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Estrada Index and Laplacian Estrada Index of Random Interdependent Graphs

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Abstract: Let G be a simple graph of order n. The Estrada index and Laplacian Estrada index of G are defined by $EE(G) = \sum_{i=1}^{n} e^{\lambda_i(A(G))}$ and $LEE(G) = \sum_{i=1}^{n} e^{\lambda_i(L(G))}$, where $\{\lambda_i(A(G))\}_{i=1}^{n}$ and $\{\lambda_i(L(G))\}_{i=1}^{n}$ are the eigenvalues of its adjacency and Laplacian matrices, respectively. In this paper, we establish almost sure upper bounds and lower bounds for random interdependent graph model, which is fairly general encompassing Erdös-Rényi random graph, random multipartite graph, and even stochastic block model. Our results unravel the non-triviality of interdependent edges between different constituting subgraphs in spectral property of interdependent graphs.

Keywords: Estrada index; Laplacian Estrada index; eigenvalue; random graph

MSC: 05C50; 15A18; 05C80

1. Introduction

We consider a simple graph $G = (V, E)$ on the vertex set $V = \{1, 2, \cdots, n\}$ with $|V| = n$ and the edge set E consisting of unordered pairs of vertices. The adjacency matrix of G is a $(0, 1)$-matrix denoted by $A(G) = (a_{ij}) \in \mathbb{R}^{n \times n}$, where $a_{ij} = a_{ji} = 1$ when i and j are adjacent, and $a_{ij} = a_{ji} = 0$ otherwise. The degree of vertex i is $d_i = \sum_{j=1}^{n} a_{ij}$, i.e., the number of its incident edges. The Laplacian matrix of G is defined to be $L(G) = D(G) - A(G) \in \mathbb{R}^{n \times n}$, where the diagonal matrix $D(G) = \text{diag}(d_1, d_2, \cdots, d_n)$ is called the degree matrix of G. Since G is undirected, both $A(G)$ and $L(G)$ are symmetric. Moreover, it follows from algebraic graph theory (e.g., [1]) that $A(G)$ has n real eigenvalues arranged in the non-increasing order $\lambda_1(A(G)) \geq \lambda_2(A(G)) \geq \cdots \geq \lambda_n(A(G))$, and $L(G)$ has n real and nonnegative eigenvalues ordered non-increasingly as $\lambda_1(L(G)) \geq \lambda_2(L(G)) \geq \cdots \geq \lambda_n(L(G)) = 0$.

The Estrada index of G, given by

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i(A(G))},$$

is a graph spectral invariant introduced by Estrada [2] in the year 2000. It has notable applications in biochemistry and complex networks including quantifying the degree of folding of long-chain molecules [3–5] and network resilience [6,7]. An important variant of Estrada index is the Laplacian Estrada index [8], which is defined by invoking Laplacian eigenvalues as

$$LEE(G) = \sum_{i=1}^{n} e^{\lambda_i(L(G))}.$$

A variety of mathematical properties, including upper and lower bounds, of these metric have been investigated; see e.g., [9–13] and references therein.

In addition to fixed graphs, Estrada and Laplacian Estrada indices have been recently investigated for classical Erdös-Rényi random graph model as well as random multipartite graphs [14–17]. These results are noteworthy in the sense that they not only contribute to the understanding of spectral theory of random networks but also presenting estimates to *EE* and *LEE* for almost all graphs (as the number of vertices goes to infinity), which are typically much sharper than previous bounds for fixed graphs. Important ramifications for distance Estrada index [18] and Gaussian Estrada index [19,20] have also been explored lately.

In this paper, we study the Estrada index and Laplacian Estrada index for the class of random interdependent graphs, which consist of m subgraphs with edges between different subgraphs appearing independently with probability p, where $p \in (0,1)$ is a constant. Formally, an interdependent graph $G_{m,n} = (G_1, G_2, \cdots, G_m, G)$ with $m = m(n) \geq 2$ is defined by a family of subgraphs $G_l = (V_l, E_l)$ for $l = 1, 2, \cdots, m$ with $|V_l| = n_l$, $n_1 + n_2 + \cdots + n_m = n$, and an m-partite graph $G = (V_1 \cup V_2 \cup \cdots \cup V_m, E)$ with E containing edges of unordered pairs between V_s and V_t with $s \neq t$. Here, G and G_l ($l = 1, \cdots, m$) are deterministic. A random interdependent graph, denoted by $G_{m,n}(p) = (G_1, G_2, \cdots, G_m, G(p))$, is a graph from the probability space $(\mathcal{S}_n, \mathcal{A}_n, \mathbb{P}_n)$ with $\mathcal{S}_n$ containing all possible interdependent graphs $G_{m,n}$, σ-algebra $\mathcal{A}_n$ being the power set of $\mathcal{S}_n$, and probability measure $\mathbb{P}_n$ defining the probability of each graph $G_{m,n}$ in $\mathcal{S}_n$ by assigning probability p for each possible edge of G independently.

