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On an Energy-Dependent Quantum System with Solutions in Terms of a Class of Hypergeometric Para-Orthogonal Polynomials on the Unit Circle

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Abstract: We study an energy-dependent potential related to the Rosen–Morse potential. We give in closed-form the expression of a system of eigenfunctions of the Schrödinger operator in terms of a class of functions associated to a family of hypergeometric para-orthogonal polynomials on the unit circle. We also present modified relations of orthogonality and an asymptotic formula. Consequently, bound state solutions can be obtained for some values of the parameters that define the model. As a particular case, we obtain the symmetric trigonometric Rosen–Morse potential for which there exists an orthogonal basis of eigenstates in a Hilbert space. By comparing the existent solutions for the symmetric trigonometric Rosen–Morse potential, an identity involving Gegenbauer polynomials is obtained.

Keywords: orthogonal polynomials; Schrödinger equation; ordinary differential equations; energy-dependent potential; hypergeometric functions; asymptotic expansions

MSC: 42C05; 33C45; 33C05; 34A05; 34E05

1. Introduction

An energy-dependent Schrödinger equation appears for the first time in relativistic quantum mechanics with the Pauli–Schrödinger equation, given by Pauli [1] in the description of the spectrum of an electron in the presence of a magnetic field. Further developments in relativistic and non relativistic quantum mechanics was made by many authors [2–10]. The list is by no means exhaustive.

These classes of quantum potentials appear frequently in many areas of quantum mechanics. A relativistic scalar particle in presence of an electromagnetic field can be studied by means of a Klein–Gordon equation with an energy dependent potential [4,8]. In [11], the authors have applied energy dependent potentials with emphasis on confining potentials to the description of heavy quark systems. Furthermore, the description of systems of N bosons bound is considered in [12] and the Hamiltonian formulation of the relativistic many-body problem in [4,5,13] also lead to energy dependent potential models. For physical applications in hydrodynamics, see [14].

Mathematical aspects of wave equations with energy-dependent potentials have been studied by several authors. The presence of an energy-dependence in the potential in the nonrelativistic context requires a modification of the underlying quantum theory, principally affecting orthogonality relation and norm [15]. An analogous modification is required in the relativistic framework [16]. An extension of the quantum mechanical formalism of systems with energy-dependent potentials to systems defined by generalized Schrödinger equations including a position-dependent mass was studied in [7]. Energy-dependence in the framework of noncommutative quantum mechanics has been recently considered in [17].

The search of solutions for energy dependent potentials in wave equations has attracted considerable attention since the appearance of Pauli's work. In the present manuscript we study a quantum system with energy dependent potential related to the Rosen–Morse trigonometric potential, used in describing the interatomic interaction of linear molecules and for describing polyatomic vibration states and energies of the NH_3 molecule [18]. It has long been known that some mathematical features of quantum systems with an energy-dependent potential have several non-trivial implications; for instance, it is necessary to modify the scalar product to guarantee the conservation of the norm [15,19]. In the present manuscript we give closed-form of solutions, modified relations of orthogonality given by a indefinite (in general) bilinear form and an asymptotic formula of the solutions. Consequently, bound state solutions can be obtained for some values of the parameters that define the model. The solutions are given in terms of a class of functions derived from a sequence of hypergeometric para-orthogonal polynomials on the unit circle. We obtain, as a particular case, the symmetric trigonometric Rosen–Morse potential. In such case, the solutions reduce to an orthogonal basis of eigenfunctions defined in a Hilbert space and are expressed in terms of the Gegenbauer or ultraspherical polynomials. By comparing this solution with other solutions given in the literature we obtain, as a consequence, an identity involving Gegenbauer polynomials. Our procedure to obtain the energy dependent potential is based in a classical technique developed by Bose in [20] to construct solvable one-variable Schrödinger potentials.

The manuscript is organized as follows. In Section 2 we give some background, notations and statement of the results, in Section 3 we give the proofs and in Section 4 we present a discussion and concluding remarks.

2. Basic Notations and Statement of the Results

Let μ be a measure on the unit circle $\mathbb{T} = \partial\mathbb{D}, \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with support consisting of an infinite number of points. We remind that $(\phi_n)_{n=0}^{\infty}$ is the sequence of orthonormal polynomials on the unit circle associated to μ (also termed as Szegő polynomials, after their introduction by G. Szegő), if

$$\int_{\mathbb{T}} \phi_m(z)\overline{\phi_n(z)}d\mu(z) = \delta_{m,n},$$

where $\phi_n(z) = \kappa_n z^n + a_{n-1}z^{n-1}+$ lower order terms and $\kappa_n > 0$.

An exposition of the theory of orthogonal polynomials systems on the unit circle is presented in the monographs [21–23]. More recent surveys in [24,25].

If P_n is a polynomial of degree n, the reciprocal polynomial P_n^* is defined as $z^n\overline{P_n(1/\overline{z})}$, or equivalently

$$P_n^*(z) = \sum_{k=0}^{n} \overline{a}_k z^{n-k} \quad \text{if} \quad P_n(z) = \sum_{k=0}^{n} a_k z^k \text{ and } a_n \neq 0.$$

Let ${}_2F_1(a, b; c; z)$ denote the Gauss hypergeometric function of the variable z with parameters $a, b, c \in \mathbb{C}, c \notin \mathbb{Z}_{\leq 0}$; cf. [26] (p. 56), given by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

for $z \in \mathbb{D}$ and for other values of $z \in \mathbb{C}$ by analytic continuation appropriately; the Pochhammer symbol is defined by $(a)_n = a(a+1)\cdots(a+n-1), (a)_0 = 1$.

For $\alpha \in \mathbb{C}$, the function z^α will be defined on the branch for which arguments are restricted between $-\pi$ and π. We also denote by $\lfloor x \rfloor$ the floor function, defined as the greatest integer less than or equal to x.

A fundamental role in this manuscript is played by the sequence of functions

$$G_n(x;\lambda,\eta) = \frac{(2\lambda)_n}{(\lambda)_n}(4z)^{-n/2}R_n(z;b), \quad \lambda > -\frac{1}{2}, \eta \in \mathbb{R}, n \in \mathbb{N}\cup\{0\}, \tag{1}$$

where λ, η and x are such that $b = \lambda + \eta\imath$ and $2x = \sqrt{z} + \frac{1}{\sqrt{z}}, z = e^{\imath\theta}, \theta \in [0, 2\pi]$.

The functions G_n were introduced in [27] and are defined from the sequence $(R_n)_{n=0}^{\infty}$ of para-orthogonal polynomials, cf. [28]

$$\frac{b+\overline{b}}{b}R_n(z;b) = S_n(z;b) + \frac{\overline{b}}{b}S_n^*(z;b), \tag{2}$$

associated to the Szegő hypergeometric polynomials, cf. [29,30]

$$S_n(z;b) = {}_2F_1(-n, b+1; b+\overline{b}+1; 1-z). \tag{3}$$

These polynomials satisfy the orthogonality relations in the unit disk through the parametrization $z = e^{2\imath\theta}, \theta \in [0, \pi]$

$$\frac{1}{\pi}\int_0^{\pi} \overline{S_m(e^{2\imath\theta};b)}S_n(e^{2\imath\theta};b)\omega(\theta;b)d\theta = \frac{n!}{(b+\overline{b}+1)_n}\delta_{m,n}, \quad \lambda > -\frac{1}{2}; m, n \in \mathbb{N}\cup\{0\} \tag{4}$$

where

$$\omega(\theta;b) = \tau^{(b)}e^{-2\eta\theta}\sin^{2\lambda}\theta,$$

here the constant

$$\tau^{(b)} = \frac{|\Gamma(b+1)|^2}{\Gamma(b+\overline{b}+1)}4^{\Re(b)}e^{\pi\Im(b)}$$

is such that the moment $\mu_0^{(b)} = \frac{1}{\pi}\int_0^{\pi}\omega(\theta;b)d\theta = 1$.

