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New Applications of the Bernardi Integral Operator

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Abstract: Let $A(p,n)$ be the class of $f(z)$ which are analytic p-valent functions in the closed unit disk $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$. The expression $B_{-m-\lambda}f(z)$ is defined by using fractional integrals of order λ for $f(z) \in A(p,n)$. When $m = 1$ and $\lambda = 0$, $B_{-1}f(z)$ becomes Bernardi integral operator. Using the fractional integral $B_{-m-\lambda}f(z)$, the subclass $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$ of $A(p,n)$ is introduced. In the present paper, we discuss some interesting properties for $f(z)$ concerning with the class $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$. Also, some interesting examples for our results will be considered.

Keywords: analytic p-valent function; Bernardi integral operator; Libera integral operator; fractional integral; gamma function; Miller–Mocanu lemma

1. Introduction

Let $A(p,n)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad , \quad n \in \mathbb{N} = \{1,2,3,\ldots\} \tag{1}$$

that are analytic p-valent functions in the closed unit disk $\overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$. For functions $f(z) \in A(p,n)$, we consider

$$B_{-1}f(z) = \frac{p+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt = z^p + \sum_{k=p+n}^{\infty}\frac{p+\gamma}{k+\gamma}a_k z^k \quad , \ \gamma \in \mathbb{N}. \tag{2}$$

If $p = 1$, for $f(z) \in A(1,n)$

$$B_{-1}f(z) = \frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt = z + \sum_{k=n+1}^{\infty}\frac{1+\gamma}{k+\gamma}a_k z^k \quad , \ \gamma \in \mathbb{N} \tag{3}$$

is considered by Bernardi [1]. Therefore, $B_{-1}f(z)$ in (3) is said to be the Bernardi integral operator. Further, if $p = 1$ and $\gamma = 1$, for $f(z) \in A(1,n)$

$$L_{-1}f(z) = \frac{2}{z}\int_0^z f(t)dt = z + \sum_{k=n+1}^{\infty}\frac{2}{k+1}a_k z^k \tag{4}$$

is defined by Libera [2]. Therefore, $L_{-1}f(z)$ in (4) is called the Libera integral operator.

For $B_{-1}f(z)$ in (2), we consider

$$B_{-2}f(z) = B_{-1}\left(B_{-1}f(z)\right) = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p+\gamma}{k+\gamma}\right)^2 a_k z^k \tag{5}$$

and

$$B_{-m}f(z) = B_{-1}\left(B_{-m+1}f(z)\right) = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p+\gamma}{k+\gamma}\right)^m a_k z^k \tag{6}$$

with $m \in \mathbb{N}$ and $B_0 f(z) = f(z)$.

From the various definitions of fractional calculus of $f(z) \in A(p,n)$ (that is, fractional integrals and fractional derivatives) given in the literature, we would like to recall here the following definitions for fractional calculus which were used by Owa [3] and Owa and Srivastava [4].

Definition 1. *The fractional integral of order λ for $f(z) \in A(p,n)$ is defined by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}dt \ , \ \ \lambda > 0 \tag{7}$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$ and Γ is the Gamma function.

With the above definitions, we know that

$$D_z^{-\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1+\lambda)} z^{p+\lambda} + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} a_k z^{k+\lambda} \tag{8}$$

for $\lambda > 0$ and $f(z) \in A(p,n)$. Using the fractional integral operator over $A(p,n)$, we consider

$$B_{-\lambda}f(z) = \frac{\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)} z^{1-\gamma-\lambda} D_z^{-\lambda}\left(z^{\gamma-1}f(z)\right) = z^p + \sum_{k=p+n}^{\infty} \frac{\Gamma(k+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(k+\gamma+\lambda)} a_k z^k, \tag{9}$$

where $0 \leq \lambda \leq 1$. If $\lambda = 0$ in (9), then $B_0 f(z) = f(z)$ and if $\lambda = 1$ in (9), then we see that

$$B_{-1}f(z) = \frac{p+\gamma}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt. \tag{10}$$

With the operator $B_{-\lambda}f(z)$ given by (9), we know

$$B_{-m-\lambda}f(z) = B_{-m}\left(B_{-\lambda}f(z)\right) = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p+\gamma}{k+\gamma}\right)^m \frac{\Gamma(k+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(k+\gamma+\lambda)} a_k z^k, \tag{11}$$

where $0 \leq \lambda \leq 1$ and $m \in \mathbb{N}$. The operator $B_{-m-\lambda}f(z)$ is a generalization of the Bernardi integral operator $B_{-1}f(z)$. From the definition of $B_{-m-\lambda}f(z)$, we know that

$$B_{-m-\lambda}f(z) = B_{-m}\left(B_{-\lambda}f(z)\right) = B_{-\lambda}\left(B_{-m}f(z)\right). \tag{12}$$

