Existence of Nonnegative Solutions of Linear Autonomous Functional Differential Equations

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Abstract: It is shown that if we exclude the existence of nontrivial small solutions, then a linear autonomous functional differential equation has a nontrivial nonnegative solution if and only if it has a nonnegative eigenfunction.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional space of real column vectors with any norm $|\cdot|$. Given $r > 0$, let $C = C([-r,0],\mathbb{R}^n)$ denote the Banach space of continuous functions from $[-r,0]$ into $\mathbb{R}^n$ with the supremum norm $\|\phi\| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$ for $\phi \in C$.

Consider the linear autonomous retarded functional differential equation

$$x'(t) = L(x_t), \tag{1}$$

where $L: C \to \mathbb{R}^n$ is a bounded linear functional and $x_t \in C$ is defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r,0]$. According to the Riesz representation theorem, L has the form

$$L(\phi) = \int_{-r}^{0} d[\eta(\theta)]\phi(\theta), \qquad \phi \in C, \tag{2}$$

where $\eta: [-r,0] \to \mathbb{R}^{n \times n}$ is a matrix function of bounded variation normalized such that η is left continuous on $(-r,0)$ and $\eta(0) = 0$. Equation (1) includes as a special case the differential equation with a single delay

$$x'(t) = Ax(t) + Bx(t - r), \tag{3}$$

where $A, B \in \mathbb{R}^{n \times n}$. By a solution of (1), we mean a function $x: [-r,\infty) \to \mathbb{R}^n$ which is continuous on $[-r,\infty)$, differentiable on $(0,\infty)$, and satisfies (1) for $t > 0$. It is well-known [1] that, for every $\phi \in C$, Equation (1) has a unique solution with initial value $x_0 = \phi$.

The characteristic equation of Equation (1) has the form

$$\det \Delta(z) = 0, \qquad \Delta(z) = zI - \int_{-r}^{0} e^{z\theta} d\eta(\theta), \tag{4}$$

where $z \in \mathbb{C}$ is a complex variable and I is the $n \times n$ identity matrix.

A solution $x = (x_1, \ldots, x_n)^T : [-r,\infty) \to \mathbb{R}^n$ of (1) is called oscillatory if all coordinate functions $x_1, \ldots, x_n$ have arbitrarily large zeros. Otherwise, x is called nonoscillatory. As usual, the superscript T indicates the transpose. If Equation (4) has a real root $\mu \in \mathbb{R}$, then there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $\Delta(\mu)v = 0$ and hence the corresponding eigensolution $x(t) = e^{\mu t}v$ is a nonoscillatory solution of (1). Thus, the existence of a real root of Equation (4) is sufficient for the existence of a nonoscillatory

solution of (1). One of the key results in the oscillation theory of functional differential equations [2] states that the existence of a real root of Equation (4) is not only sufficient, but also necessary for the existence of a nonoscillatory solution of (1).

Theorem 1. ([3], Theorem 4.1) *Equation* (1) *has a nonoscillatory solution if and only if Equation* (4) *has a real root.*

The aim of this paper is to give a similar characterization for the existence of a nontrivial nonnegative solution of Equation (1). A solution $x = (x_1, \ldots, x_n)^T \colon [-r, \infty) \to \mathbb{R}^n$ of (1) is called nonnegative if all coordinate functions $x_1, \ldots, x_n$ are nonnegative on $[-r, \infty)$, or, equivalently, $x(t) \in \mathbb{R}_+^n$ for $t \geq -r$, where $\mathbb{R}_+^n$ is the nonnegative orthant, the set of those vectors in $\mathbb{R}^n$ which have nonnegative components. If there exist $\mu \in \mathbb{R}$ and $v \in \mathbb{R}_+^n \setminus \{0\}$ such that $\Delta(\mu)v = 0$, then $x(t) = e^{\mu t}v$ is evidently a nontrivial nonnegative solution of Equation (1). Therefore it is natural to ask whether the existence of $\mu \in \mathbb{R}$ and $v \in \mathbb{R}_+^n \setminus \{0\}$ such that $\Delta(\mu)v = 0$ is necessary for the existence of a nontrivial nonnegative solution of (1). The following simple example shows that the answer in general is negative.

