \begin{cases}
\xi(c) & : \quad c \in A \setminus B \\
\omega(c) & : \quad c \in B \setminus A \\
\xi(c) \bigcup \omega(c) & : \quad c \in A \bigcap B.
\end{cases}
$$

Definition 6. *[4] The intersection of soft sets (ξ, A) and (ω, B) over Y, denoted by $(\xi, A) \widetilde{\bigcap} (\omega, B)$, is a soft set (δ, C), where $C = A \bigcap B \neq \emptyset$ and $\delta : C \to 2^Y$ is a mapping defined by $\delta(c) = \xi(c) \bigcap \omega(c)$.*

However, in this paper, we need definitions of the union and intersection for an arbitrary family of soft sets over a common universe Y and with a common set of parameters.

Definition 7. *[3] The union of a family $\{(\xi_i, A) : i \in I\}$ of soft sets over the common universe Y, denoted $\widetilde{\bigcup}_{i \in I} (\xi_i, A)$, is the soft set (η, A), where, for each $a \in A$, $\eta(a) = \bigcup_{i \in I} \xi_i(a)$.*

The intersection of a family $\{(\xi_i, A); i \in I\}$ over the common universe Y, denoted $\widetilde{\bigcap}_{i \in I} (\xi_i, A)$, is the soft set (η, A), where, for each $a \in A$, $\eta(a) = \bigcap_{i \in I} \xi_i(a)$.

Definition 8. *[33] A soft mapping between $\mathsf{SS}(Y_A)$ and $\mathsf{SS}(Z_A)$ is a mapping $f : Y \to Z$ such that the image of $(\xi, A) \in \mathsf{SS}(Y_A)$ and preimage of $(\theta, A) \in \mathsf{SS}(Z_A)$ are defined by:*

1. *$f(\xi, A) = (f(\xi), A)$, where $[f(\xi)](a) = f(\xi(a))$, $a \in A$;*
2. *$f^{\leftarrow}(\theta, A) = (f^{\leftarrow}(\theta), A)$, where $[f^{\leftarrow}(\theta)](a) = f^{\leftarrow}(\theta(a))$, $a \in A$.*

2.2. Soft Topology

Definition 9. *[5] The family τ of soft sets over Y under a set of parameters A is called a* soft topology *on Y provided that it is closed under arbitrary union and finite intersection and contains $\widetilde{Y}$ and $\widetilde{\emptyset}_Y$.*

The triple (Y, τ, A) is called a soft topological space. An element (ξ, A) is called a soft open set (resp. soft closed set*) if (ξ, A) (resp. $(\xi, A)^c$) belongs to τ.*

A soft set (ξ, A) in a soft topological space (Y, τ, A) is called a *soft neighborhood* of the soft point P_a^y in (Y, τ, A) if there exists a soft open set (η, A) such that $P_a^y \in (\eta, A) \widetilde{\subseteq} (\xi, A)$.

Definition 10. *[5] Let (ξ, A) be a soft subset of (Y, τ, A). Then,*

1. *The interior of (ξ, A), denoted by $\mathrm{Int}(\xi, A)$, is the union of all soft open sets contained in (ξ, A).*
2. *The closure of (ξ, A), denoted by $\mathrm{Cl}(\xi, A)$, is the intersection of all soft closed sets containing (ξ, A).*
3. *(ξ, A) is dense in (Y, τ, A) if its closure is equal to the set $\widetilde{Y}$. The space (Y, τ, A) is separable if it contains a countable dense soft set [17]. (Recall that a soft set (ξ, A) is countable if $\xi(a)$ is countable for each $a \in A$.)*

Definition 11. *For a subset $Z \neq \emptyset$ of (Y, τ, A), the family $\tau_Z = \{\widetilde{Z} \widetilde{\bigcap} (\xi, A) : (\xi, A) \in \tau\}$ is called a soft relative topology on Z and the triple (Z, τ_Z, A) is called a soft subspace of (Y, τ, A).*

Proposition 1. *[5] Let (Y, τ, A) be a soft topological space. Then, for each $a \in A$, the family $\tau_a = \{\xi(a) : (\xi, A) \in \tau\}$ defines a topology on Y for each $a \in A$.*

We call τ_a a *parametric topology* on Y.

Definition 12. *[5,7] Let (ξ, A) be a soft set over Y and $y \in Y$. We write:*

1. *$y \in (\xi, A)$ if $y \in \xi(a)$ for some $a \in A$; and $y \notin (\xi, A)$ if $y \notin \xi(a)$ for every $a \in A$.*

2. $y \in (\xi, A)$ if $y \in \xi(a)$ for every $a \in A$; and $y \notin (\xi, A)$ if $y \notin \xi(a)$ for some $a \in A$. In particular, $y \in \widetilde{Y}$ means $y \in Y$.

Definition 13. *[7] (Y, τ, A) is said to be:*

1. *p-soft T_0 if, for every $y \neq z \in Y$, there is a soft open set (ξ, A) such that $y \in (\xi, A)$ and $z \notin (\xi, A)$ or $z \in (\xi, A)$ and $y \notin (\xi, A)$.*
2. *p-soft T_1 if, for every $y \neq z \in Y$, there are soft open sets (ξ, A) and (ω, A) such that $y \in (\xi, A), z \notin (\xi, A), z \in (\omega, A)$ and $y \notin (\omega, A)$.*
3. *p-soft T_2 if, for every $y \neq z \in Y$, τ contains two disjoint soft open sets (ξ, A) and (ω, A) such that $y \in (\xi, A), z \notin (\xi, A), y \in (\omega, A)$ and $x \notin (\omega, A)$.*
4. *p-soft regular if for every soft closed set (η, A) and $y \in Y$ such that $y \notin (\eta, A)$, τ contains disjoint soft open sets (ξ, A) and (ω, A) such that $(\eta, A) \widetilde{\subseteq} (\xi, A)$ and $y \in (\omega, A)$.*
5. *soft normal if, for every two disjoint soft closed sets (η_1, A) and (η_2, A), τ contains two disjoint soft open sets (ξ, A) and (ω, A) such that $(\eta_1, A) \widetilde{\subseteq} (\xi, A)$ and $(\eta_2, A) \widetilde{\subseteq} (\omega, A)$.*
6. *p-soft T_3 (resp. p-soft T_4) if it is both p-soft T_1 and p-soft regular (resp. soft normal).*