From s different boundary points $z_l \ (l = 1,2,3,...,s)$ with $|z_l| = 1$, we consider

$$\alpha_s = \frac{1}{s}\sum_{l=1}^{s}\frac{B_{-m-\lambda}f(z_l)}{z_l^p}, \tag{13}$$

where $\alpha_s \in e^{i\beta}B_{-m-\lambda}f(\mathbb{U})$, $\alpha_s \neq 1$, $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ and $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. For such α_s, if $f(z) \in A(p,n)$ satisfies

$$\left|\frac{e^{i\beta}\frac{B_{-m-\lambda}f(z)}{z^p} - \alpha_s}{e^{i\beta} - \alpha_s} - 1\right| < \rho \ \ , \ \ z \in \mathbb{U} \tag{14}$$

for some real $\rho > 0$, we say that the function $f(z)$ belongs to the class $T_{p,n}\left(\alpha_s, \beta, \rho; m, \lambda\right)$.

It is clear that a function $f(z) \in A(p,n)$ belongs to the class $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$ provided that the condition

$$\left| \frac{B_{-m-\lambda} f(z)}{z^p} - 1 \right| < \rho \left| e^{i\beta} - \alpha_s \right| \quad , \ z \in \mathbb{U}, \tag{15}$$

is satisfied. If we consider the function $f(z) \in A(p,n)$ given by

$$f(z) = z^p + \left(\frac{p+n+\gamma}{p+\gamma} \right)^m \frac{\Gamma(p+\gamma)\Gamma(p+n+\gamma+\lambda)}{\Gamma(p+n+\gamma)\Gamma(p+\gamma+\lambda)} \rho (e^{i\beta} - \alpha_s) z^{p+n} \tag{16}$$

then $f(z)$ satisfies

$$\left| \frac{B_{-m-\lambda} f(z)}{z^p} - 1 \right| = \rho \left| e^{i\beta} - \alpha_s \right| |z|^m < \rho \left| e^{i\beta} - \alpha_s \right| \quad , \ z \in \mathbb{U}. \tag{17}$$

Therefore, $f(z)$ given by (16) is in the class $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$.

Discussing our problems for $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$, we have to recall here the following lemma due to Miller and Mocanu [5,6] (refining the old one in Jack [7].)

Lemma 1. *Let the function $w(z)$ given by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots \quad , \ n \in \mathbb{N} \tag{18}$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, $(0 < |z_0| < 1)$ then there exists a real number $k \geq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \tag{19}$$

and

$$Re \left(1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right) \geq k. \tag{20}$$

2. Properties of Functions Concerning with the Class $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$

We begin with a sufficient condition on a function $f(z) \in A(p,n)$ which makes it a member of $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$.

Theorem 1. *If $f(z) \in A(p,n)$ satisfies*

$$\left| \frac{B_{-m-\lambda+1} f(z)}{B_{-m-\lambda} f(z)} - 1 \right| < \frac{\left| e^{i\beta} - \alpha_s \right| n\rho}{(p+\gamma)(1 + \left| e^{i\beta} - \alpha_s \right| \rho)} \quad , z \in \mathbb{U} \tag{21}$$

for some α_s given by (13) with $\alpha_s \neq 1$ such that $z_g \in \partial\mathbb{U}$ $(g = 1, 2, 3, ..., s)$, and for some real $\rho > 1$, then

$$\left| \frac{B_{-m-\lambda} f(z)}{z^p} - 1 \right| < \rho \left| e^{i\beta} - \alpha_s \right| \quad , \ z \in \mathbb{U} \tag{22}$$

that is, $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$.

Proof. We introduce the function $w(z)$ defined by

$$w(z) = \frac{e^{i\beta} \frac{B_{-m-\lambda} f(z)}{z^p} - \alpha_s}{e^{i\beta} - \alpha_s} - 1 = \frac{e^{i\beta}}{e^{i\beta} - \alpha_s} \left\{ \sum_{k=p+n}^{\infty} \left(\frac{p+\gamma}{k+\gamma} \right)^m \frac{\Gamma(k+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(k+\gamma+\lambda)} a_k z^{k-p} \right\}. \tag{23}$$