Consider the two-dimensional system

$$
\begin{aligned}
x_1'(t) &= x_2(t-1), \\
x_2'(t) &= -x_1(t),
\end{aligned}
\tag{5}
$$

a special case of Equation (3), where

$$
A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
$$

and $r = 1$. The characteristic matrix is given by

$$
\Delta(z) = \begin{pmatrix} z & -e^{-z} \\ 1 & z \end{pmatrix}.
$$

In this case, there exist no $\mu \in \mathbb{R}$ and $v \in \mathbb{R}_+^2 \setminus \{0\}$ such that $\Delta(\mu)v = 0$. Otherwise, we obtain $0 = \det \Delta(\mu) = \mu^2 + e^{-\mu} > 0$, a contradiction. On the other hand, Equation (5) has the nontrivial nonnegative solution $x = (x_1, x_2)^T \colon [-1, \infty) \to \mathbb{R}^2$ given by

$$
x_1(t) = \begin{cases} t^2 & \text{for } t \in [-1, 0] \\ 0 & \text{for } t \in (0, \infty) \end{cases} \qquad \text{and} \qquad x_2(t) = 0 \qquad \text{for } t \in [-1, \infty).
$$

Note that the above nontrivial nonnegative solution x is a small solution in the sense of the following definition [1]. A solution x of Equation (1) is called a small solution if

$$
\lim_{t \to \infty} x(t) e^{kt} = 0 \qquad \text{for every } k \in \mathbb{R}.
$$

The zero solution is always a small solution. The question is whether there exist initial conditions $\phi \neq 0$ which generate small solutions. Such solutions are called nontrivial small solutions. A linear autonomous ordinary differential equation cannot have a nontrivial small solution. The existence of nontrivial small solutions of Equation (1) is a consequence of the fact that the phase space C is infinite dimensional. As shown in ([1], Chap. 7, Corollary 8.1), Equation (1) has no nontrivial small solutions if and only if the exponential type of the characteristic function $\det \Delta$ is equal to nr, or, equivalently, the system of eigenfunctions and generalized eigenfunctions of the generator of

Equation (1) is complete. As a corollary, we have that Equation (3) has no nontrivial small solutions if and only if $\det B \neq 0$.

In this paper, we will show that if we exclude the existence of nontrivial small solutions, then the "natural" sufficient condition $\Delta(\mu)v = 0$ for some $\mu \in \mathbb{R}$ and $v \in \mathbb{R}^n_+ \setminus \{0\}$ is necessary for the existence of a nontrivial nonnegative solution of Equation (1).

The main result and its proof are given in Section 2. In Section 3, we briefly mention some results which are relevant to our study.

2. Main Result

Our main result is the following theorem.

Theorem 2. *Suppose that Equation (1) has no nontrivial small solutions. Then Equation (1) has a nontrivial nonnegative solution if and only if there exist $\mu \in \mathbb{R}$ and $v \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\Delta(\mu)v = 0$.*

Before we present the proof of Theorem 2, we recall some facts from the decomposition theory of linear autonomous functional differential equations given in ([1], Chap. 7) and we establish two preliminary results.