Definition 14. *[7] A soft set (ξ, A) over Y is called stable provided that there is $S \subseteq Y$ such that $\xi(a) = S$ for each $a \in A$; a soft topological space (Y, τ, A) is called stable provided that all proper non null soft open sets are stable.*

A family $\{(\xi_i, A) : i \in I\}$ of soft sets in (Y, τ, A) is said be a *soft cover* of (Y, τ, A) if $\widetilde{\cup}_{i \in I}(\xi_i, A) = \widetilde{Y}$. A soft cover $\{(\xi_i, A) : i \in I\}$ is said to be *locally finite* if, for each soft point, P_a^y has a soft neighborhood intersecting only finitely many (ξ_i, A). A soft cover $\{(\xi_i, A) : i \in I\}$ is a *soft refinement* of a soft cover $\{(\omega_j, B) : j \in J\}$ if, for each (ξ_i, A), there is (ω_j, B) such that $(\xi_i, A) \widetilde{\subset} (\omega_j, B)$.

Definition 15. *A soft topological space (Y, τ, A) is said to be:*

1. *soft compact [15] provided that every soft open cover of Y has a finite subcover.*
2. *soft paracompact [20] if every soft open cover has a soft open, locally finite refinement.*
3. *soft connected [20] if it can not be expressed as a union of two disjoint soft open sets.*
4. *soft hyperconnected [30] if every nun-null soft open set is soft dense.*
5. *soft extremally disconnected [22] if $\mathrm{Cl}_\tau(\xi, A)$ is soft open for every $(\xi, A) \in \tau$.*

Theorem 1. *[15,20] A soft closed set in a soft compact (resp. soft paracompact) space is also soft compact (resp. soft paracompact).*

Definition 16. *[24] A soft mapping $f : (Y, \tau, A) \to (Z, \theta, A)$ is said to be:*

1. *soft continuous if the inverse image of each soft open set is soft open.*
2. *soft open (resp. soft closed) if the image of each soft open (resp. soft closed) set is soft open (resp. soft closed).*
3. *soft homeomorphism if it is bijective, soft continuous, and soft open.*

Definition 17. *A soft subset (ξ, A) of (Y, τ, A) is said to be:*

1. *soft semi-open [25] if $(\xi, A) \widetilde{\subseteq} \mathrm{Cl}(\mathrm{Int}((\xi, A)))$.*
2. *soft β-open [29] if $(\xi, A) \widetilde{\subseteq} \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}((\xi, A))))$.*
3. *soft pre-open [28] if $(\xi, A) \widetilde{\subseteq} \mathrm{Int}(\mathrm{Cl}((\xi, A)))$.*
4. *soft α-open [26] if $(\xi, A) \widetilde{\subseteq} \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((\xi, A))))$.*
5. *soft b-open [27] if $(\xi, A) \widetilde{\subseteq} \mathrm{Int}(\mathrm{Cl}((\xi, A))) \widetilde{\cup} \mathrm{Cl}(\mathrm{Int}((\xi, A)))$.*
6. *soft somewhere dense [30] if $\mathrm{Int}(\mathrm{Cl}((\xi, A))) \neq \widetilde{\varnothing}_Y$.*

3. Sum of Soft Topological Spaces

In this section, we introduce and study the concept of sum of soft topological spaces. Then, we investigate which properties are additive and finitely additive.

Definition 18. *A collection of two or more soft sets is said to be pairwise disjoint if the intersection of any two distinct soft sets is the null soft set.*

Proposition 2. *Let $\{(Y_i, \tau_i, A) : i \in I\}$ be a family of pairwise disjoint soft topological spaces and $Y = \bigcup_{i \in I} Y_i$. Then, the collection*

$$\tau = \{(\xi, A) \text{ over } \bigcup_{i \in I} Y_i : (\xi, A) \widetilde{\bigcap} \widetilde{Y_i} = \{(a, \xi(a) \bigcap Y_i) : a \in A\} \text{ is a soft open set in } (Y_i, \tau_i, A) \text{ for every } i \in I\}$$

defines a soft topology on Y with a fixed set of parameters A.

Proof. It is clear that $\widetilde{Y}$ and $\widetilde{\varnothing_Y}$ are members of τ. Let $(\xi_j, A) \in \tau$, where $j \in J$ be an arbitrary family. Then, $(\xi_j, A) \widetilde{\bigcap} \widetilde{Y_i} \in \tau_i$ for each $j \in J$ and $i \in I$. Thus, $\widetilde{\bigcup}_{j \in J} (\xi_j, A) \widetilde{\bigcap} \widetilde{Y_i} \in \tau_i$ for each $i \in I$. Thus, τ is closed under arbitrary unions. Let (ξ_1, A) and (ξ_2, A) be two members of τ. Then, $(\xi_1, A) \widetilde{\bigcap} \widetilde{Y_i} \in \tau_i$ and $(\xi_2, A) \widetilde{\bigcap} \widetilde{Y_i} \in \tau_i$ for each $i \in I$. Therefore, $[(\xi_1, A) \widetilde{\bigcap} (\xi_2, A)] \widetilde{\bigcap} \widetilde{Y_i} \in \tau_i$ for each $i \in I$. Thus, τ is closed under finite intersections. Hence, τ is a soft topology on Y. $\square$

Definition 19. *The soft topological space (Y, τ, A) given in the above proposition is said to be the sum of soft topological spaces and is denoted by $(\oplus_{i \in I} Y_i, \tau, A)$.*