Note that the random interdependent graph $G_{m,n}(p)$ is very general in that the random m-partite graph $G(p)$ is independent of G_l ($l = 1, 2, \cdots, m$) and no assumption is made on the topologies inside these fixed subgraphs $\{G_l\}_{l=1}^m$. It is easy to see that Erdös-Rényi random graph $G_n(p)$ is a special case with each subgraph G_l being a single vertex, and random multipartite graph is also a special case of $G_{m,n}(p)$ with each G_l being an empty graph. Moreover, the stochastic block model extensively utilized in statistics and machine learning literature (see e.g., [21,22]) can also be viewed as a special case by setting different edge probabilities in each subgraph.

The rest of the paper is organized as follows. In Section 2, we examine Estrada index and derive some upper and lower bounds for random interdependent graphs. In Section 3, we investigate the upper and lower bounds of Laplacian Estrada index for random interdependent graphs. Under certain conditions, the exact estimates for both $EE(G_{m,n}(p))$ and $LEE(G_{m,n}(p))$ are obtained. We conclude the paper in Section 4.

2. Estrada Index of Random Interdependent Graphs

In this section, we estimate the Estrada index $EE(G_{m,n}(p))$ of random interdependent graphs. Recall that $n_1, n_2, \cdots, n_m$ are the orders of subgraphs $G_1, G_2, \cdots, G_m$. We re-label these subgraphs by $\{n_{(l)}\}_{l=1}^m$ so that $n_{(1)} \geq n_{(2)} \geq \cdots \geq n_{(m)}$. Standard Landau asymptotic notations will be used here. For example, for two functions $f(n)$ and $g(n)$, $f(n) = o(g(n))$ implies that $\lim_{n \to \infty} f(n)/g(n) = 0$; $f(n) = O(g(n))$ means that $|f(n)/g(n)| \leq C$ for some constant C for sufficiently large n. A property $\mathcal{B}$ for a random graph model holds asymptotically almost surely (a.a.s.) if its probability tends to 1, namely, $\mathbb{P}_n(\mathcal{B}|\mathcal{A}_n) \to 1$ as $n \to \infty$, where $\mathcal{A}_n$ is a flow of algebras, and $\mathcal{A}_{k_1} \subseteq \mathcal{A}_{k_2}$ for each $k_1 < k_2$.

Lemma 1 (Weyl's inequality [23])**.** *Consider symmetric matrices $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{n \times n}$ and $Z \in \mathbb{R}^{n \times n}$ satisfying $X = Y + Z$. Assume that $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$, $\lambda_1(Y) \geq \lambda_2(Y) \geq \cdots \geq \lambda_n(Y)$, and $\lambda_1(Z) \geq \lambda_2(Z) \geq \cdots \geq \lambda_n(Z)$ are their eigenvalues, respectively. We have*

$$\max_{j+k=i+n} \{\lambda_j(Y) + \lambda_k(Z)\} \leq \lambda_i(X) \leq \min_{j+k=i+1} \{\lambda_j(Y) + \lambda_k(Z)\},$$

for all $i = 1, 2, \cdots, n$.

In many applications of Weyl's inequality, it suffices to use a special result $\lambda_i(Y) + \lambda_n(Z) \leq \lambda_i(X) \leq \lambda_i(Y) + \lambda_1(Z)$. We will resort to the full power of this lemma in the following sections.