In the sequel, we denote by $\widehat{S}_n$ and s_n the monic and orthonormal polynomials of degree n respectively associated to S_n. From (4) it follows that

$$\kappa_n(b) = \frac{|(b+1)_n|}{\sqrt{n!(b+\overline{b}+1)_n}}, \tag{5}$$

is the main coefficient of s_n.

It should be noted that the R_n polynomials are of hypergeometric type. According to [30] (Th. 5.1), one has

$$R_n(z;b) = {}_2F_1(-n, b; b+\overline{b}; 1-z), \quad \Re[b] > -\frac{1}{2}, \Re[b] \neq 0.$$

This last relation can also be extended to $\Re[b] = 0, \eta \neq 0$ if one takes

$$\lim_{\Re[b]\to 0}\frac{b+\overline{b}}{b}R_n(z;b) = S_n(z;\imath\eta) - S_n^*(z;\imath\eta) = n(z-1){}_2F_1(-n+1, 1+\imath\eta; 2; 1-z),$$

for $n \geq 1$.

In the present manuscript we prove the following results.

Theorem 1. *Let $\lambda > -\frac{1}{2}, \eta \in \mathbb{R}$. Then, the stationary one-dimensional Schrödinger equation with energy dependent potential*

$$\Psi''(\theta) + (E - V(\theta, E; \lambda, \eta))\Psi(\theta) = 0, \quad \theta \in (0, \pi), \tag{6}$$

where

$$\begin{aligned}
V(\theta,E;\lambda,\eta) &= -V_0(\theta;\lambda,\eta)-(-\lambda+\sqrt{E(n;\lambda,\eta)+\eta^2})V_1(\theta;\lambda,\eta),\\
V_0(\theta;\lambda,\eta) &= 2\lambda\eta\cot\theta+\lambda(1-\lambda)\frac{1}{\sin^2\theta},\\
V_1(\theta;\lambda,\eta) &= 2\eta\cot\theta,\\
E(n;\lambda,\eta) &= (n+\lambda)^2-\eta^2,
\end{aligned}$$

has by solution the system of wave functions

$$\Psi_n(\theta;\lambda,\eta)=G_n\left(\cos\theta;\lambda,\eta\right)e^{-\eta\theta}\sin^\lambda\theta, n\in\mathbb{N}\cup\{0\},$$

in $L^2[0,\pi]$.

The usual continuity equation

$$\frac{\partial P(\theta,t)}{\partial t}=-\frac{\partial J(\theta,t)}{\partial\theta},$$

where P denotes the probability density and J the probability current, governs the conservation of mass, charge, and probability of any closed system. If the potential is energy-dependent, it is necessary to modify the definition of the usual orthogonality relations in order to satisfy the continuity equation, [5,15]. More precisely, let $(\Psi_n)_{n=0}^{\infty}$ be a system of normalizable wave functions solutions of an energy dependent potential Schrödinger equation defined through the boundary value problem

$$\begin{aligned}
\Psi''(\theta)+(E-V(\theta,E))\Psi(\theta) &= 0, \quad \theta\in(a,b),\\
\Psi(a)=\Psi(b) &= 0,
\end{aligned}$$

where V is of class $C^1(I)$ with respect to the variable E, being I an open interval of the real line and $a,b\in\overline{\mathbb{R}}$.

The continuity equation read as

$$\frac{\partial P(\theta,t)}{\partial t}+\imath(V(\theta,E_{n_2})-V(\theta,E_{n_1}))\overline{\hat{\Psi}_{n_1}(\theta,t)}\hat{\Psi}_{n_2}(\theta,t)=-\frac{\partial J(\theta,t)}{\partial\theta}, \tag{7}$$

and is satisfied by the probability density P and probability current J

$$\begin{aligned}
P(\theta,t) &= \overline{\hat{\Psi}_{n_1}(\theta,t)}\hat{\Psi}_{n_2}(\theta,t),\\
J(\theta,t) &= -\imath\left(\overline{\hat{\Psi}_{n_1}(\theta,t)}\frac{\partial\hat{\Psi}_{n_2}(\theta,t)}{\partial\theta}-\frac{\partial\overline{\hat{\Psi}_{n_1}(\theta,t)}}{\partial\theta}\hat{\Psi}_{n_2}(\theta,t)\right),
\end{aligned}$$

where $\hat{\Psi}_n(\theta,t)=e^{-\imath E_n t}\Psi_n(\theta)$ is a solution to the time-dependent Schrödinger equation

$$\imath\frac{\partial\Psi(\theta,t)}{\partial t}=\left(-\frac{\partial}{\partial\theta^2}+V\left(\theta,\imath\frac{\partial}{\partial t}\right)\right)\Psi(\theta,t).$$

The orthogonality relation between two states n and m, $n\neq m$ reads as

$$\langle\Psi_n,\Psi_m\rangle=\int_a^b\Psi_n(\theta)\left(1-\frac{V(\theta,E_n)-V(\theta,E_m)}{E_n-E_m}\right)\Psi_m(\theta)d\theta=0,$$

now, by using the smooth dependence of V in relation to E one obtains

$$\langle \Psi_n, \Psi_n \rangle = \int_a^b \Psi_n^2(\theta) \left(1 - \frac{V(\theta,E)}{\partial E}\bigg|_{E=E_n}\right) d\theta.$$

In that regard, for the present quantum model we have the following relations of orthogonality. Notice that the presence of the function $\cot\theta$ in the definition implies that the associated bilinear form is not in general of a definite sign. When $|\eta| < \lambda$, we have bound states solutions.

Theorem 2. *Let $\lambda > 0, \eta \in \mathbb{R}$ and $(\Psi_n)_{n=0}^{\infty}$ be the wave functions of Theorem 1. Then, $(\Psi_n)_{n=0}^{\infty}$ satisfy the relation of orthogonality*

$$\langle \Psi_n, \Psi_m \rangle = \int_0^{\pi} \Psi_n(\theta;\lambda,\eta) \left(1 - \frac{2\eta \cot\theta}{n+m+2\lambda}\right) \Psi_m(\theta;\lambda,\eta) d\theta = c_n(\lambda,\eta)\delta_{n,m}, \tag{8}$$

where $n, m \in \mathbb{N} \cup \{0\}, c_n \in \mathbb{R}$. When $\lambda > \frac{1}{2}$ and $\eta \in \mathbb{R}$ one has

$$c_n(\lambda,\eta) = \left(1 - \frac{\eta^2}{(n+\lambda)^2}\right) \frac{\pi n!(\lambda+n)\Gamma(2\lambda+n)}{2^{2n-1}\Gamma(2\lambda+1)[(\lambda)_n]^2} \left|\frac{b}{b+n}\right|.$$

For the particular case $\eta = 0$ we obtain the symmetric trigonometric Rosen–Morse potential, cf. [31] (Prob. 12), [23,32,33] ((4.7.11) p. 81). Let $P_n^{(\lambda)}$, $\lambda > -\frac{1}{2}$ be the Gegenbauer polynomial of degree n cf. [23] (p. 80). As a consequence of Theorem 1 we obtain

Corollary 1. *The stationary one-dimensional Schrödinger equation*

$$\Psi''(\theta) + \left(E - \frac{\lambda(\lambda-1)}{\sin^2\theta}\right)\Psi(\theta) = 0, \quad \theta \in (0,\pi), \lambda > -\frac{1}{2}, \tag{9}$$

admits the energy eigenstates

$$E(n;\lambda) = (n+\lambda)^2, \quad n \in \mathbb{N} \cup \{0\}.$$

and the complete orthogonal system of wave functions

$$\Psi_n(\theta;\lambda,0) = \frac{2^{-n}(2\lambda)_n}{\binom{n+2\lambda-1}{n}(\lambda)_n} P_n^{(\lambda)}(\cos\theta)\sin^{\lambda}\theta, \quad n \in \mathbb{N} \cup \{0\},$$

in the Hilbert space $L^2[0,\pi]$ with the inner product $\langle f, g \rangle = \int_0^{\pi} f(t)g(t)dt$.