Remark 1. *The term of sum of soft topological spaces was given in [31] without a condition of pairwise disjointness. Moreover, the authors of [31] did not study the properties of additive, finitely additive and countably additive which represent the main goal of this study. In fact, this definition leads to confusion on how constructing the sum of soft topological spaces and losing some well-known properties of the sum of soft topological spaces as the following example shows:*

Example 1. *Let $A = \{a_1, a_2\}$ and let $Y_1 = \{1, 2\}$ and $Y_2 = \{1, 3\}$. Consider that $\tau_1 = \{\widetilde{\varnothing_{Y_1}}, \widetilde{Y_1}\}$ and $\tau_2 = \{\widetilde{\varnothing_{Y_2}}, \widetilde{Y_2}\}$ are two soft topologies on Y_1 and Y_2, respectively. The sum of soft topological spaces (Y_1, τ_1, A) and (Y_2, τ_2, A) does not exist according to Definition 19 because $Y_1 \bigcap Y_2 \neq \varnothing$. However, the sum of soft topologies τ_1 and τ_2 on $Y = Y_1 \bigcup Y_2 = \{1, 2, 3\}$ according to the definition given in [31] is $\tau = \{\widetilde{\varnothing_Y}, \widetilde{Y}\}$. It is clear that $\widetilde{Y_1}$ and $\widetilde{Y_2}$ do not belong to τ and this contradicts the fact that the universal sets $\widetilde{Y_1}$ and $\widetilde{Y_2}$ belong to τ, see Corollary 1. Moreover, (Y, τ, A) is soft connected and this contradicts the fact that the sum of soft topological spaces is soft disconnected, see Corollary 2.*

Proposition 3. *A soft subset (η, A) of $(\oplus_{i \in I} Y_i, \tau, A)$ is soft closed if and only if $(\eta, A) \widetilde{\bigcap} \widetilde{Y_i}$ is a soft closed subset of (Y_i, τ_i, A) for every $i \in I$.*

Proof. (η, A) is a soft closed subset of $(\oplus_{i \in I} Y_i, \tau, A) \Leftrightarrow (\eta^c, A) \widetilde{\bigcap} \widetilde{Y_i}$ is a soft open subset of (Y_i, τ_i, A) for every $i \in I \Leftrightarrow (\eta, A) \widetilde{\bigcap} \widetilde{Y_i}$ is a soft closed subset of (Y_i, τ_i, A) for every $i \in I$. $\square$

Corollary 1. *All soft sets $\widetilde{Y_i}$ are soft clopen in $(\oplus_{i \in I} Y_i, \tau, A)$.*

Corollary 2. *Every sum of soft topological spaces is soft disconnected.*

Proposition 4. *If $\{(Y_i, \tau_i, A) : i \in I\}$ is a class of pairwise disjoint soft topological spaces and X_i is a subspace of Y_i for every $i \in I$, then the soft topology of the sum of subspaces $\{(X_i, \tau_{X_i}, A) : i \in I\}$ and the soft topological subspace on $\bigcup_{i \in I} X_i$ of the sum soft topology $(\oplus_{i \in I} Y_i, \tau, A)$ coincide.*

Proof. Straightforward. $\square$

Definition 20. *A property $\mathcal{P}$ is said to be:*

1. *additive if, for any family of soft topological spaces $\{(Y_i, \tau_i, A) : i \in I\}$ with the property $\mathcal{P}$, the sum of this family $(\oplus_{i \in I} Y_i, \tau, A)$ also has property $\mathcal{P}$.*
2. *finitely additive (resp., countably additive) if, for any finite (resp., countable) family soft topological spaces with the property $\mathcal{P}$, the sum of this family $(\oplus_{i \in I} Y_i, \tau, A)$ also has property $\mathcal{P}$.*

Theorem 2. *The property of being a p-soft T_i-space is an additive property for $i = 0, 1, 2, 3, 4$.*

Proof. We prove the theorem in the case of $i = 2$. Let $y \neq z \in \oplus_{i \in I} Y_i$. Then, we have the following two cases:

1. There exists $i_0 \in I$ such that $y, z \in Y_{i_0}$.

Since (Y_{i_0}, τ_{i_0}, A) is p-soft T_2, then there exist two disjoint soft open subsets (ξ, A) and (ω, A) of (Y_{i_0}, τ_{i_0}, A) such that $y \in (\xi, A)$ and $z \in (\omega, A)$. It follows from Definition 19 that (ξ, A) and (ω, A) are disjoint soft open subsets of $(\oplus_{i \in I} Y_i, \tau, A)$.

2. There exist $i_0 \neq j_0 \in I$ such that $y \in Y_{i_0}$ and $z \in Y_{j_0}$.

Now, $\widetilde{Y_{i_0}}$ and $\widetilde{Y_{j_0}}$ are soft open subsets of (Y_{i_0}, τ_{i_0}, A) and (Y_{j_0}, τ_{j_0}, A), respectively. It follows from Definition 19 that $\widetilde{Y_{i_0}}$ and $\widetilde{Y_{j_0}}$ are disjoint soft open subsets of $(\oplus_{i \in I} Y_i, \tau, A)$.

It follows from the two cases above that $(\oplus_{i \in I} Y_i, \tau, A)$ is a p-soft T_2-space.

The theorem can be proved similarly in the cases of $i = 0, 1$.

To prove the theorem in the cases of $i = 3$ and $i = 4$, it suffices to prove the p-soft regularity and soft normality, respectively.

First, we prove the p-soft regularity property. Let (η, A) be a soft closed subset of $(\oplus_{i \in I} Y_i, \tau, A)$ such that $y \notin (\eta, A)$. It follows from Proposition 3 that $(\eta, A) \widetilde{\cap} \widetilde{Y_i}$ is soft closed in (Y_i, τ_i, A) for each $i \in I$. Since $y \in \oplus_{i \in I} Y_i$, there is only $i_0 \in I$ such that $y \in Y_{i_0}$. This implies that there are disjoint soft open subsets (ξ, A) and (ω, A) of (Y_{i_0}, τ_{i_0}, A) such that $(\eta, A) \widetilde{\cap} \widetilde{Y_{i_0}} \widetilde{\subseteq} (\xi, A)$ and $y \in (\omega, A)$. Now, $(\xi, A) \underset{i \neq i_0}{\widetilde{\cup}} \widetilde{Y_i}$ is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$ containing (η, A). The disjointness between $(\xi, A) \underset{i \neq i_0}{\widetilde{\cup}} \widetilde{Y_i}$ and (η, A) ends the proof that $(\oplus_{i \in I} Y_i, \tau, A)$ is a p-soft regular space.