It is known that Equation (1) generates in C a strongly continuous semigroup $(T(t))_{t \geq 0}$, where $T(t) \colon C \to C$ is a bounded linear operator, the so-called solution operator, defined by $T(t)\phi = x_t(\phi)$ for $t \geq 0$ and $\phi \in C$, $x_t(\phi)$ being the unique solution of (1) with initial value $x_0 = \phi$. The infinitesimal generator $A \colon \mathcal{D}(A) \to C$ of this semigroup is defined by

$$A\phi = \lim_{t \to 0+} \frac{1}{t}[T(t)\phi - \phi] \tag{6}$$

whenever the limit exists in C. It is known that

$$\mathcal{D}(A) = \{\, \phi \in C \mid \phi' \in C, \ \phi'(0) = L(\phi) \,\} \qquad \text{and} \qquad A\phi = \phi'. \tag{7}$$

The spectrum $\sigma(A)$ of the linear operator $A \colon \mathcal{D}(A) \to C$ is a point spectrum and it consists of the roots of Equation (4). In each strip $\alpha \leq |z| \leq \beta$, where $-\infty < \alpha < \beta < \infty$, Equation (4) has only a finite number of roots. Furthermore, if Λ is a finite set of characteristic roots, then C is decomposed by Λ into a direct sum

$$C = P_\Lambda \oplus Q_\Lambda, \tag{8}$$

where P_Λ is the (realified) generalized eigenspace of A associated with Λ and Q_Λ is the complementary subspace of C such that $T(t)Q_\Lambda \subset Q_\Lambda$ for $t \geq 0$. Thus, each $\phi \in C$ can be written uniquely as

$$\phi = \phi^{P_\Lambda} + \phi^{Q_\Lambda}, \qquad \text{where } \phi^{P_\Lambda} \in P_\Lambda \text{ and } \phi^{Q_\Lambda} \in Q_\Lambda. \tag{9}$$

From now on, let C_+ denote the set of nonnegative functions in C, i.e.,

$$C_+ = \{\, \phi \in C \mid \phi(\theta) \in \mathbb{R}^n_+ \text{ for all } \theta \in [-r, 0] \,\}.$$

The following lemma will play an important role in the proof of Theorem 2.

Lemma 1. *Suppose that Equation (1) has no nontrivial small solutions. If x is a nontrivial nonnegative solution of Equation (1), then its Lyapunov exponent μ given by*

$$\mu = \mu(x) = \limsup_{t \to \infty} \frac{\ln \|x_t\|}{t} \tag{10}$$

is finite, and if we let

$$\Lambda_0 = \{\, z \in \mathbb{C} \mid \det \Delta(z) = 0, \ \operatorname{Re} z = \mu \,\}, \tag{11}$$

then there exists $\psi \in P_{\Lambda_0} \cap C_+$ such that

$$\|\psi\| = 1 \qquad and \qquad x_t(\psi) = T(t)\psi \in P_{\Lambda_0} \cap C_+ \text{ for all } t \geq 0, \tag{12}$$

where P_{Λ_0} is the generalized eigenspace of the infinitesimal generator A associated with Λ_0.

Proof. Let $x = x(\phi)$ be a nontrivial nonnegative solution of Equation (1) with initial value $x_0 = \phi \in C_+$ and Lyapunov exponent $\mu = \mu(x)$. Then $x_t(\phi) = T(t)\phi \in C_+$ for all $t \geq 0$. By the assumptions, x cannot be a small solution and hence μ is finite (see [1], Chap. 7, Theorem 6.1). Define

$$\Lambda = \{\, z \in \mathbb{C} \mid \det \Delta(z) = 0, \ \operatorname{Re} z \geq \mu \,\} \tag{13}$$

so that $C = P_\Lambda \oplus Q_\Lambda$. The generalized eigenspace P_Λ can be further decomposed into the direct sum $P_\Lambda = P_{\Lambda_0} \oplus P_{\Lambda_1}$ with Λ_0 as in (11) and

$$\Lambda_1 = \{\, z \in \mathbb{C} \mid \det \Delta(z) = 0, \ \operatorname{Re} z > \mu \,\}. \tag{14}$$