Then, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and

$$\frac{B_{-m-\lambda}f(z)}{z^p} = 1 + (1 - e^{-i\beta}\alpha_s)w(z). \tag{24}$$

Noting that

$$B_{-m-\lambda+1}f(z) = \frac{\gamma}{p+\gamma}B_{-m-\lambda}f(z) + \frac{1}{p+\gamma}z(B_{-m-\lambda}f(z))', \tag{25}$$

we obtain that

$$\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)} - 1 = \frac{(1 - e^{-i\beta}\alpha_s)zw'(z)}{(p+\gamma)(1 + (1 - e^{-i\beta}\alpha_s)w(z))} \tag{26}$$

and that

$$\left|\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)} - 1\right| = \frac{1}{p+\gamma}\left|\frac{(1 - e^{-i\beta}\alpha_s)zw'(z)}{1 + (1 - e^{-i\beta}\alpha_s)w(z)}\right| < \frac{|e^{i\beta} - \alpha_s|n\rho}{(p+\gamma)(1 + |e^{i\beta} - \alpha_s|\rho)}. \tag{27}$$

by employing (21). Assume, to arrive at a contradiction, that there exists a point z_0, $(0 < |z_0| < 1)$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \tag{28}$$

Then, we can write that $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = kw(z_0), (k \geq n)$ by Lemma 1. For such a point $z_0 \in \mathbb{U}$, $f(z)$ satisfies

$$\begin{aligned}
\left|\frac{B_{-m-\lambda+1}f(z_0)}{B_{-m-\lambda}f(z_0)} - 1\right| &= \frac{1}{p+\gamma}\left|\frac{(1 - e^{-i\beta}\alpha_s)z_0 w'(z_0)}{1 + (1 - e^{-i\beta}\alpha_s)w(z_0)}\right| \\
&= \frac{1}{p+\gamma}\left|\frac{(1 - e^{-i\beta}\alpha_s)k\rho}{1 + (1 - e^{-i\beta}\alpha_s)\rho e^{i\theta}}\right| \\
&\geq \frac{|1 - e^{-i\beta}\alpha_s|n\rho}{(p+\gamma)(1 + |1 - e^{-i\beta}\alpha_s|\rho)} \\
&= \frac{|e^{i\beta} - \alpha_s|n\rho}{(p+\gamma)(1 + |e^{i\beta} - \alpha_s|\rho)}.
\end{aligned} \tag{29}$$

Since this contradicts our condition (21), we see that there is no z_0, $(0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. This shows us that

$$|w(z)| = \left|\frac{e^{i\beta}\left(\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right)}{e^{i\beta} - \alpha_s}\right| < \rho \quad , \ z \in \mathbb{U}, \tag{30}$$

that is, that

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right| < \rho\left|e^{i\beta} - \alpha_s\right| \quad , \ z \in \mathbb{U}. \tag{31}$$

This completes the proof of the theorem. □

Example 1. *We consider a function $f(z) \in A(p,n)$ given by*

$$f(z) = z^p + a_{p+n}z^{p+n} \ , z \in \mathbb{U} \tag{32}$$

with $0 < |a_{p+n}| < \frac{1}{2Q}$, where

$$Q = \left(\frac{p+\gamma}{p+n+\gamma}\right)^m \frac{\Gamma(p+n+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(p+n+\gamma+\lambda)}. \tag{33}$$

For such $f(z)$, we have

$$\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)} = \frac{z^p + \left(\frac{p+n+\gamma}{p+\gamma}\right) Qa_{p+n}z^{p+n}}{z^p + Qa_{p+n}z^{p+n}} = \frac{1 + \left(\frac{p+n+\gamma}{p+\gamma}\right) Qa_{p+n}z^n}{1 + Qa_{p+n}z^n} \tag{34}$$

that is, that

$$\left|\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)} - 1\right| = \left|\frac{\left(\frac{n}{p+\gamma}\right) Qa_{p+n}z^n}{1 + Qa_{p+n}z^n}\right| < \frac{\left(\frac{n}{p+\gamma}\right) Q|a_{p+n}|}{1 - Q|a_{p+n}|} \quad , \quad z \in \mathbb{U}. \tag{35}$$