Second, we prove the soft normality property. Let (η, A) and (δ, A) be two disjoint soft closed subsets of $(\oplus_{i \in I} Y_i, \tau, A)$. It follows from Proposition 3 that $(\eta, A) \widetilde{\cap} \widetilde{Y_i}$ and $(\delta, A) \widetilde{\cap} \widetilde{Y_i}$ are soft closed in (Y_i, τ_i, A) for each $i \in I$. Since (Y_i, τ_i, A) is soft normal for each $i \in I$, then there exist two disjoint soft open subsets (ξ_i, A) and (ω_i, A) of (Y_i, τ_i, A) such that $(\eta, A) \widetilde{\cap} \widetilde{Y_i} \widetilde{\subseteq} (\xi_i, A)$ and $(\delta, A) \widetilde{\cap} \widetilde{Y_i} \widetilde{\subseteq} (\omega_i, A)$. This implies that $(\eta, A) \widetilde{\subseteq} \underset{i \in I}{\widetilde{\cup}} (\xi_i, A)$, $(\delta, A) \widetilde{\subseteq} \underset{i \in I}{\widetilde{\cup}} (\omega_i, A)$ and $[\underset{i \in I}{\widetilde{\cup}} (\xi_i, A)] \widetilde{\cap} [\underset{i \in I}{\widetilde{\cup}} (\omega_i, A)] = \widetilde{\emptyset}$. Hence, $(\oplus_{i \in I} Y_i, \tau, A)$ is a soft normal space. $\square$

Proposition 5. *The property of being a discrete soft space is an additive property.*

Proof. Let (ξ, A) be a soft subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Then, $(\xi, A) \widetilde{\cap} \widetilde{Y_i}$ is a soft subset of (Y_i, τ_i, A) for each $i \in I$. Therefore, it is a soft open subset of (Y_i, τ_i, A) for each $i \in I$. Hence, $(\xi, A) = \underset{i \in I}{\widetilde{\cup}} [(\xi, A) \widetilde{\cap} \widetilde{Y_i}]$ is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. $\square$

In the following two examples, we show that the properties of indiscrete and door soft spaces are not additive properties. Recall that a soft topological space is a *door soft space* if each subset in it is soft open, or soft closed, or both.

Example 2. *Let $A = \{a_1, a_2\}$ and let $Y_1 = \{6, 7\}$ and $Y_2 = \{8, 9\}$. Then, $\tau_1 = \{\widetilde{\emptyset_{Y_1}}, \widetilde{Y_1}\}$ and $\tau_2 = \{\widetilde{\emptyset_{Y_2}}, \widetilde{Y_2}\}$ are two indiscrete soft topologies on Y_1 and Y_2, respectively. Now, $\tau = \{\widetilde{\emptyset_Y}, \widetilde{Y}, \widetilde{Y_1}, \widetilde{Y_2}\}$ is the sum of soft topologies τ_1 and τ_2 on $Y = Y_1 \bigcup Y_2 = \{6, 7, 8, 9\}$. Since τ is not indiscrete, then the indiscrete soft space property is not an additive property.*

Example 3. *Let $A = \{a_1, a_2\}$ and let $Y_1 = \{6\}$ and $Y_2 = \{8\}$. Then, $\tau_1 = \{\widetilde{\varnothing}_{Y_1}, \widetilde{Y}_1, \{(a_1, \{6\}), (a_2, \varnothing)\}\}$ and $\tau_2 = \{\widetilde{\varnothing}_{Y_2}, \widetilde{Y}_2, \{(a_1, \varnothing), (a_2, \{8\})\}\}$ are two door soft topologies on Y_1 and Y_2, respectively. Now, $\tau = \{\widetilde{\varnothing}_Y, \widetilde{Y}, \widetilde{Y}_1, \widetilde{Y}_2, \{(a_1, \{6\}), (a_2, \varnothing)\}, \{(a_1, \varnothing), (a_2, \{8\})\}, \{(a_1, \{6\}), (a_2, \{8\})\}\}$ is the sum of soft topologies τ_1 and τ_2 on $Y = Y_1 \cup Y_2 = \{6, 8\}$. Since τ is not a door soft topology, then the door soft topology is not an additive property.*

Proposition 6. *The property of being a soft compact space is a finitely additive property.*

Proof. Let $\{(Y_k, \tau_k, A) : k \in \{1, 2, ..., n\}\}$ be a finite family of pairwise disjoint soft compact spaces and let $(\oplus_{k=1}^n Y_k, \tau, A)$ be the sum of this family. Suppose that $\{(\xi_i, A) : i \in I\}$ is a soft open cover of $\widetilde{Y} = \overset{n}{\underset{k=1}{\widetilde{\bigcup}}} \widetilde{Y}_k$. Then, $\widetilde{Y}_k = \underset{i \in I}{\widetilde{\bigcup}} [(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_k]$ for every $k \leq n$. Since (Y_k, τ_k, A) is soft compact for every $k \leq n$, there exist finite subsets $M_1, M_2, ..., M_n$ of I such that $\widetilde{Y}_1 = \underset{i \in M_1}{\widetilde{\bigcup}} [(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_1]$, $\widetilde{Y}_2 = \underset{i \in M_2}{\widetilde{\bigcup}} [(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_2]$, ..., $\widetilde{Y}_n = \underset{i \in M_n}{\widetilde{\bigcup}} [(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_n]$. Letting $M = \overset{n}{\underset{k=1}{\bigcup}} M_k$, now, $\widetilde{Y} = \underset{i \in M}{\widetilde{\bigcup}} [(\xi_i, A) \widetilde{\bigcap} \widetilde{Y}_k]$ for every $k \leq n$. Since M is finite, then $(\oplus_{k=1}^n Y_s, \tau, A)$ is soft compact. $\square$

The following example shows that soft compactness is not an additive property.