Notice that from this corollary it follows immediately that the ground state energy for the symmetric trigonometric Rosen–Morse potential reduces to

$$E(0;\lambda,0) = \lambda^2.$$

The asymmetric trigonometric Rosen–Morse potential or Rosen–Morse I potential, cf. [32,34,35] whose associated Schrödinger equation reads as

$$\Phi''(\theta) + \left(E - \left(2\mathfrak{b}\cot\theta + \mathfrak{a}(\mathfrak{a}-1)\frac{1}{\sin^2\theta}\right)\right)\Phi(\theta) = 0, \quad \theta \in (0,\pi), \mathfrak{a} \geq \frac{3}{2}, \tag{10}$$

is among the exactly solvable potentials. Bound state solutions can be given in terms of the Jacobi polynomials with purely imaginary arguments and complex conjugate parameters, cf. [32] (pp. 296–297).

This potential has also been studied in [36,37] and solved in terms of the real Romanovski polynomials [38,39] $R_n^{(\mathfrak{a},\mathfrak{b})}$ (also known as Romanovsky–Routh or Pseudo–Jacobi polynomial as in [40]). These polynomials are defined as the polynomial solution of degree n of the differential equation

$$(1+x^2)\frac{d^2y}{dx^2}+2((1+\mathfrak{a})x+\mathfrak{b})\frac{dy}{dx}-n(n+2\mathfrak{a}+1)y=0,$$

and can be expressed in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, cf. [24] [(20.1.1) p. 509], [23] [(4.21.2) p. 62] as

$$\begin{aligned} R_n^{(\mathfrak{a},\mathfrak{b})}(x) =&(-\imath)^n P_n^{(\mathfrak{a}+\mathfrak{b}\imath,\mathfrak{a}-\mathfrak{b}\imath)}(\imath x) \\ =&(-\imath)^n \binom{n+\mathfrak{a}+\mathfrak{b}\imath}{n} {}_2F_1\left(-n,n+2\mathfrak{a}+1;1+\mathfrak{a}+\mathfrak{b}\imath;\frac{1-\imath x}{2}\right) \end{aligned} \tag{11}$$

For convenience, we will adopt the parametrization given in [41] for the solution to (10) in terms of the real Romanovski polynomials (expressed also in terms of author's parametrization), which reads

$$\begin{aligned} \Phi_n(\theta;\mathfrak{a},\mathfrak{b}) &\propto R_n^{(-\frac{2\mathfrak{b}}{n+\mathfrak{a}},1-n-\mathfrak{a})}(\cot\theta)e^{-\frac{\mathfrak{b}\theta}{n+\mathfrak{a}}}\sin^{(\mathfrak{a}+n)}\theta \\ &\propto (-\imath)^n P_n^{(-n-\mathfrak{a}-\frac{\imath\mathfrak{b}}{\mathfrak{a}+n},-n-\mathfrak{a}+\frac{\imath\mathfrak{b}}{\mathfrak{a}+n})}(\imath\cot\theta)e^{-\frac{\mathfrak{b}\theta}{n+\mathfrak{a}}}\sin^{(\mathfrak{a}+n)}\theta, \end{aligned} \tag{12}$$

for $n \geq 0$, and the corresponding energies

$$E_\Phi(n;\mathfrak{a},\mathfrak{b}) = (\mathfrak{a}+n)^2 - \frac{\mathfrak{b}^2}{(\mathfrak{a}+n)^2}. \tag{13}$$

In particular, the symmetric trigonometric Rosen–Morse is obtained by taking $\eta = 0$ in (10). In such case we have that

$$\Phi_n(\theta;\mathfrak{a},0) \propto (-\imath)^n P_n^{(-n-\mathfrak{a},-n-\mathfrak{a})}(\imath\cot\theta)\sin^{(\mathfrak{a}+n)}\theta,$$

is a solution for the Schrödinger equation associated to the symmetric Rosen–Morse potential.

By identifying the parameters $\lambda = \mathfrak{a}$ we obtain

$$\Phi_n(\theta;\lambda) \propto (-\imath)^n P_n^{(-n-\lambda+\frac{1}{2})}(\imath\cot\theta)\sin^{(\lambda+n)}\theta, \tag{14}$$

is also a solution of (9), which coincides, up to a multiplicative factor with Ψ_n, as shows the following identity

Theorem 3. *Let $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}\cup\{0\}$. Then,*

$$(-\imath)^n \frac{(\lambda+\frac{1}{2})_{\lfloor\frac{n}{2}\rfloor}}{(-n-\lambda+\frac{1}{2})_n} P_n^{(-n-\lambda+\frac{1}{2})}(\imath\cot\theta)\sin^n\theta = (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{(1-n-\lambda)_{\lfloor\frac{n}{2}\rfloor}}{(\lambda)_n}P_n^{(\lambda)}(\cos\theta).$$

For those values of λ, say $\lambda = \lambda_0$, for which we have a zero or pole in the expressions $\frac{(\lambda+\frac{1}{2})_{\lfloor\frac{n}{2}\rfloor}}{(-n-\lambda+\frac{1}{2})_n}$ or $\frac{(1-n-\lambda)_{\lfloor\frac{n}{2}\rfloor}}{(\lambda)_n}$ the formula may be interpreted as a limit when $\lambda \to \lambda_0$. In such cases, the limit exists and is finite.

For the next result, let $\epsilon_1, \epsilon_2 > 0$ be fixed quantities sufficiently small so that the interval $[\epsilon_1, \pi-\epsilon_2]$ lies wholly in $(0,\pi)$. We have that

Theorem 4. *Let $\lambda > -\frac{1}{2}, \eta \in \mathbb{R}$. Then,*

$$2^{n-1}\Psi_n(\theta;\lambda,\eta) = \Re\left[\frac{\Gamma(\lambda)}{\Gamma(b)} e^{\imath(n\theta+\eta\ln n)}(1-e^{-2\imath\theta})^{-\bar{b}}\right] e^{-\eta\theta}\sin^{\lambda}\theta + o(1), \quad \textit{as} \quad n\to\infty.$$

The bound for the error holds uniformly in $\theta \in [\epsilon_1, \pi-\epsilon_2]$.

2.1. Some Basic Facts about the Functions G_n

We recall that a polynomial $P_n(z) = \sum_{j=0}^{n} a_j z^j, a_j \in \mathbb{C}$ is conjugate reciprocal if P_n satisfies the identity,

$$P_n(z) = P_n^*(z),$$

that is, $\overline{a_j} = a_{n-j}, j = 0, 1, \ldots, n$. From (1), it follows that $G_n, n \geq 0$ takes real values.