Theorem 1. *Let $G_{m,n}(p)$ be a random interdependent graph in $(\mathcal{S}_n, \mathcal{A}_n, \mathbb{P}_n)$. We have*

$$e^{np-n_{(1)}p}(e^{O(\sqrt{n})} + o(1)) \le EE(G_{m,n}(p)) \le e^{np+n_{(1)}}(e^{O(\sqrt{n})} + o(1)) \quad a.a.s.$$

Proof. By the construction of random interdependent graph, the adjacency matrix $A(G_{m,n}(p))$ satisfies the following relations:

$$A(G_{m,n}(p)) = \overline{A} + A(G(p)), \tag{1}$$

and

$$A(G_n(p)) = A(G(p)) + \tilde{A}, \tag{2}$$

where $\overline{A} = \text{diag}(A(G_1), A(G_2), \cdots, A(G_m))$ and $\tilde{A} = \text{diag}(A(G_{n_1}(p)), A(G_{n_2}(p)), \cdots, A(G_{n_m}(p)))$ are two n-dimensional block diagonal matrices.

We will first estimate $EE(G(p))$ and the eigenvalues of $A(G(p))$. By using (2) and Lemma 1, we have

$$\lambda_1(A(G_n(p))) - \lambda_1(\tilde{A}) \le \lambda_1(A(G(p))) \le \lambda_1(A(G_n(p))) - \lambda_n(\tilde{A}).$$

For Erdös-Rényi random graph $G_n(p)$, it is known that [14] $\lambda_1(A(G_n(p))) = np + O(\sqrt{n})$ and $\lambda_i(A(G_n(p))) = O(\sqrt{n})$ for $i = 2, 3, \cdots, n$ a.a.s. Therefore, we derive

$$np - n_{(1)}p + O(\sqrt{n}) \le \lambda_1(A(G(p))) \le np + O(\sqrt{n}), \quad \text{a.a.s.}$$

For $\lambda_i(A(G(p)))$, $i = 2, 3, \cdots, n - m + 1$, it follows from Lemma 1 that

$$\begin{aligned}
\lambda_{i+m-1}(A(G_n(p))) - \lambda_m(\tilde{A}) &= \lambda_{i+m-1}(A(G_n(p))) + \lambda_{n-m+1}(-\tilde{A}) \\
&\le \lambda_i(A(G(p))) \\
&\le \lambda_i(A(G_n(p))) + \lambda_1(-\tilde{A}) = \lambda_i(A(G_n(p))) - \lambda_n(\tilde{A}).
\end{aligned}$$

Similarly, we have $\lambda_i(A(G(p))) = O(\sqrt{n})$ a.a.s. for $i = 2, 3, \cdots, n - m + 1$. A further application of Lemma 1 and decomposition (2) yields

$$\lambda_i(A(G_n(p))) - \lambda_1(\tilde{A}) \le \lambda_i(A(G(p))) \le \lambda_i(A(G_n(p))) - \lambda_n(\tilde{A})$$

for $i = n - m + 2, n - m + 3, \cdots, n$. Hence, $-n_{(1)}p + O(\sqrt{n}) \le \lambda_i(A(G(p))) \le O(\sqrt{n})$ a.a.s. for $i = n - m + 2, n - m + 3, \cdots, n$. Combining the above discussion, we arrive at

$$e^{np-n_{(1)}p}(e^{O(\sqrt{n})} + o(1)) \le EE(G(p)) \le e^{np}(e^{O(\sqrt{n})} + o(1)) \quad \text{a.a.s.} \tag{3}$$

It is well known that Estrada index $EE(G)$ can be interpreted as weighted sum of closed walks of all lengths in G, and hence it changes increasingly with respect to edge addition. More results on the perturbation of Estrada index have been reported in [24]. In light of (1) and (3), we obtain

$$e^{np-n_{(1)}p}(e^{O(\sqrt{n})} + o(1)) \le EE(G(p)) \le EE(G_{m,n}(p)). \tag{4}$$

On the other hand, if G_l ($l = 1, 2, \cdots, m$) are complete graphs, the Estrada index $EE(G_{m,n}(p))$ will attain its maximum. In this case, the eigenvalues of $\overline{A}$ satisfy $\lambda_1, \lambda_2, \cdots, \lambda_m \in \{n_1 - 1, n_2 - 1, \cdots, n_m - 1\}$ and $\lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_n = -1$. Therefore, using (1) and Lemma 1, we have

$$\lambda_1(A(G_{m,n}(p))) \le \lambda_1(\overline{A}) + \lambda_1(A(G(p))) \le n_{(1)} - 1 + np + O(\sqrt{n}) \quad \text{a.a.s.}$$