Now, we consider five boundary points such that

$$z_1 = e^{-i\frac{arg(a_{p+n})}{n}} \tag{36}$$

$$z_2 = e^{i\frac{\pi-6arg(a_{p+n})}{6n}} \tag{37}$$

$$z_3 = e^{i\frac{\pi-4arg(a_{p+n})}{4n}} \tag{38}$$

$$z_4 = e^{i\frac{\pi-3arg(a_{p+n})}{3n}} \tag{39}$$

and

$$z_5 = e^{i\frac{\pi-2arg(a_{p+n})}{2n}}. \tag{40}$$

For these five boundary points, we know that

$$\frac{B_{-m-\lambda}f(z_1)}{z_1^p} = 1 + Qa_{p+n}e^{-iarg(a_{p+n})} = 1 + Q|a_{p+n}|, \tag{41}$$

$$\frac{B_{-m-\lambda}f(z_2)}{z_2^p} = 1 + Qa_{p+n}e^{i\left(\frac{\pi}{6} - arg(a_{p+n})\right)} = 1 + \frac{\sqrt{3}+i}{2}Q|a_{p+n}|, \tag{42}$$

$$\frac{B_{-m-\lambda}f(z_3)}{z_3^p} = 1 + Qa_{p+n}e^{i\left(\frac{\pi}{4} - arg(a_{p+n})\right)} = 1 + \frac{\sqrt{2}(1+i)}{2}Q|a_{p+n}|, \tag{43}$$

$$\frac{B_{-m-\lambda}f(z_4)}{z_4^p} = 1 + Qa_{p+n}e^{i\left(\frac{\pi}{3} - arg(a_{p+n})\right)} = 1 + \frac{1+\sqrt{3}i}{2}Q|a_{p+n}|, \tag{44}$$

and

$$\frac{B_{-m-\lambda}f(z_5)}{z_5^p} = 1 + Qa_{p+n}e^{i\left(\frac{\pi}{2} - arg(a_{p+n})\right)} = 1 + iQ|a_{p+n}|. \tag{45}$$

Thus α_5 is given by

$$\alpha_5 = \frac{1}{5}\sum_{l=1}^{5} \frac{B_{-m-\lambda}f(z_l)}{z_l^p} = 1 + \frac{(3+\sqrt{2}+\sqrt{3})(1+i)}{10}Q|a_{p+n}|. \tag{46}$$

This gives us that

$$\left|1 - e^{-i\beta}\alpha_5\right| = \frac{\sqrt{2}(3+\sqrt{2}+\sqrt{3})}{10}Q|a_{p+n}| \tag{47}$$

with $\beta = 0$. For such α_5 and β, we take $\rho > 1$ with

$$\frac{\left(\frac{n}{p+\gamma}\right) Q|a_{p+n}|}{1 - Q|a_{p+n}|} \leq \frac{|e^{i\beta} - \alpha_5|n\rho}{(p+\gamma)(1 + |e^{i\beta} - \alpha_5|\rho)}. \tag{48}$$

It follows from the above that

$$\rho \geq \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})(1-2Q|a_{p+n}|)} > \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})} > 1. \tag{49}$$

For such α_5 *and* $\rho > 1$, $f(z)$ *satisfies*

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right| < Q|a_{p+n}| \leq \rho|e^{i\beta} - \alpha_5| \quad , \quad z \in \mathbb{U}. \tag{50}$$

Our next result reads as follows.

Theorem 2. *If* $f(z) \in A(p,n)$ *satisfies*

$$\left|\left(\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)} - 1\right)\left(\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right)\right| < \frac{\left|e^{i\beta} - \alpha_s\right|^2 n\rho^2}{(p+\gamma)(1+\left|e^{i\beta}-\alpha_s\right|\rho)} \quad , \quad z \in \mathbb{U} \tag{51}$$

for some α_s *defined by (13) with* $\alpha_s \neq 1$ *such that* $z_g \in \partial\mathbb{U}$ $(g = 1,2,3,...,s)$, *and for some real* $\rho > 1$, *then*

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right| < \rho\left|e^{i\beta} - \alpha_s\right| \quad , \quad z \in \mathbb{U} \tag{52}$$

that is, $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$ *.*

Proof. Define a function $w(z)$ by (23). Using (24) and (26), we have

$$\left|\left(\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)} - 1\right)\left(\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right)\right| = \left|\frac{\left(1-e^{-i\beta}\alpha_s\right)^2 zw(z)w'(z)}{(p+\gamma)(1+\left(1-e^{-i\beta}\alpha_s\right)w(z))}\right|. \tag{53}$$

We suppose that there exists a point z_0, $(0 < |z_0| < 1)$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \tag{54}$$