Let $\lambda > 0$, define the sequences

$$\alpha_{n+1}^{(\lambda)} = \frac{1}{4}\frac{n(n+2\lambda-1)}{(n+\lambda-1)(n+\lambda)}, \quad \beta_n^{(\lambda,\eta)} = \frac{\eta}{\lambda+n-1}, \quad n \geq 1,$$

it follows from [42] (Section 2) that the functions G_n satisfy the recurrence relation

$$G_{n+1}(\cos\theta;\lambda,\eta) = (\cos\theta - \beta_{n+1}^{(\lambda,\eta)}\sin\theta)G_n(\cos\theta;\lambda,\eta) - \alpha_{n+1}^{(\lambda)}G_{n-1}(\cos\theta;\lambda,\eta), \quad \theta\in[0,\pi], \tag{15}$$

where $G_0(\cos\theta;\lambda,\eta) = 1$ and $G_1(\cos\theta;\lambda,\eta) = \cos\theta - \beta_1^{(\lambda,\eta)}\sin\theta$.

Let $\lambda > \frac{1}{2}$ and $n, m \in \mathbb{N}\cup\{0\}$. It follows from [42] (Th. 5.2),

$$\begin{aligned} \int_0^{\pi} G_{2n}(\cos\theta;\lambda,\eta)\, G_{2m+1}(\cos\theta;\lambda,\eta)\, \frac{\omega(\theta;b)}{\sin\theta} d\theta &= 0, \\ \int_0^{\pi} G_{2n}(\cos\theta;\lambda,\eta)\, G_{2m}(\cos\theta;\lambda,\eta)\, \omega(\theta;b) d\theta &= \tau^{(b)}\delta_{2n}^{(b)}\delta_{n,m}, \\ \int_0^{\pi} G_{2n+1}(\cos\theta;\lambda,\eta)\, G_{2m+1}(\cos\theta;\lambda,\eta)\, \omega(\theta;b) d\theta &= \tau^{(b)}\delta_{2n+1}^{(b)}\delta_{n,m}, \end{aligned} \tag{16}$$

where

$$\delta_n^{(b)} = \frac{\pi n!(\lambda+n)\Gamma(2\lambda+n)}{2^{2\lambda+2n-1}e^{\eta\pi}|\Gamma(b+n+1)|^2}\frac{1}{[(\lambda)_n]^2}[(\Re[(b)_n])^2 + (\Im[(b)_n])^2].$$

Let μ be a finite positive Borel measure supported in $[-\pi,\pi)$ and $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\nu_s$ its Lebesgue decomposition. Recall that for $z \in \mathbb{D}$, the Szegő function, cf. [25] (§2.4 p. 143, Part I), $D(d\mu;z)$ is defined as

$$D(d\mu;z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{\imath\theta}+z}{e^{\imath\theta}-z}\log(w(\theta))d\theta\right),$$

whenever μ is the Szegő class. For almost every $t \in [-\pi,\pi)$, the boundary value $D(d\mu;e^{\imath t})$ is defined as the radial limit:

$$D(d\mu;e^{\imath t}) = \lim_{r\uparrow 1} D(d\mu;re^{\imath t}).$$

From [30] (Th. 4.3), the Szegő function $D(\omega(\theta)d\theta;z)$, (or $D(z;b)$ for short) reads as

$$D(z;b) = \frac{|\Gamma(b+1)|}{\sqrt{\Gamma(b+\bar{b}+1)}}(1-z)^b. \tag{17}$$

The function $G_n(x)$ satisfy the differential equation, [27] (Th. 2.2)

$$(1-x^2)y'' - ((2\lambda+1)x - 2\eta\sqrt{1-x^2})y' + n\left(n + 2\lambda + \frac{2\eta x}{\sqrt{1-x^2}}\right)y = 0, \quad x \in (-1,1), n \in \mathbb{N} \cup \{0\}. \tag{18}$$

When $\eta = 0$, we obtain the differential equation that defines the ultraspherical polynomials cf. [23] (p. 80). Notice that from (1) and (2), G_n reduces to a polynomial. Therefore, G_n coincides, up to a constant factor, with the n degree Gegenbauer polynomial.

Remark 1. *We remark that in [27] (Th. 2.2), the term m should be corrected in the last summand of the left hand side of the differential equation.*

The Schrödinger Invariant of a Second Order Differential Equation

Quantum systems with energy-dependent potentials have been studied following several approaches such as supersymmetric quantum mechanics, Darboux transformations, exceptional orthogonal polynomials, among others, see [7] for a review. In this subsection we summarize the method we followed, introduced by Bose in [20] in order to construct one-variable Schrödinger solvable potentials.

Let us have a second order differential equation

$$u''(x) + p(x)u'(x) + q(x)u(x) = 0, \quad x \in I, \tag{19}$$

being I an open subset of the real line and where p and q are functions defined on I. A straightforward calculation shows that the middle term in (19) can be eliminated by taking the substitution

$$u = ve^{-\frac{1}{2}\int p(t)dt}.$$

Under the above substitution, the Equation (19) transforms to the canonical form

$$v''(x) + I(x)v = 0,$$

where I is given by

$$I(x) = q(x) - \frac{1}{2}p'(x) - \frac{1}{4}(p(x))^2, \tag{20}$$

the term I is named by Milson in [43] as the Bose invariant.

By applying now the transformation $v = \sqrt{x'}\psi, x' = \frac{dx}{d\theta}$, we obtain the normal form

$$\psi''(\theta) + I_S(\theta)\psi(\theta) = 0,$$

being

$$I_S = (x')^2 I(x) + \frac{1}{2}\{x,\theta\},$$

the Schrödinger invariant, named by Bose in [20] and $\{x,\theta\}$ is the Schwartzian derivative

$$\{x,\theta\} = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2.$$

3. Proof of the Results

Proof of Theorem 1. Under the transformation

$$y = v\left(1-x^2\right)^{-\frac{\lambda}{2}-\frac{1}{4}} e^{\eta \arccos(x)}, \quad x \in (-1,1), \tag{21}$$

the Equation (18) reduces to its normal form

$$v'' + I(x)v = 0, \tag{22}$$

where

$$I(x) = \frac{2 - 4\eta^2 + 4\lambda + 4n^2 + 8\lambda n + x^2\left(4\eta^2 - 4(\lambda+n)^2 + 1\right) + 8\eta n x\sqrt{1-x^2}}{4\left(x^2-1\right)^2} + \frac{8\eta\lambda x\sqrt{1-x^2}}{4\left(x^2-1\right)^2}.$$

By applying the transformations to (22)

$$\begin{aligned} x &= \cos\theta, \quad \theta \in (0,\pi), \\ v &= (x')^{1/2}\Psi, \end{aligned} \tag{23}$$

we obtain the Schrödinger normal form

$$\Psi'' + (\kappa(n;\lambda,\eta) - V(\theta;n,\lambda,\eta))\Psi = 0, \quad \theta \in (0,\pi), \tag{24}$$

where $\kappa(n;\lambda,\eta) = (n+\lambda)^2 - \eta^2$ and

$$\begin{aligned} V(\theta;n,\lambda,\eta) &= -V_0(\theta;\lambda,\eta) - nV_1(\theta;\lambda,\eta), \\ V_0(\theta;\lambda,\eta) &= 2\lambda\eta\cot\theta + \lambda(1-\lambda)\frac{1}{\sin^2\theta}, \\ V_1(\theta;\lambda,\eta) &= 2\eta\cot\theta. \end{aligned}$$

By identifying $\kappa(n;\lambda,\eta)$ with the energy term we get

$$E(n;\lambda,\eta) = (n+\lambda)^2 - \eta^2, \tag{25}$$

therefore

$$n = -\lambda + \sqrt{E+\eta^2}. \tag{26}$$

Substituting (25) and (26) into (24) we obtain (6).