For $\lambda_i(A(G_{m,n}(p)))$, $i = m+2, m+3, \cdots, n$, similarly,

$$\lambda_i(A(G_{m,n}(p))) \leq \lambda_{m+1}(\overline{A}) + \lambda_{i-m}(A(G(p))) \leq -1 + O(\sqrt{n}) \quad \text{a.a.s.,}$$

and for $\lambda_i(A(G_{m,n}(p)))$, $i = 2, 3, \cdots, m+1$, we have

$$\lambda_i(A(G_{m,n}(p))) \leq \lambda_1(\overline{A}) + \lambda_i(A(G(p))) \leq n_{(1)} - 1 + O(\sqrt{n}) \quad \text{a.a.s.}$$

It follows from the above comments and the definition of Estrada index that

$$\begin{aligned} EE(G_{m,n}(p)) &\leq e^{n_{(1)}+np+O(\sqrt{n})} + (n-m-1)e^{O(\sqrt{n})} + me^{n_{(1)}+O(\sqrt{n})} \\ &= e^{np+n_{(1)}}(e^{O(\sqrt{n})} + o(1)) \quad \text{a.a.s.,} \end{aligned} \tag{5}$$

where the last equality holds since

$$\frac{e^{n_{(1)}+np+O(\sqrt{n})}}{e^{np+n_{(1)}}} + \frac{(n-m-1)e^{O(\sqrt{n})}}{e^{np+n_{(1)}}} + \frac{me^{n_{(1)}+O(\sqrt{n})}}{e^{np+n_{(1)}}} = e^{O(\sqrt{n})} + o(1) \quad \text{a.a.s.}$$

The theorem thus follows from (4) and (5). □

Corollary 1. *Let $G_{m,n}(p)$ be a random interdependent graph in $(\mathcal{S}_n, \mathcal{A}_n, \mathbb{P}_n)$. If $n_{(1)} = O(\sqrt{n})$, we have*

$$EE(G_{m,n}(p)) = e^{np}(e^{O(\sqrt{n})} + o(1)) \quad \text{a.a.s.}$$

Proof. Recall that p is a constant. If $n_{(1)} = O(\sqrt{n})$, then $EE(G_{m,n}(p)) = e^{np}(e^{O(\sqrt{n})} + o(1))$ a.a.s. by using (4) and (5). □

Corollary 1 gives an exact estimate for Estrada index of random interdependent graphs. This result reveals that $EE(G_{m,n}(p))$ only depends on the inter-subgraph edge probability p and is independent of the intra-subgraph architectures. This highlights that the inter-network connections in such graph models have an essential role.

In Table 1 we show the theoretical and experimental values for $EE(G_{m,n}(p))$ with $m = 2$ subgraphs, $n_{(1)} = n/2$ and $p = 0.1$.

Table 1. $EE(G_{m,n}(p))$ for $m = 2$, $n_{(1)} = n/2$ and $p = 0.1$.

$EE(G_{m,n}(p))$	Lower Bound	Upper Bound	Numerical Calculation
$n = 2000$	$e^{100}(e^{O(\sqrt{2000})} + o(1))$	$e^{1200}(e^{O(\sqrt{2000})} + o(1))$	e^{912}
$n = 4000$	$e^{200}(e^{O(\sqrt{4000})} + o(1))$	$e^{2400}(e^{O(\sqrt{4000})} + o(1))$	e^{1843}
$n = 6000$	$e^{300}(e^{O(\sqrt{6000})} + o(1))$	$e^{3600}(e^{O(\sqrt{6000})} + o(1))$	e^{2975}
$n = 8000$	$e^{400}(e^{O(\sqrt{8000})} + o(1))$	$e^{4800}(e^{O(\sqrt{8000})} + o(1))$	e^{4036}

3. Laplacian Estrada Index of Random Interdependent Graphs

In this section, we consider Laplacian Estrada index $LEE(G_{m,n}(p))$ for random interdependent graphs. The following result in regard to the Laplacian eigenvalues of random multipartite graphs has been essentially proved in [16]. We rephrase it as follows.