Then, Lemma 1 leads us that $w(z_0) = \rho e^{i\theta}$, $(0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = kw(z_0)$, $(k \geq n)$. It follows from the above that

$$\begin{aligned}\left|\left(\frac{B_{-m-\lambda+1}f(z_0)}{B_{-m-\lambda}f(z_0)} - 1\right)\left(\frac{B_{-m-\lambda}f(z_0)}{z_0^p} - 1\right)\right| &= \left|\frac{\left(1-e^{-i\beta}\alpha_s\right)^2 z_0 w(z_0)w'(z_0)}{(p+\gamma)(1+\left(1-e^{-i\beta}\alpha_s\right)w(z_0))}\right| \\ &= \frac{\left|e^{i\beta}-\alpha_s\right|^2 \rho^2 k}{(p+\gamma)\left|1+(1-e^{-i\beta}\alpha_s)\rho e^{i\theta}\right|} \\ &\geq \frac{\left|e^{i\beta}-\alpha_s\right|^2 n\rho^2}{(p+\gamma)(1+\left|e^{i\beta}-\alpha_s\right|\rho)}.\end{aligned} \tag{55}$$

This contradicts our condition (51) for $f(z)$. Therefore, there is no z_0, $(0 < |z_0| < 1)$ such that $|w(z_0)| = \rho > 1$. This means that

$$\left|\left(\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right)\right| < \rho|e^{i\beta} - \alpha_s| \quad , \quad z \in \mathbb{U}. \tag{56}$$

□

Example 2. *Consider a function $f(z)$ given by (32) with $0 < |a_{p+n}| < \frac{1}{Q}$, where Q is given by (33). For this function $f(z)$, we have*

$$\left|\left(\frac{B_{-m-\lambda+1}f(z)}{B_{-m-\lambda}f(z)}-1\right)\left(\frac{B_{-m-\lambda}f(z)}{z^p}-1\right)\right| = \left|\frac{nQ^2a_{p+n}^2z^{2n}}{(p+\gamma)(1+Qa_{p+n}z^n)}\right|$$

$$< \frac{nQ^2|a_{p+n}|^2}{(p+\gamma)(1-Q|a_{p+n}|)} \quad , \quad z \in \mathbb{U}. \tag{57}$$

Consider five boundary points z_1, z_2, z_3, z_4 and z_5 in Example 1. Then, we have

$$\left|1-e^{-i\beta}\alpha_5\right| = \frac{\sqrt{2}(3+\sqrt{2}+\sqrt{3})}{10}Q|a_{p+n}| \tag{58}$$

with $\beta = 0$. With such α_5 and β, we take $\rho > 1$ by

$$\frac{nQ^2|a_{p+n}|^2}{(p+\gamma)(1-Q|a_{p+n}|)} \leq \frac{\left|e^{i\beta}-\alpha_5\right|^2 n\rho^2}{(p+\gamma)(1+\left|e^{i\beta}-\alpha_5\right|\rho)}. \tag{59}$$

Then, this ρ satisfies

$$\rho \geq \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})Q|a_{p+n}|} > 1. \tag{60}$$

For such α_5 and ρ, we know that

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p}-1\right| < \rho|e^{i\beta}-\alpha_5| \quad , \quad z \in \mathbb{U}. \tag{61}$$

Next, we derive the following result.

Theorem 3. *If $f(z) \in A(p,n)$ satisfies*

$$\left|\frac{B_{-m-\lambda+g}f(z)}{z^p}-1\right| < \rho|e^{i\beta}-\alpha_s|\left(\frac{p+n+\gamma}{p+\gamma}\right) \quad , \quad z \in \mathbb{U} \tag{62}$$

for some α_s defined by (13) with $\alpha_s \neq 1$, $g = 1, 2, 3, ..., m$, and for some real $\rho > 1$, then

$$\left|\frac{B_{-m-\lambda+g-1}f(z)}{z^p}-1\right| < \rho\left|e^{i\beta}-\alpha_s\right| \quad , \quad z \in \mathbb{U}. \tag{63}$$

Proof. Define the function $w(z)$ by

$$w(z) = \frac{e^{i\beta}\frac{B_{-m-\lambda+g-1}f(z)}{z^p}-\alpha_s}{e^{i\beta}-\alpha_s}-1$$

$$= \frac{e^{i\beta}}{e^{i\beta}-\alpha_s}\left\{\sum_{k=p+n}^{\infty}\left(\frac{p+\gamma}{k+\gamma}\right)^{m-g}\frac{\Gamma(k+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(k+\gamma+\lambda)}a_kz^{k-p}\right\}. \tag{64}$$

It follows from the above that

$$B_{-m-\lambda+g-1}f(z) = z^p + (1-e^{-i\beta}\alpha_s)z^pw(z). \tag{65}$$

By the definition of $B_{-m-\lambda+g}f(z)$, we know that

$$B_{-m-\lambda+g}f(z) = \frac{z^{1-\gamma}}{p+\gamma}\left(z^\gamma B_{-m-\lambda+g-1}f(z)\right)' = z^p\left\{1+(1-e^{-i\beta}\alpha_s)w(z)\left(1+\frac{zw'(z)}{(p+\gamma)w(z)}\right)\right\}. \tag{66}$$

