mathematics

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Perturbation Theory for Quasinilpotents in Banach Algebras

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Received: 21 June 2020; Accepted: 12 July 2020; Published: 15 July 2020

Abstract: In this paper, we prove the following result by perturbation technique. If q is a quasinilpotent element of a Banach algebra and spectrum of $p+q$ for any other quasinilpotent p contains at most n values then $q^n = 0$. Applications to C* algebras are given.

Keywords: perturbation theory; socle; quasinilpotent elements; finite spectrum

1. Introduction

The perturbation technique is often useful for characterization of quasinilpotents of Banach algebras in the Jacobson radical and socle. For example, J. Zemánek gave a characterizations of Jacobson radical via perturbation by quasinilpotents [1]. A. Katavolos and C. Stamatopoulos gave some equivalent conditions for the identity of the set of quasinilpotents and Jacobson radical [2]. B. Aupetit gave some characterizations of Jacobson radicals and socles from the view of the cardinality of the spectrum, and also the root distribution of some special analytic multifuntion ([3,4]). In his monograph ([5], Theorem 5.6.10), he proved that in an infinite dimensional Banach space, there are two quasinilpotent compact operators, such that the spectrum of the sum of these two operators is infinite. The first author and Y. Turovskii gave a similar result for Calkin algebra in Hilbert space in ([6], Corollary 4.12). A more detailed analysis will be given in this paper. At the same time, inspired by Katavolos and Stamatopoulos's work, the author and X. Wang proved that if the perturbation of quasinilpotents has only one point in the spectrum in the Banach algebra, then all the quasinilpotent elements would be in a Jacobson radical ([7], Theorem 3.1). For the finite spectrum case, the first author gave a characterization of elements (especially for quasinilpotents) in the socle in [8]. In this paper, we develop the perturbation technique for characterization of nilpotency of quasinilpotents of a Banach algebra and give applications to non-commutativity of Banach algebras.

In what follows, all spaces and algebras are taken over the field $\mathbb{C}$ of complex numbers and all the algebras are unital. Let $\mathcal{A}$ be a unital Banach algebra. For an element a in $\mathcal{A}$, let $\sigma(a)$ mean the spectrum of a, and $\widehat{\sigma}(a)$, the polynomially convex hull of $\sigma(a)$, mean the full spectrum of a. The cardinality of $\sigma(a)$ is denoted by $\#\sigma(a)$; a is *quasinilpotent* (*finite-spectrum*, or *scattered*), if $\sigma(a)$ is $\{0\}$ (finite, or at most countable). The set of all quasinilpotent (finite-spectrum or scattered) elements in $\mathcal{A}$ is denoted by $\mathcal{Q}(A)$ ($\mathcal{F}(A)$ or $S(\mathcal{A})$). By $\mathrm{Rad}(\mathcal{A})$ we denote the *Jacobson radical* of $\mathcal{A}$. The *socle*, that is, the sum of all minimal one-sided ideals of $\mathcal{A}$, is denoted by $\mathrm{Soc}(\mathcal{A})$. We will also use $\mathrm{kh}(\mathrm{soc}A)$ to denote the *Kernel–Hull closure* of the socle of $\mathcal{A}$. One can find the definition and properties in ([5], Section 5.7).

2. Finite Spectrum Problem

The following lemma gave a characterization of socle from the perturbation theory.

Lemma 1 ([8], Theorem 1)**.** *] Let $\mathcal{A}$ be a semi-simple unital Banach algebra; then the following statements are equivalent:*

(i) $x \in \mathrm{Soc}(\mathcal{A})$*;*
(ii) $\widehat{\sigma}(x+a)\backslash\widehat{\sigma}(a)$ *is finite for every* $a \in \mathcal{A}$*.*

In order to calculate easily, we can modify Lemma 1 to the following.