Lemma 2. *For $i = 1, 2, \cdots, n-1$,*

$$np - n_{(1)}p + o(n) \leq \lambda_i(L(G(p))) \leq np + o(n) \quad \text{a.a.s.,}$$

and

$$-n_{(1)}p + o(n) \leq \lambda_n(L(G(p))) \leq o(n) \quad a.a.s.$$

Theorem 2. *Let $G_{m,n}(p)$ be a random interdependent graph in $(\mathcal{S}_n, \mathcal{A}_n, \mathbb{P}_n)$. We have*

$$e^{np-n_{(1)}p}((n-1)e^{o(n)} + o(1)) \leq LEE(G_{m,n}(p))$$
$$\leq e^{np+n_{(1)}}(me^{-n_{(1)}+o(n)} + (n-m)e^{o(n)}) \quad a.a.s.$$

Proof. By the definition of random interdependent graphs, we observe that

$$L(G_{m,n}(p)) = \overline{L} + L(G(p)), \tag{6}$$

where $\overline{L} = \text{diag}(L(G_1), L(G_2), \cdots, L(G_m))$ is a block diagonal matrix with component blocks representing the Laplacian matrices of subgraphs.

To estimate $LEE(G_{m,n}(p))$, we need to bound the eigenvalues of $L(G_{m,n}(p))$. Note that the Laplacian eigenvalues vary monotonically with respect to edge addition or removal; see e.g., the interlacing theorem ([1] Theorem 7.1.5). Therefore, in view of Lemma 2, we have the following lower bounds

$$np - n_{(1)}p + o(n) \leq \lambda_i(L(G(p))) \leq \lambda_i(L(G_{m,n}(p))) \quad \text{a.a.s.} \tag{7}$$

for $i = 1, 2, \cdots, n-1$, and

$$-n_{(1)}p + o(n) \leq \lambda_n(L(G(p))) \leq \lambda_n(L(G_{m,n}(p))) \quad \text{a.a.s.} \tag{8}$$

We obtained (7) and (8) by thinking all subgraphs $G_1, \cdots, G_m$ as empty. On the other hand, if all these subgraphs are complete graphs, then $\lambda_1(\overline{L}), \cdots, \lambda_{n-m}(\overline{L}) \in \{(n_1 - 1) \cdot n_1, (n_2 - 1) \cdot n_2, \cdots, (n_m - 1) \cdot n_m\}$, and $\lambda_{n-m+1}(\overline{L}) = \lambda_{n-m+2}(\overline{L}) = \cdots = \lambda_n(\overline{L}) = 0$. Here, we used the multiset representation $a_1 \cdot n_1$, meaning the multiplicity of element n_1 is a_1, etc. Therefore, by using Lemmas 1 and 2, and (6), we have

$$\lambda_i(L(G_{m,n}(p))) \leq \lambda_i(\overline{L}) + \lambda_1(L(G(p))) = np + o(n) \quad \text{a.a.s.} \tag{9}$$

for $i = n - m + 1, n - m + 2, \cdots, n$.

Recall that $n_{(1)} \geq n_{(2)} \geq \cdots \geq n_{(m)}$. We define $\langle n_{(1)} \rangle = \{1, 2, \cdots, n_{(1)} - 1\}$, $\langle n_{(2)} \rangle = \{n_{(1)}, n_{(1)} + 1, \cdots, n_{(1)} + n_{(2)} - 2\}, \cdots, \langle n_{(m)} \rangle = \{n_{(1)} + n_{(2)} + \cdots + n_{(m-1)} - m + 2, n_{(1)} + n_{(2)} + \cdots + n_{(m-1)} - m + 3 \cdots, n_{(1)} + n_{(2)} + \cdots + n_{(m)} - m\}$. For $i \in \langle n_{(j)} \rangle, j = 1, 2, \cdots, m$, it follows from Lemmas 1 and 2 that

$$\lambda_i(L(G_{m,n}(p))) \leq \lambda_i(\overline{L}) + \lambda_1(L(G(p))) \leq n_{(j)} + np + o(n) \quad \text{a.a.s.} \tag{10}$$

By the definition of Laplacian Estrada index and the estimates (7)–(10),

$$(n-1)e^{np-n_{(1)}p+o(n)} + e^{-n_{(1)}p+o(n)}$$
$$\leq LEE(G_{m,n}(p))$$
$$\leq me^{np+o(n)} + (n_{(1)} - 1)e^{n_{(1)}+np+o(n)} + \cdots + (n_{(m)} - 1)e^{n_{(m)}+np+o(n)} \quad \text{a.a.s.}$$