Thus, writing $Q = Q_\Lambda$, $P = P_\Lambda$, $P_0 = P_{\Lambda_0}$ and $P_1 = P_{\Lambda_1}$ for brevity, we have that $C = P_0 \oplus P_1 \oplus Q$ and hence

$$x_t(\phi) = x_t^{P_0}(\phi) + x_t^{P_1}(\phi) + x_t^{Q}(\phi) = T(t)\phi^{P_0} + T(t)\phi^{P_1} + T(t)\phi^{Q} \qquad \text{for } t \geq 0, \tag{15}$$

where $\phi^{P_0} \in P_0$, $\phi^{P_1} \in P_1$ and $\phi^Q \in Q$. All three subspaces P_0, P_1 and Q are invariant under the solution semigroup. The generalized eigenspaces P_0 and P_1 are finite-dimensional and the solutions starting from P_0 and P_1 can be extended backward to all $t < 0$. As a consequence, on P_0 and P_1 the solution semigroup can be extended to a group. It is known that for every $\epsilon > 0$ there exists $K = K(\epsilon) > 0$ such that the following exponential estimates hold:

$$\begin{aligned}
\|T(t)\phi^Q\| &\leq K e^{(\mu - \epsilon)t} \|\phi^Q\|, & t &\geq 0, \\
\|T(t)\phi^{P_0}\| &\leq K(1 + |t|)^{m-1} e^{\mu t} \|\phi^{P_0}\|, & t &\in (-\infty, \infty), \\
\|T(t)\phi^{P_1}\| &\leq K e^{(\mu + \epsilon)t} \|\phi^{P_1}\|, & t &\leq 0,
\end{aligned} \tag{16}$$

where m is the maximum of the ascents of the characteristic roots from Λ_0 (see ([1], Sec. 7.6) and ([4], Equations (3.16) and (3.17))). Replacing ϕ^{P_1} with $T(t)\phi^{P_1}$, $t \geq 0$, in the last inequality and using the group property $T(-t)T(t) = T(0) = I$ on P_1, we find that

$$\|T(t)\phi^{P_1}\| \geq K^{-1} e^{(\mu + \epsilon)t} \|\phi^{P_1}\|, \qquad t \geq 0. \tag{17}$$

This implies that $\phi^{P_1} = 0$. Otherwise, the last inequality, combined with (15) and the previous exponential estimates, would imply that $\mu = \mu(x) \geq \mu + \epsilon$, a contradiction. We claim that $\phi^{P_0} \neq 0$. Otherwise, by virtue of (15), we have that $x_t(\phi) = T(t)\phi^Q$ for $t \geq 0$, which, together with the exponential estimate on Q, implies that $\mu = \mu(x) \leq \mu - \epsilon$, a contradiction. From the exponential estimate on P_0, we find that

$$\|T(t)\phi^{P_0}\| \geq K^{-1}(1 + t)^{1-m} e^{\mu t} \|\phi^{P_0}\|, \qquad t \geq 0. \tag{18}$$

Since $\phi^{P_0} \neq 0$, this, together with the exponential estimate on Q, implies that

$$\frac{\|x_t^Q(\phi)\|}{\|x_t^{P_0}(\phi)\|} = \frac{\|T(t)\phi^Q\|}{\|T(t)\phi^{P_0}\|} \longrightarrow 0 \qquad \text{as } t \to \infty.$$

Hence

$$\frac{\|x_t(\phi)\|}{\|x_t^{P_0}(\phi)\|} = \frac{\|x_t^{P_0}(\phi) + x_t^Q(\phi)\|}{\|x_t^{P_0}\|} \longrightarrow 1 \qquad \text{as } t \to \infty$$

and

$$\frac{\|x_t^Q(\phi)\|}{\|x_t(\phi)\|} \longrightarrow 0 \qquad \text{as } t \to \infty.$$

In particular,

$$\frac{\|x_n^{P_0}(\phi)\|}{\|x_n(\phi)\|} \longrightarrow 1 \qquad \text{as } n \to \infty.$$

Thus, $\{\|x_n(\phi)\|^{-1} x_n^{P_0}(\phi)\}_{n=1}^\infty$ is a bounded sequence in the finite-dimensional and hence closed subspace P_0 of C. Therefore there exist $\psi \in P_0$ with $\|\psi\| = 1$ and a sequence $n_k \to \infty$ such that

$$\frac{x_{n_k}^{P_0}(\phi)}{\|x_{n_k}(\phi)\|} \longrightarrow \psi \qquad \text{as } k \to \infty.$$