Lemma 2. *Let $\mathcal{A}$ be a semi-simple unital Banach algebra; then the following statements are equivalent:*

(i) $x \in \mathrm{Soc}(\mathcal{A})$*;*
(ii) $\sigma(x+a)\backslash\widehat{\sigma}(a)$ *is finite for every* $a \in \mathcal{A}$*.*

Proof. In fact, it is known that under the condition of Lemma 1, $\widehat{\sigma}(x+a)\backslash\widehat{\sigma}(a) = \sigma(x+a)\backslash\widehat{\sigma}(a)$. For example, one can see the proof at the end of Section 3.1 of [6]. To make the reader easier, we would like to give the details here. This follows from the fact that if a compact set in $\mathbb{C}$ is the union of a polynomially convex set K and a countable set Z, then it is polynomially convex. Indeed, let $\lambda \in \widehat{K \cup Z}$. If $\lambda \in \widehat{K}$, then clearly $\lambda \in K$. Assume now that $\lambda \notin K$. As K is polynomially convex, there is a polynomial p such that

$$\max_{\mu \in K} |p(\mu)| < |p(\lambda)| \leq \max_{\mu \in Z} |p(\mu)|.$$

Let $Z' = \{\mu \in Z : |p(\lambda)| \leq |p(\mu)|\}$. It is easy to see that Z' is a compact set and $\lambda \in \widehat{Z'}$. But every countable compact set in $\mathbb{C}$ is polynomially convex. Thus, $\lambda \in Z' \subset Z$. □

Thus, the following problem might be interesting. Will the change be uniform?

Problem 1. *Let $\mathcal{A}$ be a semi-simple unital Banach algebra, $x \in \mathrm{Soc}(\mathcal{A})$. Is there a (fixed) positive integer $Prank(x)$, such that $\#(\sigma(x+a)\backslash\widehat{\sigma}(a)) \leq Prank(x)$, for every $a \in \mathcal{A}$?*

The answer is no.

There are two examples in Hilbert space. Let H be a Hilbert space, with the basis $\{e_i\}_{i=0}^{\infty}$. $\mathcal{A} = B(H)$. For every positive integer n, let $H_n = span\{e_0, e_1, ..., e_{n-1}\}$. Then $H = H_n \oplus H_n^{\perp}$, and $B(H) \supseteq B(H_n) \oplus B\left(H_n^{\perp}\right)$. The first example is for rank one operators.

Example 1. *Let p_0 be the projection on $span\{e_0\}$. Then $p_0 \in \mathrm{Soc}\mathcal{A}$. Let $a_n \in B(H_n)$, such that $a_n(e_i) = e_{i-1}$ for $i = 1, ..., n-1$, and $a_n(e_0) = e_{n-1}$. Then a_n is the primitive permutation matrix, and it is well known that $\sigma(a_n) \in \{\lambda \in \mathbb{C} : \lambda^n = 1\}$ and $\sigma(a_n)$ has at most n points. Let $A_n = a_n \oplus 0$; then $\sigma(A_n) \in \{\lambda \in \mathbb{C} : \lambda^n = 1\} \cup \{0\}$ and $\widehat{\sigma}(A_n)) = \sigma(A_n)$ is finite. Let us calculate the spectrum of $p_0 + A_n$. In fact, we only need to consider the matrix*

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is $p(\lambda) = \lambda^n - \lambda^{n-1} - 1$. It is clear that no eigenvalue of this matrix is in $\{\lambda \in \mathbb{C} : \lambda^n = 1\} \cup \{0\}$. Next please note that $p(\lambda)$ has n different roots. If not, then $p(\lambda)$ and $p'(\lambda) = n\lambda^{n-1} - (n-1)\lambda^{n-2}$ has at least one same root, but it is easy to check that it is impossible. Thus, for any $n > 0$, there is an operator A_n, such that $\#(\sigma(p_0 + A_n)\backslash\widehat{\sigma}(A_n))$ is n.

Another example is for quasinilpotent operators.