Example 4. *Let $A = \{a_1, a_2\}$ and let $Y_n = \{2n - 1, 2n\}$, where n belongs to the set $\mathcal{N}$ of natural numbers. Consider the discrete soft topology τ_n on Y_n. Now, $\{(Y_n, \tau_n, A) : n \in \mathcal{N}\}$ is a family of pairwise disjoint soft compact spaces. It follows from Proposition 5 that the sum of these soft spaces $(\oplus_{n \in \mathcal{N}} Y_n, \tau, A)$ is soft discrete. Obviously, $(\oplus_{n \in \mathcal{N}} Y_n, \tau, A)$ is not soft compact. Hence, soft compactness is not an additive property.*

Proposition 7. *If the sum of soft topological spaces $(\oplus_{i \in I} Y_i, \tau, A)$ is soft compact, then we have the following two assertions that are true:*

1. *all (Y_i, τ_i, A) are soft compact.*
2. *the index set I is finite.*

Proof. 1. From Corollary 1, (Y_i, τ_i, A) is a soft closed subspace of $(\oplus_{i \in I} Y_i, \tau, A)$ for each $i \in I$. It follows from Theorem 1 that (Y_i, τ_i, A) is soft compact for each $i \in I$.

2. Let $(\oplus_{i \in I} Y_i, \tau, A)$ be the sum of soft topological spaces. Then, $\Lambda = \{Y_i : i \in I\}$ is a soft open cover of $\widetilde{Y} = \underset{i \in I}{\widetilde{\bigcup}} \widetilde{Y}_i$. It is clear that Λ does not have a finite subcover. This contradicts the fact that $(\oplus_{i \in I} Y_i, \tau, A)$ is soft compact. Hence, it must be that I is finite. $\square$

Remark 2. *It is clear that the soft topological spaces (Y_1, τ_1, A) and (Y_2, τ_2, A) given in Example 2 are soft hyperconnected. Moreover, they are soft connected. However, the sum of (Y_1, τ_1, A) and (Y_2, τ_2, A) is neither soft hyperconnected nor soft connected. This means that the properties of soft hyperconnected and soft connected are not finite additive.*

Similarly to the proof of Proposition 6, we prove the following:

Proposition 8. *The property of being a soft Lindelöf space is a countably additive property.*

Definition 21. *A soft topological space (Y, τ, A) is said to be soft locally compact if every soft point P_a^y has a soft compact neighborhood.*

Theorem 3. *The sum of soft topological spaces $(\oplus_{i \in I} Y_i, \tau, A)$ is soft locally compact if and only if all soft spaces (Y_i, τ_i, A) are soft locally compact.*

Proof. *Necessity*: Let $P_a^{y_i}$ be a soft point in (Y_i, τ_i, A). Then, there is a soft compact neighborhood (ξ, A) of $P_a^{y_i}$ in $(\oplus_{i \in I} Y_i, \tau, A)$. Since (Y_i, A) is soft closed, then $(\xi, A) \widetilde{\cap} (Y_i, A)$ is a soft compact set in $(\oplus_{i \in I} Y_i, \tau, A)$. Therefore, $(\xi, A) \widetilde{\cap} (Y_i, A)$ is a soft compact set in (Y_i, τ_i, A). Since $P_a^{y_i} \in \text{Int}_\tau(\xi, A)$, then $P_a^{y_i} \in \text{Int}_{\tau_i}[(\xi, A) \widetilde{\cap} (Y_i, A)]$. Hence, $(\xi, A) \widetilde{\cap} (Y_i, A)$ is a soft compact neighborhood of $P_a^{y_i}$ in (Y_i, τ_i, A), as required.

Sufficiency: Let P_a^y be a soft point in $(\oplus_{i \in I} Y_i, \tau, A)$. Then, there is Y_i such that $P_a^y \in \widetilde{Y_i}$. Since (Y_i, τ_i, A) is soft locally compact, then there is a soft compact neighborhood (ξ, A) of P_a^y in (Y_i, τ_i, A). Since $Y_i \cap Y_j = \emptyset$ for each $i \neq j$, then (ξ, A) is a soft compact neighborhood of P_a^y in (Y_i, τ_i, A). $\square$

Theorem 4. *The sum of soft topological spaces $(\oplus_{i \in I} Y_i, \tau, A)$ is soft paracompact if and only if all soft spaces (Y_i, τ_i, A) are soft paracompact.*

Proof. *Necessity*: From Corollary 1, (Y_i, τ_i, A) is a soft closed subspace of $(\oplus_{i \in I} Y_i, \tau, A)$ for each $i \in I$. It follows from Theorem 1 that (Y_i, τ_i, A) is soft paracompact for each $i \in I$.

Sufficiency: Suppose that $\Lambda = \{(\xi_j, A) : j \in J\}$ is a soft open cover of $(\oplus_{i \in I} Y_i, \tau, A)$. Then, $\mathcal{U} = \{(\xi_j, A) \widetilde{\cap} \widetilde{Y_i} : j \in J\}$ is a soft open cover of (Y_i, τ_i, A) for each $i \in I$. By hypothesis, there is a $\mathcal{V}_i$ such that $\mathcal{V}_i = \{(\xi_s, A) \widetilde{\cap} \widetilde{Y_i} : s \in S \subseteq J\}$ is a locally finite soft open refinement of $\mathcal{U}$. Now, $\bigcup_{i \in I} \mathcal{V}_i$ is a soft open refinement of Λ. Since the family of (Y_i, τ_i, A) is pairwise disjoint, then $\bigcup_{i \in I} \mathcal{V}_i$ is locally finite as well. Hence, $(\oplus_{i \in I} Y_i, \tau, A)$ is soft paracompact. $\square$

Definition 22. *Let $\{f_i : (Y_i, \tau_i, A) \to (Z_i, \theta_i, B) : i \in I\}$ be a family of soft mappings. Then, we define a soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A) \to (\oplus_{i \in I} Z_i, \theta, B)$ as follows: For each soft subsets (ξ, A) and (ω, B) of $(\oplus_{i \in I} Y_i, \tau, A)$ and $(\oplus_{i \in I} Z_i, \theta, B)$, respectively, we have:*

1. $f(\xi, A) = \widetilde{\bigcup_{i \in I}} f_i([(\xi, A) \widetilde{\cap} \widetilde{Y_i}])$; and
2. $f^{-1}(\omega, B) = \widetilde{\bigcup_{i \in I}} f_i^{-1}([(\omega, B) \widetilde{\cap} \widetilde{Z_i}])$.