Since G_n is a solution of (18), it follows from (21) and (23) that

$$\Psi_n(\theta;\lambda,\eta) = G_n\left(\cos\theta;\lambda,\eta\right)e^{-\eta\theta}\sin^{\lambda}\theta, \quad n \in \mathbb{N}\cup\{0\},$$

is a solution of (24). □

Proof of Theorem 2. On the one hand, by applying iterated integration to the left hand side of (7) and taking into account the definition of P we obtain

$$\begin{aligned} &\int_0^{\pi}\int_0^{t}\left(\frac{\partial P(\theta,t')}{\partial t'} + \imath(V(\theta,E_m) - V(\theta,E_n))\overline{\hat{\Psi}_n(\theta,t')}\hat{\Psi}_m(\theta,t')\right)dt'd\theta = \\ &\left(e^{\imath(E_n-E_m)t} - 1\right)\int_0^{\pi}\Psi_n(\theta)\left(1 - \frac{V(\theta,E_n) - V(\theta,E_m)}{E_n - E_m}\right)\Psi_m(\theta)d\theta. \end{aligned} \tag{27}$$

On the other hand, for the right hand side of (7) we have

$$\int_0^{\pi}\int_0^{t}\frac{\partial J(\theta,t')}{\partial\theta}dt'd\theta = \int_0^{t}\int_0^{\pi}\frac{\partial J(\theta,t')}{\partial\theta}d\theta dt' \tag{28}$$

Now, from the definition of J and the fact that $\Psi_n(0;\lambda,\eta) = \Psi_n(\pi;\lambda,\eta) = 0$ when $\lambda > 0$ one has

$$\int_0^\pi \frac{\partial J(\theta,t')}{\partial\theta} d\theta = 0. \tag{29}$$

Hence, from (27), (28) and (29)

$$\int_0^\pi \Psi_n(\theta;\lambda,\eta)\left(1 - \frac{V(\theta,E_n) - V(\theta,E_m)}{E_n - E_m}\right)\Psi_m(\theta;\lambda,\eta)d\theta = 0, \quad n \neq m.$$

By substituting the value of V given in Theorem 1 we obtain (8).

To evaluate the numerical value of the constant c_n, notice that if $\lambda > \frac{1}{2}, \eta \in \mathbb{R}$, from the recurrence relations (15) and (16) we have

$$\int_0^\pi \Psi_n^2(\theta;\lambda,\eta)\cot\theta d\theta = \frac{\eta}{\lambda+n}\int_0^\pi \Psi_n^2(\theta;\lambda,\eta)d\theta,$$

hence,

$$c_n(\lambda,\eta) = \left(1 - \frac{\eta^2}{(n+\lambda)^2}\right)\frac{\pi n!(\lambda+n)\Gamma(2\lambda+n)}{2^{2n-1}\Gamma(2\lambda+1)[(\lambda)_n]^2}\left|\frac{b}{b+n}\right|.$$

This completes the proof of the theorem. □

Proof of Corollary 1. From Theorem 1, the eigenstates are given by the system of real functions

$$\Psi_n(\theta;\lambda) = G_n(\cos\theta;\lambda,0)\sin^\lambda\theta, \quad n \in \mathbb{N}\cup\{0\},$$

From (18), G_n reduces, up to a multiplicative constant c_n factor, to the Gegenbauer polynomial of degree n, cf. [23] (§4.7 p. 80). To find the multiplicative constant c_n, note that from (1), (2) and (3) we deduce that

$$G_n(1;\lambda,0) = 2^{-n}\frac{(2\lambda)_n}{(\lambda)_n} = c_n P_n^{(\lambda)}(1),$$

therefore, from [23] [(4.7.3) p. 80] one obtains

$$c_n = \frac{2^{-n}(2\lambda)_n}{\binom{n+2\lambda-1}{n}(\lambda)_n}.$$

The corresponding energy eigenstates read as

$$E(n;\lambda) = (n+\lambda)^2, \quad n \in \mathbb{N}\cup\{0\}.$$

From the orthogonality relation for Gegenbauer polynomials

$$\int_{-1}^1 P_n^{(\lambda)}(x)(1-x^2)^\lambda dx,$$

one has that $\Psi_n(\theta;\lambda)$ is an orthogonal system in the Hilbert space $L_2[0,\pi]$ with the scalar product

$$\langle f,g\rangle = \int_0^\pi f(t)g(t)dt.$$

□

Proof of Theorem 3. From Corollary 1 and the relation (14) we have that Φ_n and Ψ_n are solutions of the differential equation

$$\Psi''(\theta) + \left(E - \frac{\lambda(\lambda-1)}{\sin^2\theta}\right)\Psi(\theta) = 0, \quad \theta \in (0,\pi), \lambda \geq \frac{3}{2},$$

hence, the transformations (21) and (23) give that

$$y_1(x) = (x')^{\frac{1}{2}}(1-x^2)^{-\frac{\lambda}{2}-\frac{1}{4}}\Psi_n(\theta;\lambda), \tag{30}$$

$$y_2(x) = (x')^{\frac{1}{2}}(1-x^2)^{-\frac{\lambda}{2}-\frac{1}{4}}\Phi_n\theta;\lambda), \tag{31}$$

are solutions of the differential equation

$$(1-x^2)y'' - (2\lambda+1)xy' + n\,(n+2\lambda)\,y = 0, \quad x \in (-1,1), \lambda \geq \frac{3}{2}, \tag{32}$$

notice that y_1 reduces, up to a constant factor, to the Gegenbauer polynomial of degree n.

From (14) and (31) one has

$$y_2(x) = (-\imath)^n P_n^{(-n-\lambda+\frac{1}{2})}\left(\imath\frac{x}{\sqrt{1-x^2}}\right)(1-x^2)^{\frac{n}{2}}, \tag{33}$$

and from [44] [(6.4.12) p. 303],

$$P_n^{(-n-\lambda+\frac{1}{2})}\left(\imath\frac{x}{\sqrt{1-x^2}}\right) = \frac{(-n-\lambda+\frac{1}{2})_n(2\imath)^n}{n!}\frac{x^n}{(1-x^2)^{\frac{n}{2}}}{}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; \lambda+\frac{1}{2}; \frac{x^2-1}{x^2}\right),$$

hence, from (33)

$$(1-x^2)^{\frac{n}{2}}P_n^{(-n-\lambda+\frac{1}{2})}\left(\imath\frac{x}{\sqrt{1-x^2}}\right) = \frac{(-n-\lambda+\frac{1}{2})_n(2\imath)^n}{n!}x^n{}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; \lambda+\frac{1}{2}; \frac{x^2-1}{x^2}\right). \tag{34}$$

Notice that the relation (34) defines the left hand side as a polynomial of degree n whose coefficients are rational functions of the variable λ varying in $\mathbb{C}$.