Our condition implies that

$$\left|\frac{B_{-m-\lambda+g}f(z)}{z^p}-1\right| = \left|(1-e^{-i\beta}\alpha_s)w(z)\left(1+\frac{zw'(z)}{(p+\gamma)w(z)}\right)\right| < \rho\left|e^{i\beta}-\alpha_s\right|\left(\frac{p+n+\gamma}{p+\gamma}\right) \quad (67)$$

for all $z \in \mathbb{U}$. Suppose that there exists a point z_0, $(0<|z_0|<1)$ such that

$$\max\{|w(z)|; |z| \leq |z_0|\} = |w(z_0)| = \rho > 1. \quad (68)$$

Then, Lemma 1 says that $w(z_0) = \rho e^{i\theta}, (0 \leq \theta \leq 2\pi)$ and $z_0 w'(z_0) = kw(z_0), (k \geq n)$. Therefore, we have

$$\begin{aligned}\left|\frac{B_{-m-\lambda+g}f(z_0)}{z_0^p}-1\right| &= \left|(1-e^{-i\beta}\alpha_s)w(z_0)\left(1+\frac{z_0w'(z_0)}{(p+\gamma)w(z_0)}\right)\right| \\ &= \rho\left|e^{i\beta}-\alpha_s\right|\left(1+\frac{k}{p+\gamma}\right) \geq \rho\left|e^{i\beta}-\alpha_s\right|\left(\frac{p+n+\gamma}{p+\gamma}\right) \quad (69)\end{aligned}$$

which contradicts the inequality (67). This means that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0| = \rho > 1$. Thus we know that

$$|w(z)| = \left|\frac{e^{i\beta}\frac{B_{-m-\lambda+g-1}f(z)}{z^p}-\alpha_s}{e^{i\beta}-\alpha_s}-1\right| = \left|\frac{e^{i\beta}}{e^{i\beta}-\alpha_s}\left(\frac{B_{-m-\lambda+g-1}f(z)}{z^p}-1\right)\right| < \rho. \quad (70)$$

This completes the proof of the theorem. □

Theorem 3 implies the following one.

Theorem 4. *If $f(z) \in A(p,n)$ satisfies*

$$\left|\frac{B_{-m-\lambda+g}f(z)}{z^p}-1\right| < \rho\left|e^{i\beta}-\alpha_s\right|\left(\frac{p+n+\gamma}{p+\gamma}\right)^g \quad , \quad z \in \mathbb{U} \quad (71)$$

for some α_s given by (13) with $\alpha_s \neq 1$, $g = 1,2,3,...,m$, and for some real $\rho > 1$, then

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p}-1\right| < \rho\left|e^{i\beta}-\alpha_s\right| \quad , \quad z \in \mathbb{U} \quad (72)$$

or, equivalently, $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$.

Proof. By means of Theorem 3, we see that if $f(z)$ satisfies the inequality (71), then

$$\left|\frac{B_{-m-\lambda+g-1}f(z)}{z^p}-1\right| < \rho\left|e^{i\beta}-\alpha_s\right|\left(\frac{p+n+\gamma}{p+\gamma}\right)^{g-1} \quad , \quad z \in \mathbb{U} \quad (73)$$

Similarly, we have

$$\left|\frac{B_{-m-\lambda+g-2}f(z)}{z^p}-1\right| < \rho\left|e^{i\beta}-\alpha_s\right|\left(\frac{p+n+\gamma}{p+\gamma}\right)^{g-2} \quad , \quad z \in \mathbb{U}. \quad (74)$$

Continuing this consideration, we obtain that

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p}-1\right| < \rho\left|e^{i\beta}-\alpha_s\right| \quad , \quad z \in \mathbb{U}. \quad (75)$$

□

Example 3. *Consider the function*

$$f(z) = z^p + a_{p+n}z^{p+n} \quad , \quad z \in \mathbb{U} \tag{76}$$

which satisfies

$$B_{-m-\lambda+g}f(z) = z^p + \frac{\Gamma(p+n+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(p+n+\gamma+\lambda)}\left(\frac{p+\gamma}{p+n+\gamma}\right)^{m-g} a_{p+n}z^{p+n}. \tag{77}$$