Theorem 5. *A soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A) \to (\oplus_{i \in I} Z_i, \theta, B)$ is soft continuous (resp. soft open, soft closed) if and only if every soft mapping $f_i : (Y_i, \tau_i, A) \to (Z_i, \theta_i, B)$ is soft continuous (resp. soft open, soft closed).*

Proof. We merely give a proof for the theorem in the case of soft continuity and one can prove the cases between parentheses similarly.

Necessity: Suppose that a soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A) \to (\oplus_{i \in I} Z_i, \theta, B)$ is soft continuous. Taking an arbitrary soft map $f_j : (Y_j, \tau_j, A) \to (Z_j, \theta_j, B)$, where $j \in I$. Let (ξ, B) be a soft open subset of (Z_j, θ_j, B). Then, (ξ, B) is a soft open subset of $(\oplus_{i \in I} Z_i, \theta, B)$. By assumption, $f^{-1}(\xi, B)$ is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Since $(\xi, B) \widetilde{\cap} \widetilde{Z_i} = \widetilde{\emptyset_{Z_i}}$ for each $i \neq j$, then $f^{-1}(\xi, B) = f_j^{-1}(\xi, B)$. Therefore, $f_j^{-1}(\xi, B)$ is a soft open subset of (Y_j, τ_j, A), as required.

Sufficiency: Suppose that $f_i : (Y_i, \tau_i, A) \to (Z_i, \theta_i, B)$ is soft continuous for every $i \in I$ and let (ω, B) be a soft open subset of $(\oplus_{i \in I} Z_i, \theta, B)$. Now, $(\omega, B) \widetilde{\cap} \widetilde{Z_i}$ is a soft open subset of (Z_i, θ_i, B) for every $i \in I$. By assumption, $f_i^{-1}[(\omega, B) \widetilde{\cap} \widetilde{Z_i}]$ is a soft open subset of (Y_i, θ_i, A) for every $i \in I$. Therefore, $\widetilde{\bigcup_{i \in I}} f_i^{-1}[(\omega, B) \widetilde{\cap} \widetilde{Z_i}]$ is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Since $f^{-1}(\omega, B) = \widetilde{\bigcup_{i \in I}} f_i^{-1}[(\omega, B) \widetilde{\cap} \widetilde{Z_i}]$, then $f^{-1}(\omega, B)$ is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$, as required. $\square$

Corollary 3. *A soft mapping $f : (\oplus_{i \in I} Y_i, \tau, A) \to (\oplus_{i \in I} Z_i, \theta, B)$ is soft homeomorphism if and only if every soft mapping $f_i : (Y_i, \tau_i, A) \to (Z_i, \theta_i, B)$ is soft homeomorphism.*

Theorem 6. *Let Int_{τ_i} and Int_τ (resp. Cl_{τ_i} and Cl_τ, l_{τ_i} and l_τ) be the soft interior (resp. soft closure, soft limit) points of a soft set $(\eta, A) \widetilde{\subseteq} \widetilde{Y_i}$ in (Y_i, τ_i, A) and $(\oplus_{i \in I} Y_i, \tau, A)$, respectively. Then:*

1. $\text{Int}_{\tau_i}(\eta, A) = \text{Int}_\tau(\eta, A)$.

2.　$\text{Cl}_{\tau_i}(\eta, A) = \text{Cl}_\tau(\eta, A)$.
3.　$l_{\tau_i}(\eta, A) = l_\tau(\eta, A)$.

Proof. 1. $\text{Int}_{\tau_i}(\eta, A) = \widetilde{\bigcup}\{(\xi, A) \widetilde{\subseteq} \widetilde{Y}_i : (\xi, A) \in \tau_i \text{ and } (\xi, A) \widetilde{\subseteq} (\eta, A)\}$
$= \widetilde{\bigcup}\{(\xi, A) \widetilde{\subseteq} \widetilde{\oplus_{i \in I} Y_i} : (\xi, A) \in \tau \; (\xi, A) \widetilde{\subseteq} (\eta, A)\}$ [Since $Y_i \cap Y_j = \varnothing, \forall i \neq j$]
$= \text{Int}_\tau(\eta, A)$.
　　2, 3. Following similar above arguments, results 2 and 3 are satisfied.　□

Corollary 4. *Let b_{τ_i} and b_τ be the soft boundary of a soft set $(\eta, A) \widetilde{\subseteq} \widetilde{Y}_i$ in (Y_i, τ_i, A) and $(\oplus_{i \in I} Y_i, \tau, A)$, respectively. Then, $b_{\tau_i}(\eta, A) = b_\tau(\eta, A)$.*

Proof. $b_{\tau_i}(\eta, A) = \text{Cl}_{\tau_i}(\eta, A) \setminus \text{Int}_{\tau_i}(\eta, A)$
$\qquad\qquad\quad = \text{Cl}_\tau(\eta, A) \setminus \text{Int}_\tau(\eta, A)$ (by the above theorem)
$\qquad\qquad\quad = b_\tau(\eta, A)$.　□