Now, (33) is a solution to (32). Since this solution is a polynomial of degree n, it follows from [23] (Th. 4.2.2 p. 61) that

$$y_2 = const.y_1, \tag{35}$$

when $\Re[\lambda] \geq \frac{3}{2}$. Using formula [44] [(6.4.12) p. 303], by comparing $P_{2k}^{(-n-\lambda-1)}(0)$ with $P_{2k}^{(\lambda)}(0)$ and $\left(P_{2k+1}^{(-n-\lambda-1)}\right)'(0)$ with $\left(P_{2k+1}^{(\lambda)}\right)'(0)$, from (35) we obtain

$$(-\imath)^n\frac{(\lambda+\frac{1}{2})_{\lfloor\frac{n}{2}\rfloor}}{(-n-\lambda+\frac{1}{2})_n}P_n^{(-n-\lambda+\frac{1}{2})}(\imath\cot\theta)\sin^n\theta = (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{(1-n-\lambda)_{\lfloor\frac{n}{2}\rfloor}}{(\lambda)_n}P_n^{(\lambda)}(\cos\theta), \quad \Re[\lambda] \geq \frac{3}{2}. \tag{36}$$

Let us consider n fixed. Since the coefficients of y_2 and y_1 are rational functions of λ and are equal when $\Re[\lambda] \in [\frac{3}{2}, +\infty)$, it follows from [45] (Th. 17.1 p. 369) that the relation is valid for $\lambda \in \mathbb{C}$, with exception of a finite number of special values of λ. Notice that from [44] [(6.4.12) p. 303], the zeros of the functions $(-n-\lambda+\frac{1}{2})_n$ and $(\lambda)_n$ are removable singularities of the left hand side and right hand side respectively of (36). Furthermore, when λ is a zero of $(\lambda+\frac{1}{2})_{\lfloor\frac{n}{2}\rfloor}$ or $(1-n-\lambda)_{\lfloor\frac{n}{2}\rfloor}$ we have simple poles in the main coefficients of the left hand side or right hand side accordingly. For these values of λ_0, the formula may be interpreted as a limit as $\lambda \to \lambda_0$. This completes the proof of the theorem. □

To prove Theorem 4 a preliminary lemma is necessary.

Lemma 1. *Let $\lambda > -\frac{1}{2}$ and $\eta \in \mathbb{R}$ be fixed and define for $\theta \in (0, \pi)$,*

$$\begin{aligned} a_n(\theta) &= \frac{D^{-1}(0;b)}{\pi} \int_0^{\pi} \frac{D^{-1}(e^{\imath\theta'};b) - D^{-1}(e^{\imath\theta};b)}{1 - e^{\imath(\theta-\theta')}} \overline{s_n^*(e^{\imath\theta'};b)} \omega(\theta';b) d\theta', \\ b_n(\theta) &= \frac{D^{-1}(0;b)}{\pi} \int_0^{\pi} \frac{D^{-1}(e^{\imath\theta'};b) - D^{-1}(e^{\imath\theta};b)}{1 - e^{\imath(\theta-\theta')}} \overline{s_n(e^{\imath\theta'};b)} \omega(\theta';b) d\theta'. \end{aligned}$$

Then,

$$\lim_{n\to\infty} a_n(\theta) = 0, \quad \forall \theta \in [\epsilon_1, \pi - \epsilon_2],$$
$$\lim_{n\to\infty} b_n(\theta) = 0, \quad \forall \theta \in [\epsilon_1, \pi - \epsilon_2],$$

being $\epsilon_1, \epsilon_2 > 0$ any fixed quantities sufficiently small so that $[\epsilon_1, \pi - \epsilon_2] \subset (0, \pi)$.

Proof. Let us denote

$$K(\theta', \theta) = D^{-1}(0;b) \left(\frac{D^{-1}(e^{\imath\theta'};b) - D^{-1}(e^{\imath\theta};b)}{1 - e^{\imath(\theta-\theta')}} \right).$$

From (17), we have that for $\theta \in (0, \pi)$ fixed, $K(\theta', \theta)$ is continuous as a function of $\theta' \in [0, \pi]$, hence

$$K(\theta', \theta) \in L^2(\omega(\theta';b)d\theta'),$$

therefore, from Lemma [21] (Lem. 4.2 p. 220)

$$\lim_{n\to\infty} a_n(\theta) = 0, \tag{37}$$

pointwise in the interval $(0, \pi)$.

Let $\epsilon_1, \epsilon_2 > 0$ be any fixed quantities sufficiently small so that $[\epsilon_1, \pi - \epsilon_2] \subset (0, \pi)$. Since $K(\theta', \theta)$ is continuous in the compact set $[0, \pi] \times [\epsilon_1, \pi - \epsilon_2]$, the Heine–Cantor Theorem cf. [46] (Th. 2 p. 201) implies that $K(\theta', \theta)$ is uniformly continuous in $[0, \pi] \times [\epsilon_1, \pi - \epsilon_2]$. Hence, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\theta_1, \theta_2 \in [\epsilon_1, \pi - \epsilon_2]$, $|\theta_1 - \theta_2| < \delta$, then

$$|K(\theta', \theta_1) - K(\theta', \theta_2)| < \epsilon, \quad \forall \theta' \in [0, \pi],$$

therefore,

$$|a_n(\theta_1) - a_n(\theta_2)| \le \left(\frac{1}{\pi} \int_0^{\pi} |K(\theta', \theta_1) - K(\theta', \theta_2)|^2 \omega(\theta';b) d\theta' \right)^{\frac{1}{2}} < \epsilon, \quad \forall n \in \mathbb{N} \cup \{0\},$$

this shows that the family $(a_n(\theta))_{n\ge0}, \theta \in [\epsilon_1, \pi - \epsilon_2]$ is equicontinuous.

On the other hand, we have also that there exists $M > 0$ such that

$$|a_n(\theta)| \le \frac{1}{\pi} \int_0^{\pi} |K(\theta', \theta)|^2 \omega(\theta';b) d\theta' \le M, \quad \forall \theta \in [\epsilon_1, \pi - \epsilon_2], \forall n \in \mathbb{N} \cup \{0\},$$

this shows that the family $(a_n(\theta))_{n\ge0}, \theta \in [\epsilon_1, \pi - \epsilon_2]$ is uniformly bounded.

From Arzela's Theorem, cf. [47] (p. 54) it follows that the family of functions $(a_n)_{n\ge0}$ is compact in $C[\epsilon_1, \pi - \epsilon_2]$ equipped the uniform norm. Therefore, from (37) every uniform convergent subsequence of $(a_n)_{n\ge0}$ converges to the same limit, hence for the whole sequence we have

$$\lim_{n\to\infty} a_n(\theta) = 0, \quad \forall \theta \in [\epsilon_1, \pi - \epsilon_2].$$

By using a similar argument, we also conclude that

$$\lim_{n\to\infty} b_n(\theta) = 0, \quad \forall\theta \in [\epsilon_1, \pi - \epsilon_2].$$

□

Proof of Theorem 4. From (2)–(4) it follows that

$$\frac{b+\bar{b}}{b} R_n(z;b) = \frac{(b+1)_n}{(\bar{b}+b+1)_n}\widehat{S}_n(z;b) + \frac{\bar{b}}{b}\frac{(\bar{b}+1)_n}{(\bar{b}+b+1)_n}\widehat{S}^*_n(z;b), \quad n \geq 1,$$

hence, from (1)

$$G_n(\cos\theta;\lambda,\eta) = \frac{2^{-n-1}(2\lambda)_n}{(\lambda)_n\kappa_n(b)}\left(\frac{b(b+1)_n}{\lambda(\bar{b}+b+1)_n}z^{-\frac{n}{2}}s_n(z;b) + \frac{\bar{b}(\bar{b}+1)_n}{\lambda(\bar{b}+b+1)_n}z^{-\frac{n}{2}}s^*_n(z;b)\right), z = e^{2\imath\theta}, \theta \in [0,\pi],$$

where κ_n is given by (5).