It follows from (77) that

$$\left|\frac{B_{-m-\lambda+g}f(z)}{z^p} - 1\right| = \frac{\Gamma(p+n+\gamma)\Gamma(p+\gamma+\lambda)}{\Gamma(p+\gamma)\Gamma(p+n+\gamma+\lambda)}\left(\frac{p+\gamma}{p+n+\gamma}\right)^{m-g} |a_{p+n}||z|^n$$

$$< Q\left(\frac{p+n+\gamma}{p+\gamma}\right)^g |a_{p+n}| \quad , \quad z \in \mathbb{U}, \tag{78}$$

where Q is given by (33). Now, we consider the five boundary points z_1, z_2, z_3, z_4 and z_5 as in Example 1. Then we see

$$|e^{i\beta} - \alpha_5| = \frac{\sqrt{2}(3+\sqrt{2}+\sqrt{3})}{10} Q|a_{p+n}| \tag{79}$$

where $\beta = 0$. With the above relation (79), we consider $\rho > 1$ such that

$$Q\left(\frac{p+n+\gamma}{p+\gamma}\right)^g |a_{p+n}| \le \rho|e^{i\beta} - \alpha_5|, \tag{80}$$

that is, ρ satisfies

$$\rho \ge \frac{10}{\sqrt{2}(3+\sqrt{2}+\sqrt{3})}\left(\frac{p+n+\gamma}{p+\gamma}\right)^g > 1.$$

Thus, we have that

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p} - 1\right| \le Q|a_{p+n}| \tag{81}$$

$$\le \rho|e^{i\beta} - \alpha_5|\left(\frac{p+\gamma}{p+n+\gamma}\right)^g$$

$$< \rho|e^{i\beta} - \alpha_5| \quad , \quad z \in \mathbb{U}. \tag{82}$$

Remark 1. *If we take $\gamma = 1$ in the results of this section, then these results correspond to applications of the Libera integral operator as introduced by Libera [2].*

Let us write that

$$B_{-m-\lambda}f(z) = L_{-m-\lambda}f(z) = z^p + \sum_{k=p+n}^{\infty}\left(\frac{p+1}{k+1}\right)^m \frac{\Gamma(p+1+\lambda)k!}{\Gamma(k+1+\lambda)p!}a_k z^k \tag{83}$$

for $\gamma = 1$ in (11). Then Theorem 1 says that if $f(z) \in A(p,n)$ satisfies

$$\left|\frac{L_{-m-\lambda+1}f(z)}{L_{-m-\lambda}f(z)} - 1\right| < \frac{|e^{i\beta} - \alpha_s|n\rho}{(p+1)(1+|e^{i\beta} - \alpha_s|\rho)} \quad , z \in \mathbb{U}, \tag{84}$$

for some α_s given by

$$\alpha_s = \frac{1}{s}\sum_{l=1}^{s}\frac{L_{-m-\lambda}f(z_l)}{z_l^p} \quad , \quad z_l \in \overline{\mathbb{U}}, \tag{85}$$

where $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$, *and for some real* $\rho > 1$, *then*

$$\left| \frac{L_{-m-\lambda} f(z)}{z^p} - 1 \right| < |e^{i\beta} - \alpha_s|\rho \quad , \quad z \in \mathbb{U}. \tag{86}$$

For another result, we consider again the Libera integral operator with $\gamma = 1$.

3. Application of Carathéodory Lemma

In this section, we will apply Carathéodory Lemma for coefficients of functions $f(z) \in A(p, n)$.

In 1907, Carathéodory [8] gave the following result.

Lemma 2. *Let a function* $g(z)$ *given by*

$$g(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \tag{87}$$

be analytic in $\mathbb{U}$ *and* $Reg(z) > 0, z \in \mathbb{U}$. *Then* $g(z)$ *satisfies*

$$|c_k| \le 2 \quad , \quad (k = 1, 2, 3, ...). \tag{88}$$

The inequality (88) is sharp for each k.

Applying the above lemma, we derive the following thorem.