Example 2. *Let $q_0(e_1) = e_0$, and $q_0(e_i) = 0$, for every $i \neq 1$. Then clearly q_0 is nilpotent. For every $n > 2$, let $b_n(e_i) = e_{i-1}$ for $i = 2, ..., n-1$, $b_n(e_1) = 0$, and $b_n(e_0) = e_{n-1}$. Then $q_0|_{H_n} + b_n$ is a primitive permutation matrix, and at the same time b_n is nilpotent. Let $B_n = b_n \oplus 0$; then the number of spectrum of $q_0 + B_n$ is $n+1$, and q_0, B_0 is nilpotent.*

In the paper [9], we gave a characterization of quasinilpotent elements which are in the socle in semi-simple Banach algebras.

Lemma 3. *Let $\mathcal{A}$ be a semi-simple unital Banach algebra; then a quasinilpotent element $q \in \mathcal{A}$ is in the socle if and only if $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{F}(\mathcal{A})$, where the $\mathcal{Q}(\mathcal{A})$ (or $\mathcal{F}(\mathcal{A})$) means the set of all the quasinilpotent (or finite-spectrum) elements in $\mathcal{A}$.*

Now we will give a more clear characterization of quasinilpotent elements with $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A})$, where $\mathcal{I}_n(\mathcal{A})$ means the set of elements with at most n-points spectra. It is trivial if $\mathcal{A}$ is finite-dimensional. Please note that $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A}) \subset \mathcal{F}(\mathcal{A})$l then $q \in \mathrm{Soc}(\mathcal{A})$ by Lemma 3. Thus, q must be nilpotent. But in fact we can show that $q^n = 0$.

In the next theorem, the main technical tool we used is the Jacobson density theorem in Sinclair's form: if $\xi_1, \cdots, \xi_n$ and $\eta_1, \cdots, \eta_n$ are two linearly independent systems of vectors in the underlying space X_π of a strictly irreducible representation $\pi \in \mathrm{Irr}(\mathcal{A})$ of a Banach algebra $\mathcal{A}$, then there is an invertible element $a \in \mathcal{A}^{-1}$ such that

$$\pi(a)\xi_i = \eta_i$$

for $i = 1, \cdots, n$, where $\mathcal{A}^{-1}$ means the set of all invertible elements in $\mathcal{A}$. One can see [3], Theorem 4.2.5, and [3], Corollary 4.2.6.

Recall that a set of vectors is linearly independent if all of its finite subsets are linearly independent. For instance, it is easy to show (see [2], observation on page 161) that if T is a quasinilpotent operator on a Banach space X and $x \in X$, then the set $\{x, Tx, T^2x, \cdots\}\backslash\{0\}$ is linearly independent.

Theorem 1. *Let $\mathcal{A}$ be a semi-simple unital Banach algebra. If there is a nonzero quasinilpotent element $q \in \mathcal{A}$ with $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A})$, then $q^n = 0$.*

Proof of Theorem 1. If for every irreducible representation π, $\pi(q^n) = 0$, then $q^n \in Rad(\mathcal{A}) = 0$. Thus, we can assume there is an irreducible representation π on a Banach space X and an $x \in X$ such $\pi(q^n)x \neq 0$. In this case, the set $S = \{x_1, x_2, \cdots, x_{n+1}, \cdots\}\backslash\{0\}$ is linearly independent and contains $x_1, x_2, \cdots, x_{n+1}$, where $x_k = \pi(q^{k-1})x$ by ([2], (the observation in page 161)). If $x_{n+2} \in S$, we can find an invertible $a \in \mathcal{A}$ such that

$$\pi(a)x_1 = x_1,$$
$$\pi(a)x_i = [\frac{i}{2}]x_{i-1} + (-1)^{i+1}x_i,\ 2 \leq i \leq n+1. \tag{1}$$

We have

$$\pi(a^{-1})x_i = \sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{i+1}x_i \tag{2}$$

for all $1 \leq i \leq n+1$. This result is obvious for $n = 1$ and 2. Suppose $2 < m < n+1$ and the result is true for m; that is,

$$\pi(a^{-1})x_m = \sum_{k=1}^{m-1}(-1)^{[\frac{m-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{m+1}x_m.$$