It remains to show that ψ has the desired properties. Since $\psi \in P_0$ and P_0 is invariant under the solution semigroup, we have that $x_t(\psi) = T(t)\psi \in P_0$ for $t \geq 0$. As shown before, $x_t^Q(\phi) = o(\|x_t(\phi)\|)$ as $t \to \infty$. Hence

$$\frac{x_{n_k}(\phi)}{\|x_{n_k}(\phi)\|} = \frac{x_{n_k}^{P_0}(\phi)}{\|x_{n_k}(\phi)\|} + \frac{x_{n_k}^Q(\phi)}{\|x_{n_k}(\phi)\|} \longrightarrow \psi \qquad \text{as } k \to \infty.$$

From this, using the nonnegativity of $x(\phi)$, the continuity and the semigroup property of $T(t)$, we find that

$$T(t)\psi = \lim_{k\to\infty} \frac{T(t)x_{n_k}(\phi)}{\|x_{n_k}(\phi)\|} = \lim_{k\to\infty} \frac{T(t)T(n_k)(\phi)}{\|x_{n_k}(\phi)\|} = \lim_{k\to\infty} \frac{T(t+n_k)(\phi)}{\|x_{n_k}(\phi)\|} = \lim_{k\to\infty} \frac{x_{t+n_k}(\phi)}{\|x_{n_k}(\phi)\|} \in C_+$$

for all $t \geq 0$. □

Let X be a real Banach space. A subset $K \subset X$ is called a cone if the following three conditions hold:

(i) K is a nonempty, convex and closed subset of X,
(ii) $tK \subset K$ for all $t \geq 0$, where $tK = \{tx \mid x \in K\}$,
(iii) $K \cap (-K) = \{0\}$, where $-K = \{-x \mid x \in K\}$.

In the proof of Theorem 2, we will need the following result which gives a necessary and sufficient condition for the existence of a nontrivial orbit of a linear invertible map $M\colon X \to X$ which lies in a given cone K. By an orbit starting from $x \in X$, we mean the sequence of iterates $\{M^m(x)\}_{m=0}^\infty$. As usual, $M^0 = I$, the identity on X.

Lemma 2. *Let K be a cone in a finite-dimensional real Banach space X. Suppose that $M\colon X \to X$ is a linear invertible operator. Then M has an orbit belonging to $K \setminus \{0\}$ if and only if M has a positive eigenvalue with an eigenvector in K.*

In the special case $X = \mathbb{R}^n$, Lemma 2 was proved in ([5], Theorem 3). Here we give a different argument which is valid in general finite-dimensional Banach spaces.

Proof of Lemma 2. If $v \in K$ is an eigenvector of M corresponding to a positive eigenvalue ρ, then $M^m(v) = \rho^m v \in K \setminus \{0\}$. Thus, the orbit starting from v belongs to $K \setminus \{0\}$.

Now suppose that M has an orbit starting from $y \in X$ which lies in $K \setminus \{0\}$. In particular, $y = M^0(y) \in K \setminus \{0\}$. Without loss of generality, we may (and do) assume that $\|y\| = 1$. Otherwise, we replace y with $\|y\|^{-1}y$ and use the cone property (ii). Let

$$S = \{ x \in K \mid \|x\| \leq 1 \text{ and } M^m(x) \in K \text{ for all } m = 0, 1, \dots \}.$$

Evidently, S is a convex closed subset of X and $y \in S$. Define an operator $F \colon S \to X$ by

$$F(x) = \frac{(1 - \|x\|)y + \|x\|M(x)}{\|(1 - \|x\|)y + \|x\|M(x)\|} \qquad \text{for } x \in S.$$