From [21] [(2.7) p. 200] we have that

$$\lim_{n\to\infty} \kappa_n(b) = \delta^{-1}(b), \tag{38}$$

where $\delta(b) = e^{\frac{1}{2\pi}\int_0^\pi \log(\omega(\theta))d\theta} > 0$ is the solution of the Szegő extremum problem, [21] (p. 200 & Th. 2.5 p. 204).

From the relation [21] (p. 206)

$$\delta^{-2}(b) = \Delta(0;b),$$

where

$$\Delta(z;b) = D^{-1}(z;b)\delta^{-1}(b),$$

cf. [21] [(3.4) p. 209], one finds that

$$\delta(b) = D(0;b).$$

The representation [21] [(4.6) p. 220] gives

$$e^{-\imath n\theta}s_n(e^{2\imath\theta};b) = \frac{\kappa_n(b)e^{\imath n\theta}\overline{D^{-1}(e^{2\imath\theta};b)} - a_n(\theta)e^{\imath n\theta}\overline{D^{-1}(e^{2\imath\theta};b)} + b_n(\theta)e^{-\imath n\theta}D^{-1}(e^{2\imath\theta};b)}{\delta(b)(|\kappa_n(b) - a_n(\theta)|^2 + |b_n(\theta)|^2)}, \tag{39}$$

with a_n and b_n defined as in Lemma 1.

Taking into account the representation

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)},$$

from [26] [(4) p. 47] one has

$$\frac{(\gamma_1)_n}{(\gamma_2)_n} = \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_1)}n^{\gamma_1-\gamma_2}\left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{as} \quad n \to \infty. \tag{40}$$

Hence, from (38)–(40)

$$2^{n+1}\Psi_n(\theta;\lambda,\eta) = \left(\frac{2\Gamma(\lambda)}{\Gamma(b)}n^{\imath\eta}e^{\imath n\theta}(1-e^{-2\imath\theta})^{-\bar{b}} + \frac{2\Gamma(\lambda)}{\Gamma(\bar{b})}n^{-\imath\eta}e^{-\imath n\theta}(1-e^{2\imath\theta})^{-b}\right)e^{-\eta\theta}\sin^\lambda\theta + r_n(\theta).$$

Now, Lemma 1 gives us that,

$$r_n(\theta) = \Re\left[\left(-\frac{1}{\Gamma(b)}a_n(\theta)e^{\imath n\theta}(1-e^{-2\imath\theta})^{-\bar{b}} + \frac{1}{\Gamma(b)}b_n(\theta)e^{-\imath n\theta}(1-e^{-2\imath\theta})^{-b}\right)O(1)\right] + O\left(\frac{1}{n}\right) = o(1), \quad \text{as} \quad n\to\infty,$$

uniformly in $\theta \in [\epsilon_1, \pi-\epsilon_2]$, being $\epsilon_1, \epsilon_2 > 0$ any fixed quantities sufficiently small so that $[\epsilon_1, \pi-\epsilon_2] \subset (0,\pi)$. This completes the proof of the theorem. □

4. Discussion

The present work is devoted to the study an energy-dependent potential related to the Rosen–Morse potential. The system is obtained by the addition of a potential term which depends on the function $\cot\theta$ and an energy dependence through a square root.

In order to show some numerical comparisons we will identify accordingly the parameters λ, η of the quantum model given by Theorem 1 and the parameters $\mathfrak{a}, \mathfrak{b}$ of the asymmetric trigonometric Rosen–Morse model, following the form given in [41]. In effect, by identifying the parameters one has $\mathfrak{b} = -\lambda\eta, \mathfrak{a} = \lambda$. Consequently, the quantum system defined by Theorem 1, in terms of the parameters $\mathfrak{a}$ and $\mathfrak{b}$ reads as,

$$\Psi''(\theta) + \left(E - \left(2\mathfrak{b}\cot\theta + \mathfrak{a}(\mathfrak{a}-1)\frac{1}{\sin^2\theta} + 2\frac{\mathfrak{b}}{\mathfrak{a}}\left(-\mathfrak{a} + \sqrt{E + \left(\frac{\mathfrak{b}}{\mathfrak{a}}\right)^2}\right)\cot\theta\right)\right)\Psi(\theta) = 0, \quad \theta \in (0,\pi), \mathfrak{a} \geq -\frac{1}{2}, \mathfrak{a} \neq 0,$$

which has, when $n \in \mathbb{N}\cup\{0\}$, the system of solutions

$$\Psi_n\left(\theta; \mathfrak{a}, -\frac{\mathfrak{b}}{\mathfrak{a}}\right) = G_n\left(\cos\theta; \mathfrak{a}, -\frac{\mathfrak{b}}{\mathfrak{a}}\right)e^{\frac{\mathfrak{b}}{\mathfrak{a}}\theta}\sin^{\mathfrak{a}}\theta = \frac{(2\mathfrak{a})_n 2^{-n}}{(\mathfrak{a})_n}\times (\cos\theta + \imath\sin\theta)^{-n}{}_2F_1\left(-n, \mathfrak{a} - \imath\frac{\mathfrak{b}}{\mathfrak{a}}; 2\mathfrak{a}; 1 - (\cos\theta + \imath\sin\theta)^2\right)e^{\frac{\mathfrak{b}}{\mathfrak{a}}\theta}\sin^{\mathfrak{a}}\theta, \quad \mathfrak{a} \geq -\frac{1}{2}, \mathfrak{a} \neq 0, \tag{41}$$

and the corresponding energy levels,

$$E_\Psi\left(n; \mathfrak{a}, -\frac{\mathfrak{b}}{\mathfrak{a}}\right) = (n+\mathfrak{a})^2 - \left(\frac{\mathfrak{b}}{\mathfrak{a}}\right)^2. \tag{42}$$

On the other hand, following Theorem 3, we will multiply by an adequate numerical constant γ_n the expression that defines the Φ_n functions (12),

$$\gamma_n = (-1)^{\lfloor\frac{n}{2}\rfloor}\frac{2^{-n}(2\mathfrak{a})_n(\mathfrak{a}+\frac{1}{2})_{\lfloor\frac{n}{2}\rfloor}}{\binom{n+2\mathfrak{a}-1}{n}(-n-\mathfrak{a}+\frac{1}{2})_n(1-n-\mathfrak{a})_{\lfloor\frac{n}{2}\rfloor}}\frac{(1-2n-2\mathfrak{a})_n}{(1-n-\mathfrak{a})_n}.$$

By (11) and (12), we have that the rescaled function $\Phi_n, n \in \mathbb{N}\cup\{0\}$ can be expressed as

$$\Phi_n(\theta; \mathfrak{a}, \mathfrak{b}) = \gamma_n(-\imath)^n P_n^{\left(-n-\mathfrak{a}-\frac{\imath\mathfrak{b}}{\mathfrak{a}+n}, -n-\mathfrak{a}+\frac{\imath\mathfrak{b}}{\mathfrak{a}+n}\right)}(\imath\cot\theta)e^{-\frac{\mathfrak{b}\theta}{n+\mathfrak{a}}}\sin^{(\mathfrak{a}+n)}\theta = \gamma_n(-\imath)^n\binom{-\mathfrak{a}-\frac{\imath\mathfrak{b}}{\mathfrak{a}+n}}{n}{}_2F_1\left(-n, 1-n-2\mathfrak{a}; 1-n-\mathfrak{a}-\frac{\imath\mathfrak{b}}{\mathfrak{a}+n}; \frac{1-\imath\cot\theta}{2}\right)e^{-\frac{\mathfrak{b}\theta}{n+\mathfrak{a}}}\sin^{(\mathfrak{a}+n)}\theta. \tag{43}$$

Notice that, by virtue of Corollary 1 and [23] [(4.7.1) p. 80] one has now that

$$\Phi_n(\theta;\mathfrak{a},0) = \Psi_n(\theta;\mathfrak{a},0), \quad \mathfrak{a} > -\frac{1}{2}, \mathfrak{a} \neq 0,$$

we recover in particular, the symmetric trigonometric Rosen–Morse oscillator.