Theorem 7. *A soft set $(\eta, A) \widetilde{\subseteq} \widetilde{\oplus_{i \in I} Y_i}$ is soft semi-open (resp. soft pre-open, soft α-open, soft b-open, soft β-open) in $(\oplus_{i \in I} Y_i, \tau, A)$ if and only if all $(\eta, A) \widetilde{\cap} \widetilde{Y}_i$ are soft semi-open (resp. soft pre-open, soft α-open, soft b-open, soft β-open) in (Y_i, τ_i, A).*

Proof. We give a proof for the theorem in the case of soft semi-open sets and one can prove the cases between parentheses similarly.
　　Necessity: Let (η, A) be a soft semi-open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Then, $(\eta, A) \widetilde{\subseteq} \text{Cl}_\tau(\text{Int}_\tau((\eta, A)))$. Now, $(\eta, A) \widetilde{\cap} \widetilde{Y}_i \widetilde{\subseteq} \text{Cl}_\tau(\text{Int}_\tau((\eta, A))) \widetilde{\cap} \widetilde{Y}_i$. Since $\widetilde{Y}_i$ is soft open in $(\oplus_{i \in I} Y_i, \tau, A)$, then
$\text{Cl}_\tau(\text{Int}_\tau((\eta, A))) \widetilde{\cap} \widetilde{Y}_i \widetilde{\subseteq} \text{Cl}_\tau[\text{Int}_\tau((\eta, A)) \widetilde{\cap} \widetilde{Y}_i] = \text{Cl}_\tau[\text{Int}_\tau[(\eta, A) \widetilde{\cap} \widetilde{Y}_i]]$. From Theorem 6, we have $\text{Cl}_\tau[\text{Int}_\tau[(\eta, A) \widetilde{\cap} \widetilde{Y}_i]] = \text{Cl}_{\tau_i}[\text{Int}_{\tau_i}[(\eta, A) \widetilde{\cap} \widetilde{Y}_i]]$. Thus, $(\eta, A) \widetilde{\cap} \widetilde{Y}_i \widetilde{\subseteq} \text{Cl}_{\tau_i}[\text{Int}_{\tau_i}[(\eta, A) \widetilde{\cap} \widetilde{Y}_i]]$. Hence, $(\eta, A) \widetilde{\cap} \widetilde{Y}_i$ is a soft semi-open set in (Y_i, τ_i, A).
　　Sufficiency: Let $(\eta, A) \widetilde{\cap} \widetilde{Y}_i$ be a soft semi-open subset of (Y_i, τ_i, A). Then, $(\eta, A) \widetilde{\cap} \widetilde{Y}_i \widetilde{\subseteq} \text{Cl}_{\tau_i}(\text{Int}_{\tau_i}((\eta, A) \widetilde{\cap} \widetilde{Y}_i))$. It follows from Theorem 6 that $(\eta, A) \widetilde{\cap} \widetilde{Y}_i \widetilde{\subseteq} \text{Cl}_\tau(\text{Int}_\tau((\eta, A) \widetilde{\cap} \widetilde{Y}_i))$. Thus, $(\eta, A) \widetilde{\cap} \widetilde{Y}_i$ is soft semi-open in (Y_i, τ_i, A). It is well known that $\underset{i \in I}{\widetilde{\bigcup}} (\eta, A) \widetilde{\cap} \widetilde{Y}_i = (\eta, A)$ is a soft semi-open subset of $(\oplus_{i \in I} Y_i, \tau, A)$.　□

Lemma 1. *Let (η, A) be a soft subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Then, the collection $\{(\eta, A) \widetilde{\cap} \widetilde{Y}_i : i \in I\}$ is locally finite.*

Proof. For each $P_a^x \in \{(\eta, A) \widetilde{\cap} \widetilde{Y}_i : i \in I\}$, there is a soft open subset $\widetilde{Y}_j$ of $(\oplus_{i \in I} Y_i, \tau, A)$ such that $P_a^x \in \widetilde{Y}_j$. Since $(\eta, A) \widetilde{\cap} \widetilde{Y}_j$ is the only member of $\{(\eta, A) \widetilde{\cap} \widetilde{Y}_i : i \in I\}$ such that $[(\eta, A) \widetilde{\cap} \widetilde{Y}_j] \widetilde{\cap} \widetilde{Y}_j \neq \widetilde{\varnothing}_{Y_j}$, the desired result holds.　□

Theorem 8. *$(\eta, A) \widetilde{\subseteq} \widetilde{\oplus_{i \in I} Y_i}$ is a soft dense subset of $(\oplus_{i \in I} Y_i, \tau, A)$ if and only if all $(\eta, A) \widetilde{\cap} \widetilde{Y}_i$ are a soft dense subset of (Y_i, τ_i, A).*

Proof. *Necessity*: First, we prove that $(\eta, A) \widetilde{\cap} \widetilde{Y}_i \neq \widetilde{\varnothing}_{Y_i}$ for each $i \in I$. Suppose that there exists $j \in I$ such that $(\eta, A) \widetilde{\cap} \widetilde{Y}_j = \widetilde{\varnothing}_{Y_j}$. Then, $(\eta, A) \widetilde{\subseteq} [\widetilde{Y}_j]^c$. Therefore, $\text{Cl}_\tau(\eta, A) \widetilde{\subseteq} [\widetilde{Y}_j]^c$. However, this contradicts that (η, A) is a soft dense subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Second, it is clear that $(\eta, A) = \underset{i \in I}{\widetilde{\bigcup}} [(\eta, A) \widetilde{\cap} \widetilde{Y}_i]$. Since the collection $\{(\eta, A) \widetilde{\cap} \widetilde{Y}_i : i \in I\}$ is locally finite, then $\text{Cl}_\tau(\eta, A) = \underset{i \in I}{\widetilde{\bigcup}} \text{Cl}_\tau[(\eta, A) \widetilde{\cap} \widetilde{Y}_i]$. This implies that $\text{Cl}_\tau[(\eta, A) \widetilde{\cap} \widetilde{Y}_i] = \text{Cl}_{\tau_i}[(\eta, A) \widetilde{\cap} \widetilde{Y}_i] = \widetilde{Y}_i$ for each $i \in I$.
　　Sufficiency: The proof follows from the fact that $\text{Cl}_\tau(\eta, A) = \underset{i \in I}{\widetilde{\bigcup}} \text{Cl}_{\tau_i}[(\eta, A) \widetilde{\cap} \widetilde{Y}_i] = \underset{i \in I}{\widetilde{\bigcup}} \widetilde{Y}_i = \oplus_{i \in I} Y_i$.　□