With a further look into the above lower and upper bounds, we have

$$\frac{(n-1)e^{np-n_{(1)}p+o(n)} + e^{-n_{(1)}p+o(n)}}{e^{np-n_{(1)}p}} = (n-1)e^{o(n)} + o(1)$$

and

$$\frac{me^{np+o(n)} + (n_{(1)} - 1)e^{n_{(1)}+np+o(n)} + \cdots + (n_{(m)} - 1)e^{n_{(m)}+np+o(n)}}{e^{np+n_{(1)}}}$$

$$\leq \frac{me^{np+o(n)} + (n-m)e^{n_{(1)}+np+o(n)}}{e^{np+n_{(1)}}}$$

$$\leq me^{-n_{(1)}+o(n)} + (n-m)e^{o(n)}.$$

Hence,

$$e^{np-n_{(1)}p}((n-1)e^{o(n)} + o(1)) \leq LEE(G_{m,n}(p))$$
$$\leq e^{np+n_{(1)}}(me^{-n_{(1)}+o(n)} + (n-m)e^{o(n)}) \quad \text{a.a.s.}$$

The proof is complete. □

Corollary 2. *Let $G_{m,n}(p)$ be a random interdependent graph in $(\mathcal{S}_n, \mathcal{A}_n, \mathbb{P}_n)$. If $n_{(1)} = o(n)$, we have*

$$LEE(G_{m,n}(p)) = e^{np}(e^{o(n)} + o(1)) \quad \text{a.a.s.}$$

Proof. If $n_{(1)} = o(n)$, by Theorem 2 we have

$$e^{np}((n-1+o(1))e^{o(n)} + o(1)) \leq LEE(G_{m,n}(p))$$
$$\leq e^{np+o(n)}(me^{o(n)} + (n-m)e^{o(n)})$$
$$\leq e^{np+o(n)}(ne^{o(n)} + o(1)).$$

Since $n = e^{o(n)}$, we obtain $LEE(G_{m,n}(p)) = e^{np}(e^{o(n)} + o(1))$ a.a.s. The proof is complete. □

Similar to Corollary 1, the Laplacian Estrada index $LEE(G_{m,n}(p))$ only relies on the inter-subgraph edge probability p and is independent of the intra-subgraph topologies. If more information is available for the structure of subgraphs $G_1, \cdots, G_m$, we may be able to derive sharper bounds for the (Laplacian) Estrada index of these interdependent graphs.

In Table 2 we show the theoretical and experimental values for $LEE(G_{m,n}(p))$ with $m = 2$ subgraphs, $n_{(1)} = n/2$ and $p = 0.1$.

Table 2. $LEE(G_{m,n}(p))$ for $m = 2$, $n_{(1)} = n/2$ and $p = 0.1$.

$LEE(G_{m,n}(p))$	Lower Bound	Upper Bound	Numerical Calculation
$n = 2000$	$e^{100}(1999e^{o(2000)} + o(1))$	$e^{1200}(1998e^{o(2000)} + o(1))$	e^{935}
$n = 4000$	$e^{200}(3999e^{o(4000)} + o(1))$	$e^{2400}(3998e^{o(4000)} + o(1))$	e^{1880}
$n = 6000$	$e^{300}(5999e^{o(6000)} + o(1))$	$e^{3600}(5998e^{o(6000)} + o(1))$	e^{3127}
$n = 8000$	$e^{400}(7999e^{o(8000)} + o(1))$	$e^{4800}(7998e^{o(8000)} + o(1))$	e^{4206}

4. Conclusions

We have studied the Estrada index and Laplacian Estrada index of a class of random interdependent graphs $G_{m,n}(p)$. Some lower bounds and upper bounds are determined in the asymptotically almost sure limit as the number of vertices n tends to infinity. Our results have key connotations for such interdependent graphs: the interdependent edges between different subgraphs seem to play a key role in their spectral properties.

Although the random interdependent graph takes Erdös-Rényi random graph, random multipartite graph, and even stochastic block model as special cases, it does not cover the

edge-independent random graph in general [25]. It seems interesting to further extend the results in this paper to random edge-independent interdependent graphs. It would also be desirable to explore more general spectral properties for such interdependent graphs. We leave these as future work.

Conflicts of Interest: The author declares no conflict of interest.

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