The cone property (iii), the fact that $y \neq 0$ and the invertibility of M imply that F is well-defined. Evidently, F is continuous on S. The definition of S and the cone properties (i) and (ii) imply that $F(S) \subset S$. By Brouwer's fixed point theorem, there exists $v \in S$ such that $F(v) = v$. Since $\|v\| = \|F(v)\| = 1$, it follows that

$$v = F(v) = \frac{M(v)}{\|M(v)\|}.$$

Hence $M(v) = \rho v$, where $\rho = \|M(v)\|$. Since $v \neq 0$ and M is invertible, we have that $M(v) \neq 0$ and hence $\rho = \|M(v)\| > 0$. Thus, ρ is a positive eigenvalue of M and $v \in K$ is a corresponding eigenvector. $\square$

Now we can give a proof of Theorem 2 which follows similar lines as the proof of a Perron type theorem for positive solutions of a perturbed system of nonautonomous linear functional differential equations in [6].

Proof of Theorem 2. As noted before, if there exist $\mu \in \mathbb{R}$ and $v \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\Delta(\mu)v = 0$, then $x(t) = e^{\mu t}v$ is a nontrivial nonnegative solution of (1).

Now suppose that Equation (1) has a nontrivial nonnegative solution x. By Lemma 1, the Lyapunov exponent $\mu = \mu(x)$ is finite. Let Λ_0 be the spectral set defined by (11). As noted before, the associated generalized eigenspace $P_0 = P_{\Lambda_0}$ of (1) is finite-dimensional and invariant under the solution semigroup $(T(t))_{t \geq 0}$ with infinitesimal generator A given by (7). Since $P_0 \subset \mathcal{D}(A)$ is finite-dimensional, it is a closed subspace of C and therefore we can define the subspace semigroup $(T_0(t))_{t \geq 0}$ on P_0 by $T_0(t) = T(t)|_{P_0}$, the restriction of $T(t)$ to P_0 ([7], Paragraph I.5.12). Its generator is $A_0 = A|_{P_0}$ with domain $\mathcal{D}(A_0) = P_0$ ([7], Paragraph II.2.3) and $\sigma(A_0) = \Lambda_0$. Since $\dim P_0 < \infty$, the generator $A_0 \colon P_0 \to P_0$ is bounded and therefore $T_0(t) = e^{tA_0}$ for $t \geq 0$ ([8], Chap. I, Sec. 1.1). According to the spectral mapping theorem [8], we have that

$$\sigma(T_0(t)) = e^{t\sigma(A_0)} = \{ e^{zt} \mid z \in \Lambda_0 \}, \qquad t \geq 0. \tag{19}$$

Define $K = P_0 \cap C_+$. By Lemma 1, there exists $\psi \in K \setminus \{0\}$ such that $T_0(t)\psi \in K$ for all $t \geq 0$. Since $0 \notin \sigma(T_0(t))$, we have that $T_0(t) \colon P_0 \to P_0$ is invertible for $t \geq 0$. Hence

$$T_0(t)\psi \in K \setminus \{0\}, \qquad t \geq 0. \tag{20}$$

Evidently, K is a cone in P_0. Let $\{t_k\}_{k=1}^{\infty}$ be a sequence of positive numbers such that $t_k \to 0$ as $k \to \infty$. For every fixed k, consider the linear operator $M = T_0(t_k)$ in P_0. By the semigroup property, we have that $M^m = T_0(mt_k)$ for $m = 0, 1, 2, \dots$. This, together with (20), implies that M has an orbit which belongs to $K \setminus \{0\}$. By the application of Lemma 2, we conclude that $M = T_0(t_k)$ has a positive eigenvalue ρ_k with an eigenvector $\psi_k \in K$. Without loss of generality, we may assume that $\|\psi_k\| = 1$.