Let $\pi(a^{-1})$ act on two sides of the Equation (1). We have

$$\begin{aligned}\pi(a^{-1})x_{m+1} =&(-1)^{m+1}[\frac{m+1}{2}]\pi(a^{-1})x_m + (-1)^{m+2}x_{m+1}\\ =&\sum_{k=1}^{m-1}(-1)^{[\frac{m-1}{2}]+[\frac{k}{2}]+m+1}[\frac{m+1}{2}][\frac{m}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + [\frac{m+1}{2}]x_m + (-1)^{m+2}x_{m+1}\\ =&\sum_{k=1}^{m}(-1)^{[\frac{m}{2}]+[\frac{k}{2}]}[\frac{m+1}{2}][\frac{m}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{m+2}x_{m+1}.\end{aligned}$$

Thus, Equation (2) holds for all $1 \leq i \leq n+1$ by the induction hypothesis.

Let $p = a^{-1}qa$. Then p is quasinilpotent. We can calculate that

$$\pi(q+p)x_1 = x_1,$$

$$\begin{aligned}\pi(q+p)x_i =&x_{i+1} + \pi(a^{-1})([\frac{i}{2}]x_i + (-1)^{i+1}x_{i+1})\\ =&x_{i+1} + [\frac{i}{2}](\sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{i+1}x_i)\\ &+ (-1)^{i+1}(\sum_{k=1}^{i}(-1)^{[\frac{i}{2}]+[\frac{k}{2}]}[\frac{i+1}{2}][\frac{i}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{i+2}x_{i+1})\\ =&[\frac{i}{2}](\sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k) + (-1)^{i+1}[\frac{i}{2}]x_i\\ &+ (-1)^{i+1}(\sum_{k=1}^{i-1}(-1)^{[\frac{i}{2}]+[\frac{k}{2}]}[\frac{i+1}{2}][\frac{i}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k) + (-1)^{i+1}[\frac{i+1}{2}]x_i\\ =&\sum_{k=1}^{i-1}((-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}] + (-1)^{i+1+[\frac{i}{2}]+[\frac{k}{2}]}[\frac{i+1}{2}])[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+1}{2}]x_k + (-1)^{i+1}ix_i\\ =&\sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}i[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+1}{2}]x_k + (-1)^{i+1}ix_i, \quad 2 \leq i \leq n+1.\end{aligned}$$

We can always find some real numbers $\alpha_{i,j}$, $j = 1, \cdots, i$ such that

$$\pi(q+p)(\alpha_{i,1}x_1 + \alpha_{i,2}x_2 + \cdots + \alpha_{i,i}x_i) = (-1)^{i+1}i(\alpha_{i,1}x_1 + \alpha_{i,2}x_2 + \cdots + \alpha_{i,i}x_i).$$

Thus
$$\{(-1)^{i+1}i : i \in \mathbb{N}, 1 \le i \le n+1\} \subset \sigma(\pi(q+p)) \subset \sigma(q+p).$$

If $x_{n+2} = 0$, we can find an invertible $b \in \mathcal{A}$ such that

$$\pi(b)x_1 = x_1,\ \pi(b)x_{n+1} = (-1)^{n+2}(n+1)x_n + x_{n+1},$$

$$\pi(b)x_i = [\frac{i}{2}]x_{i-1} + (-1)^{i+1}x_i,\ 2 \le i \le n.$$

We also have

$$\pi(b^{-1})x_i = \sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{i+1}x_i$$