In Figures 1 and 2 we plotted the wave functions for several toy values of the parameters. In Figures 3 and 4 we plotted the densities $\left(1 + \frac{\mathfrak{b}\cot\theta}{\mathfrak{a}(n+\mathfrak{a})}\right)|\Psi_n(\theta;\mathfrak{a},\mathfrak{b})|^2$ (see Theorem 2) and $|\Phi_n(\theta;\mathfrak{a},\mathfrak{b})|^2$ for the same values of the parameters $\mathfrak{a}, \mathfrak{b}$ with n fixed ($n = 5$). As can be appreciated, by fixing $\mathfrak{b}$ and making $\mathfrak{a}$ variable, the abscissas of the local maxima of the densities tend to be localized at the same points in both models, as $\mathfrak{a}$ increases. These points correspond to the regions where it is most likely the particle to be found. It should be also noticed from the expressions of the energies (13) and (42) that

$$\lim_{\mathfrak{a}\to\infty} E_\Psi(n;\mathfrak{a},\mathfrak{b}) - E_\Phi(n;\mathfrak{a},\mathfrak{b}) = 0,$$

for $\mathfrak{b}$ and n fixed. It could be interesting the further study of these facts.

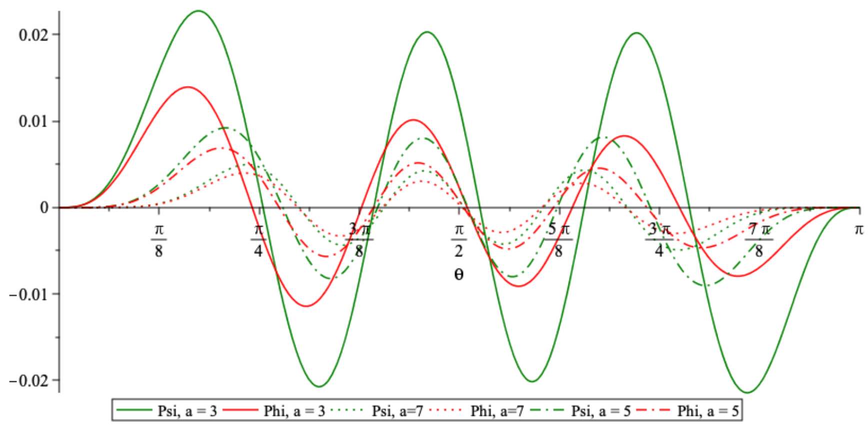

Figure 1. $\Psi_n(\theta;\mathfrak{a},\mathfrak{b})$ in green and $\Phi_n(\theta;\mathfrak{a},\mathfrak{b})$ in red with $n = 5$, $\mathfrak{b} = 1$ and $\mathfrak{a} = 3, 5, 7$.

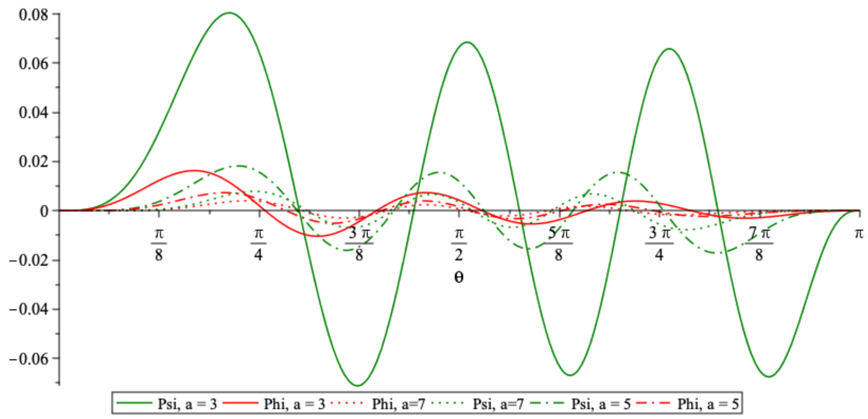

Figure 2. $\Psi_n(\theta;\mathfrak{a},\mathfrak{b})$ in green and $\Phi_n(\theta;\mathfrak{a},\mathfrak{b})$ in red with $n = 5$, $\mathfrak{b} = 3$ and $\mathfrak{a} = 3, 5, 7$.

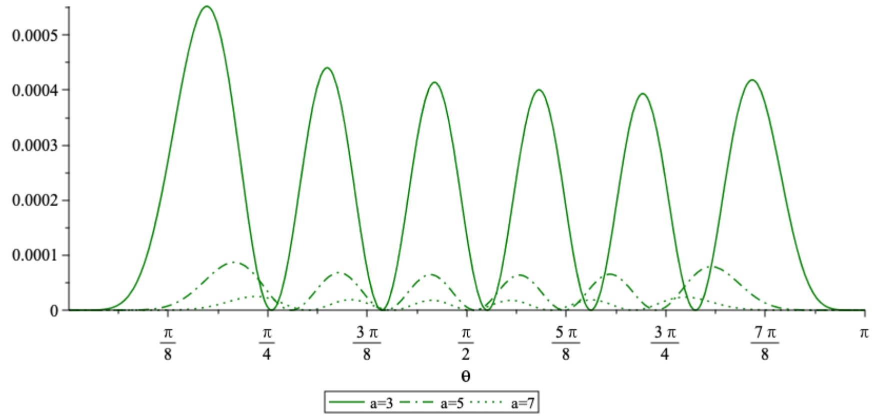

Figure 3. Density $\left(1 + \frac{\mathfrak{b}\cot\theta}{\mathfrak{a}(n+\mathfrak{a})}\right) |\Psi_n(\theta; \mathfrak{a}, \mathfrak{b})|^2$ with $n = 5$, $\mathfrak{b} = 1$ and $\mathfrak{a} = 3, 5, 7$.

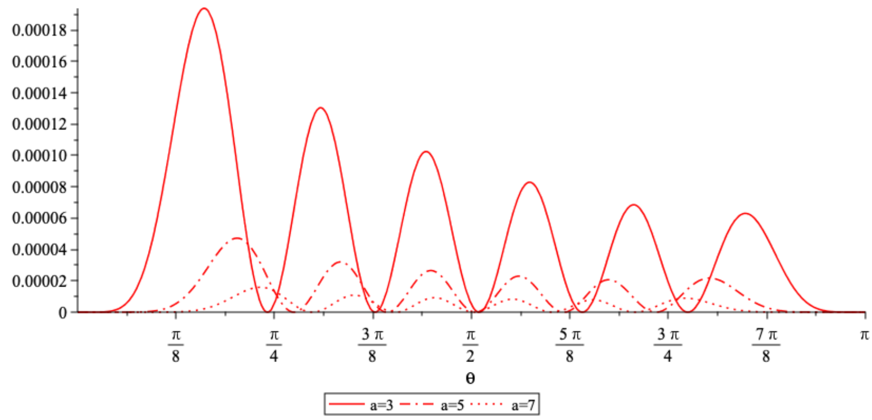

Figure 4. Density $|\Phi_n(\theta; \mathfrak{a}, \mathfrak{b})|^2$ with $n = 5$, $\mathfrak{b} = 1$ and $\mathfrak{a} = 3, 5, 7$.

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