Proposition 9. *The property of being a soft separable space is a countably additive property.*

Proof. Let $\{(Y_i, \tau_i, A) : i = 1, 2, ...\}$ be a countable family of pairwise disjoint soft topological spaces such that there exists a countable soft dense subset (ξ_i, A) of (Y_i, τ_i, A) for each $i \in \mathbb{N}$. Set $(\omega, A) = \widetilde{\bigcup}_{i=1}^{\infty} (\xi_i, A)$. Obviously, (ω, A) is countable. In addition, $\mathrm{Cl}_{\tau}(\omega, A) = \widetilde{\bigcup}_{i=1}^{\infty} \mathrm{Cl}_{\tau_i}(\xi_i, A) = \widetilde{\bigcup}_{i=1}^{\infty} \widetilde{Y}_i = \oplus_{i \in \mathbb{N}} Y_i$. Thus, (ω, A) is soft dense. Thus, $(\oplus_{i \in \mathbb{N}} Y_i, \tau, A)$ is soft separable. Hence, the desired result is proved. □

Theorem 9. $(\oplus_{i \in I} Y_i, \tau, A)$ *is soft extremally disconnected if and only if all* (Y_i, τ_i, A) *are soft extremally disconnected.*

Proof. *Necessity*: Let (ξ, A) be a soft open subset of (Y_i, τ_i, A). Then, it is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. By hypothesis, $\mathrm{Cl}_{\tau}(\xi, A)$ is soft open. Since $\mathrm{Cl}_{\tau}(\xi, A) = \mathrm{Cl}_{\tau_i}(\xi, A)$, then $\mathrm{Cl}_{\tau_i}(\xi, A)$ is a soft open subset of (Y_i, τ_i, A). Hence, (Y_i, τ_i, A) is soft extremally disconnected.

Sufficiency: Let (ξ, A) be a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Then, $(\xi, A) = \bigcup_{i \in I}[(\xi, A) \widetilde{\cap} \widetilde{Y}_i]$. It follows from Lemma 1 that $\mathrm{Cl}_{\tau}(\xi, A) = \bigcup_{i \in I} \mathrm{Cl}_{\tau}[(\xi, A) \widetilde{\cap} \widetilde{Y}_i] = \bigcup_{i \in I} \mathrm{Cl}_{\tau_i}[(\xi, A) \widetilde{\cap} \widetilde{Y}_i]$. Since $\mathrm{Cl}_{\tau_i}[(\xi, A) \widetilde{\cap} \widetilde{Y}_i]$ is a soft open subset of (Y_i, τ_i, A) for each $i \in I$, then $\bigcup_{i \in I} \mathrm{Cl}_{\tau_i}[(\xi, A) \widetilde{\cap} \widetilde{Y}_i]$ is a soft open subset of $(\oplus_{i \in I} Y_i, \tau, A)$. Thus, $\mathrm{Cl}_{\tau}(\xi, A)$ is soft open. Hence, $(\oplus_{i \in I} Y_i, \tau, A)$ is soft extremally disconnected. □

The other path of this study is the answer of the following two questions:

1. Under what conditions can a soft topological space represent the sum of soft topological spaces?
2. If a soft topological space represents the sum of soft topological spaces, what is the maximum number of these soft topological spaces?

The following results answer these questions.

Theorem 10. *If (Y, τ, A) is stable soft disconnected, then it represents the sum of two soft topological spaces.*

Proof. Since (Y, τ, A) is soft disconnected, then it contains at least a proper soft clopen set (ξ, A). Since (Y, τ, A) is stable, then $\xi(a) = X \subseteq Y$ for each $a \in A$. Thus, the two soft subspaces (X, τ_X, A) and (X^c, τ_{X^c}, A) are soft topological spaces such that (Y, τ, A) is their sum. □

Corollary 5. *If (Y, τ, A) contains m stable soft clopen sets, then (Y, τ, A) represents the sum of m soft topological spaces.*

Theorem 11. *If $Y_i = \bigcup_{a \in A} (\xi(a) \cap Y_i)$ for a soft subset (ξ, A) of (Y, τ, A). Then, the maximum partition of $\{Y_i : i \in I\}$ for Y represents the maximum number of soft topological spaces of the sum (Y, τ, A).*

Proof. Straightforward. □

4. Conclusions

The study of soft topological spaces is of great importance because it provides a general frame that consists of parameterized classical topological spaces. The aim of the present work is to study the sum of topological spaces in the soft setting. Our results mainly investigate invariant properties between soft topological spaces and their sum. Thus, we define additive and finitely additive properties. In this regard, we demonstrate some additive properties such as p-soft T_i, $i = 0, 1, 2, 3, 4$, soft paracompactness, soft extremally disconnectedness, and soft continuity. With the help of illustrative examples, we show that the properties of soft compact and soft separable spaces are finitely additive and the properties of soft hyperconnected, soft indiscrete, and door soft spaces are not additive. We made use of

interchangeability of soft interior and soft closure operators between soft topological spaces and their sum to obtain some results related to some important generalized soft open sets. We complete this study by investigating the necessary conditions for a soft topological space to represent the sum of some soft topological spaces.

In the upcoming studies, we plan to examine more notions with respect to additive properties such as e-soft separation axioms and w-soft separation axioms. In addition, we are going to introduce and discuss the concept of sum of soft topological spaces on the contexts of ordered soft topological spaces and fuzzy soft topological spaces. Finally, we hope that this work will help the researchers who are interested in soft topology to study additive properties as a new characteristic of the concepts.

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