for $1 \le i \le n$ by the induction hypothesis and

$$\pi(b^{-1})x_{n+1} = (-1)^{n+1}(n+1)\pi(b^{-1})x_n + x_{n+1}.$$

Let $r = b^{-1}qb$. Then r is quasinilpotent. We can calculate that

$$\pi(q+r)x_1 = x_1,$$

$$\begin{aligned}
\pi(q+r)x_i =& x_{i+1} + \pi(b^{-1})([\frac{i}{2}]x_i + (-1)^{i+1}x_{i+1})\\
=& x_{i+1} + [\frac{i}{2}](\sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{i+1}x_i)\\
&+ (-1)^{i+1}(\sum_{k=1}^{i}(-1)^{[\frac{i}{2}]+[\frac{k}{2}]}[\frac{i+1}{2}][\frac{i}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k + (-1)^{i+2}x_{i+1})\\
=& [\frac{i}{2}](\sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k) + (-1)^{i+1}[\frac{i}{2}]x_i\\
&+ (-1)^{i+1}(\sum_{k=1}^{i-1}(-1)^{[\frac{i}{2}]+[\frac{k}{2}]}[\frac{i+1}{2}][\frac{i}{2}]\cdots[\frac{k+2}{2}][\frac{k+1}{2}]x_k) + (-1)^{i+1}[\frac{i+1}{2}]x_i\\
=& \sum_{k=1}^{i-1}((-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}[\frac{i}{2}] + (-1)^{i+1+[\frac{i}{2}]+[\frac{k}{2}]}[\frac{i+1}{2}])[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+1}{2}]x_k + (-1)^{i+1}ix_i\\
=& \sum_{k=1}^{i-1}(-1)^{[\frac{i-1}{2}]+[\frac{k}{2}]}i[\frac{i}{2}][\frac{i-1}{2}]\cdots[\frac{k+1}{2}]x_k + (-1)^{i+1}ix_i, \qquad 2 \le i \le n.
\end{aligned}$$

$$\pi(p+r)x_{n+1} = (n+1)^2(\pi(b^{-1})x_n) + (-1)^{n+2}(n+1)x_{n+1}.$$

We can always find some real numbers $\beta_{i,j}$, $j = 1, \cdots, i$ such that

$$\pi(q+r)(\beta_{i,1}x_1 + \beta_{i,2}x_2 + \cdots + \beta_{i,i}x_i) = (-1)^{i+1}i(\beta_{i,1}x_1 + \beta_{i,2}x_2 + \cdots + \beta_{i,i}x_i).$$

Thus
$$\{(-1)^{i+1}i : i \in \mathbb{N}, 1 \le i \le n+1\} \subset \sigma(\pi(q+r)) \subset \sigma(q+r).$$

Both of these conclusions contradict that $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A})$. Thus, we have $q^n = 0$. $\square$

Remark 1. *Theorem 1 is a generalization of Theorem 3.1 in [7], which is the case $n = 1$.*

Remark 2. *The reverse of Theorem 1 does not hold. For example, we can consider*

$$A = M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

Then for any quasinilpotent element $q \in A$, we have that q is nilpotent and $q^2 = 0$, but we can easily find $p \in A$ is nilpotent, such that $\#\sigma(p + q) > 2$.

3. Noncommutativity

It is well known that a C* algebra is commutative if and only if there is no nonzero quasinilpotent element. Now we want to use quasinilpotent elements to character C* algebra which is nearly commutative, following Behncke's way in [10].

Theorem 2. *Let A be a unital C* algebra. If $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{F}(\mathcal{A})$, then any irreducible representation π of $\mathcal{A}$ is finite dimensional.*

Proof of Theorem 2. Note that if $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{F}(\mathcal{A})$, then for every $q \in \mathcal{Q}(\mathcal{A})$, q is nilpotent by Lemma 3. Thus, there is no ∞-nilpotent element in the sense of [10]. Thus, we can get the result by [10], Theorem 2. $\square$

Remark 3. *If we have $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A})$, for some n in the condition of Theorem 2, then $q^n = 0$, for every quasinilpotent element by Theorem 1. Then the irreducible representation is n dimensional by [10], Theorem 1.*

Remark 4. *The reverse of Theorem 2 does not hold. For example, let $\mathcal{A} = M_2(\mathbb{C})^\infty$. Then the irreducible representation of $\mathcal{A}$ is 2 dimensional. Then we can find two nilpotent elements a, b, but $a + b$ has infinite spectrum.*

Problem 2. *In the view of Theorem 2, we know that under the condition of $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{F}(\mathcal{A})$, the C* algebra must be a CCR algebra. One can find the definition of CCR algebra in [11], Definition 1.5.1. Thus, there is a problem: which kind of CCR algebra has the property $\mathcal{Q}(\mathcal{A}) + \mathcal{Q}(\mathcal{A}) \subset \mathcal{F}(\mathcal{A})$?*