Otherwise, we replace ψ_k with $\|\psi_k\|^{-1}\psi_k$ which belongs to K by the cone property (ii). By virtue of (19), $\rho_k = e^{zt_k}$ for some characteristic root z with $\mathrm{Re}\, z = \mu$. From this, using the positivity of ρ_k, we find that

$$\rho_k = |\rho_k| = |e^{zt_k}| = e^{t_k \,\mathrm{Re}\, z} = e^{\mu t_k}.$$

Hence

$$T_0(t_k)\psi_k = e^{\mu t_k}\psi_k, \qquad \psi_k \in K, \quad \|\psi_k\| = 1 \tag{21}$$

for $k = 1, 2, \ldots$. Since $\{\psi_k\}_{k=1}^{\infty}$ is a bounded sequence in the finite-dimension Banach space P_0, there exists a subsequence $\{\psi_{k_j}\}_{j=1}^{\infty}$ of $\{\psi_k\}_{k=1}^{\infty}$ such that the limit

$$\chi = \lim_{j \to \infty} \psi_{k_j} \tag{22}$$

exists in P_0. Evidently, $\|\chi\| = 1$. Since K is closed subset of P_0, we have that $\chi \in K$. From (21), we find that

$$\frac{T_0(t_{k_j})\psi_{k_j} - \psi_{k_j}}{t_{k_j}} = \frac{e^{\mu t_{k_j}} - 1}{t_{k_j}}\psi_{k_j}$$

for $j = 1, 2, \ldots$. From this, letting $j \to \infty$, using (22) and the fact that

$$\frac{d^+}{dt}\bigg|_{t=0} T_0(t) = \frac{d^+}{dt}\bigg|_{t=0} e^{tA_0} = A_0,$$

we obtain

$$\chi' = A_0\chi = \mu\chi.$$

Hence $\chi(\theta) = \chi(0)e^{\mu\theta}$ for $\theta \in [-r, 0]$. Since $0 \neq \chi \in K \subset C_+$, we have that $v := \chi(0) \in \mathbb{R}_+^n \setminus \{0\}$. Finally, $\chi \in P_0 \subset \mathcal{D}(A)$ implies that $\chi'(0) = L(\chi)$ which is equivalent to $\Delta(\mu)v = 0$. $\qquad\square$

3. Discussion

The basic oscillation theorem for differential equations with constant coefficients and several delays was obtained by Arino and Győri [9]. A generalization to a class of linear differential equations with distributed delays was given by Győri and Krisztin [3]. Krisztin [10] showed that linear functional differential equations of mixed type may have nonoscillatory solutions in spite of the nonexistence of a real root of the characteristic equation. Henry [11] proved that small solutions of linear autonomous retarded functional differential equations must vanish after some time. Henry's theorem was improved by Verduyn Lunel [12]. Further information on small solutions and the completness of the eigenfunctions and generalized eigenfunctions of linear autonomous functional differential equations can be found in the monographs by Hale and Verduyn Lunel [1] and Dieckman et al. [13]. Small solutions for nonlinear equations were studied for the first time by Mallet Paret [14], who showed that they do not exist on the attractor of certain nonlinear scalar delay differential equations. For later results, see the papers by Cao [15], Arino [16], Cooke and Verduyn Lunel [17], Mallet Paret and Sell [18], Braverman et al. [19], Garab [20] and the references therein.

The main result of this paper, Theorem 2, is closely related to Theorem 1.5 from our recent work [6], which shows that if K is a cone in $\mathbb{R}^n$, then Equation (1) has a positive solution with respect to the partial order induced by K if and only if Equation (4) has a real root with a positive eigenfunction. In the case $K = \mathbb{R}_+^n$, Theorem 2 improves ([6], Theorem 1.5) in the sense that while the latter theorem applies only to positive solutions, Theorem 2 provides a similar conclusion for the larger class of nontrivial nonnegative solutions. It should be noted that ([6], Theorem 1.5) is a corollary of a more general Perron type theorem for positive solutions of a perturbed system of functional differential equation with a long proof. We believe that the above short proof of Theorem 2 can be of interest.

Conflicts of Interest: The author declares no conflict of interest.

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