Theorem 5. *If* $f(z) \in A(p, n)$ *is in the class* $T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$, *then*

$$|a_k| \le \frac{2|e^{i\beta} - \alpha_s|\rho}{R} \quad , \quad (k = p + n, p + n + 1, ...), \tag{89}$$

where

$$R = \left(\frac{p + \gamma}{k + \gamma} \right)^m \frac{\Gamma(k + \gamma)\Gamma(p + \gamma + \lambda)}{\Gamma(p + \gamma)\Gamma(k + \gamma + \lambda)}. \tag{90}$$

The result is sharp for $f(z)$ *given by*

$$B_{-m-\lambda} f(z) = z^p \frac{e^{i\theta} - (1 + 2|e^{i\beta} - \alpha_s|\rho)z}{e^{i\theta} - z}. \tag{91}$$

Proof. For $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$, we see that

$$\left| \frac{B_{-m-\lambda} f(z)}{z^p} - 1 \right| < |e^{i\beta} - \alpha_s|\rho \quad , \quad z \in \mathbb{U}. \tag{92}$$

If we define a function $g(z)$ with $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$ by

$$g(z) = \frac{\frac{B_{-m-\lambda} f(z)}{z^p} - (1 - |e^{i\beta} - \alpha_s|\rho)}{|e^{i\beta} - \alpha_s|\rho} \quad , z \in \mathbb{U}, \tag{93}$$

then $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$ and $Reg(z) > 0, z \in \mathbb{U}$. Also, $g(z)$ has the following power series expansion:

$$g(z) = 1 + \sum_{k=p+n}^{\infty} \frac{R}{|e^{i\beta} - \alpha_s|\rho} a_k z^{k-p}.$$

Therefore, by applying Lemma 2 to $g(z)$, we obtain

$$\frac{R}{|e^{i\beta}-\alpha_s|\rho}|a_k| \leq 2 \quad , \quad (k=p+n,p+n+1,...). \tag{94}$$

This shows the coefficient inequalities (89). Note that

$$g(z)=\frac{e^{i\theta}+z}{e^{i\theta}-z}=1+\sum_{k=1}^{\infty}2e^{i\theta}z^k=1+\sum_{k=1}^{\infty}c_kz^k \tag{95}$$

is analytic in $\mathbb{U}$, $g(0)=1$, $Reg(z)>0, (z\in\mathbb{U})$ and $|c_k|=2 \;\;,\;\; (k=1,2,3,...)$. Therefore, considering $f(z)$ such that

$$g(z)=\frac{\frac{B_{-m-\lambda}f(z)}{z^p}-(1-|e^{i\beta}-\alpha_s|\rho)}{|e^{i\beta}-\alpha_s|\rho}=\frac{e^{i\theta}+z}{e^{i\theta}-z}, \tag{96}$$

we have

$$B_{-m-\lambda}f(z)=z^p\frac{e^{i\theta}-(1+2|e^{i\beta}-\alpha_s|\rho)z}{e^{i\theta}-z}. \tag{97}$$

This completes the proof of the theorem. □

Remark 2. *If we take $\gamma=1$ in Theorem 5, then we get the following result for the Libera integral operator.*

If $f(z)\in A(p,n)$ satisfies

$$\left|\frac{L_{-m-\lambda}f(z)}{z^p}-1\right|<|e^{i\beta}-\alpha_s|\rho \quad , \quad z\in\mathbb{U}, \tag{98}$$

then

$$|a_k|\leq\frac{2|e^{i\beta}-\alpha_s|\rho}{R_0} \quad , \quad (k=p+n,p+n+1,...), \tag{99}$$

where

$$R_0=\left(\frac{p+1}{k+1}\right)^m\frac{\Gamma(p+1+\lambda)k!}{\Gamma(k+1+\lambda)p!}. \tag{100}$$

The result is sharp for $f(z)$ given by

$$L_{-m-\lambda}f(z)=z^p\frac{e^{i\theta}-(1+2|e^{i\beta}-\alpha_s|\rho)z}{e^{i\theta}-z}. \tag{101}$$

Finally, we derive

Theorem 6. *If $f(z)\in A(p,n)$ satisfies*

$$\sum_{k=p+n}^{\infty}R|a_k|\leq|e^{i\beta}-\alpha_s|\rho, \tag{102}$$

then $f(z)\in T_{p,n}(\alpha_s,\beta,\rho;m,\lambda)$, where R is given by (90).

Proof. For $f(z)\in A(p,n)$, we consider

$$\left|\frac{B_{-m-\lambda}f(z)}{z^p}-1\right|=\left|\sum_{k=p+n}^{\infty}Ra_kz^k\right|$$

$$<\sum_{k=p+n}^{\infty}R|a_k|\leq|e^{i\beta}-\alpha_s|\rho \quad , \quad z\in\mathbb{U}. \tag{103}$$

Therefore, if $f(z) \in A(p,n)$ satisfies (102), then we know $f(z) \in T_{p,n}(\alpha_s, \beta, \rho; m, \lambda)$. $\square$

Remark 3. *Letting $\gamma = 1$ in Theorem 6, we have the result concerning with the Libera integral operator $L_{-m-\lambda}f(z)$.*

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