Now we turn to the case that the socle in a Banach algebra is finite dimensional. One can find some interesting examples and theorems for this kind of Banach algebra in [12]. We have the following result:

Theorem 3. *Let $\mathcal{A}$ be a semi-simple unital Banach algebra with finite dimensional socle. If there is a nonzero quasinilpotent element $q \in \mathcal{A}$ with $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A})$ for some n. Then for any primitive ideal P with q not being in P, we have that $\mathcal{A}/P$ is isomorphic to $M_{m_p}(\mathbb{C})$ for some $m_p \in \mathbb{N}$.*

Proof of Theorem 3. Without loss of generality, we can assume that $dim(Soc(\mathcal{A})) \leq n^2$ for some $n \in \mathbb{N}$. It is well known that it is necessary to prove that for any irreducible representation $\pi : \mathcal{A} \to B(X)$ with $\pi(q) \neq 0$, then $dimX \leq n$ by ([5], Chapter IV Exercise 13). Now we assume that $dimX > n$. Note that $q + \mathcal{Q}(\mathcal{A}) \subset \mathcal{I}_n(\mathcal{A})$, so it is known that $q^n = 0$ by Theorem 1. We can pick $x \in X$ such that $\pi(q)x \neq 0$, $\pi(q)^{k-1} \neq 0$, $\pi(q)^k = 0$ for some $1 < k \leq n$. Let $V_1 = span\{x, \pi(q)x, ..., \pi(q)^{k-1}\}$. Note that V_1 is an invariant subspace of $\pi(q)$; we can consider $\pi(q)$: $X/V_1 \to X/V_1$. If $\pi(q)$= 0, then clearly we can find $V \subset X$, which is a finite dimensional invariant subspace of $\pi(q)$ and $dimV > n$. If $\pi(q)$ is not zero, then it is also nilpotent and so we can find a subspace $V_2 \subset X$, such that V_1 is a proper subset of V_2, and V_2 is an invariant subspace of $\pi(q)$ by the same method. If $dimV_2 \leq n$, then we can continuous this process and finally we can find that $V \subset X$, V is a finite dimensional invariant subspace of $\pi(q)$, and $dimV > n$.

Let $q_1 = \pi(q)|_V$, $q_1(x) = y \neq 0$ and $\{e_i : i = 0, ..., m-1\}$ be a basis of V, where $dimV = m > n$. For any $\xi \in V$, we can find $a \in \mathcal{A}$, such that $\pi(a)(e_0) = x$, $\pi(a)(e_i) = 0$, for $i = 1, ..., m-1$ by Jacobson's density theorem ([5], Theorem 4.2.5). At the same time, we can also find $b \in \mathcal{A}$, such that V is the invariant subspace of $\pi(b)$ with $\pi(b)y = \xi$ by Jacobson density theorem. Thus, $\pi(AqA)$ has a subspace W, such that V is the invariant subspace of W, and $W|_V$ has all the one rank matrices on V. Hence $dimW > n^2$, so is $\pi(AqA)$. Note $q \in \mathrm{Soc}(\mathcal{A})$, so we have $AqA \subset \mathrm{Soc}(\mathcal{A})$. Recall $dim(\mathrm{Soc}(\mathcal{A})) \leq n^2$; then $dim(AqA) \leq n^2$. It is a contradiction. $\square$

Remark 5. *The reverse of Theorem 3 does not hold. For example, let $\mathcal{A} = C(X)$, where $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Then it is a semisimple commutative Banach algebra, such that every primitive ideal has codimension 1, but the socle is not finite dimensional.*

Author Contributions: conceptualization, P.C.; methodology, X.W. ang and P.C.; validation, X.W. and P.C.; formal analysis, X.W. and P.C.; investigation, X.W. and P.C.; writing–original draft preparation, P.C.; writing–review and editing, X.W. and P.C.; supervision, P.C.; project administration, P.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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