mathematics

MDPI

Article

A Pinching Theorem for Compact Minimal Submanifolds in Warped Products $I \times_f \mathbb{S}^m(c)$

Xin Zhan * and Zhonghua Hou

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China; zhh@mail.dlut.edu.cn

* Correspondence: zhanxin@mail.dlut.edu.cn

Received: 4 August 2020; Accepted: 24 August 2020; Published: 28 August 2020

Abstract: Let $\mathbb{S}^m(c)$ be a Euclidean sphere of curvature $c > 0$ and $\mathbb{R}$ be a Euclidean line. We prove a pinching theorem for compact minimal submanifolds immersed in Riemannian warped products of the type $I \times_f \mathbb{S}^m(c)$, where $f : I \to \mathbb{R}^+$ is a smooth positive function on an open interval I of $\mathbb{R}$. This allows us to generalize Chen-Cui's pinching theorem from Riemannian products $\mathbb{S}^m(c) \times \mathbb{R}$ to Riemannian warped products $I \times_f \mathbb{S}^m(c)$.

Keywords: Riemannian warped product; DDVV Conjecture; pinching theorem

1. Introduction

Let $\mathbb{M}^{n+p}(c)$ $(c \neq 0)$ be an $(n+p)$-dimensional real space form with constant sectional curvature c and M^n be an $n(\geq 2)$-dimensional immersed connected submanifold of $\mathbb{M}^{n+p}(c)$. Denote by H the mean curvature of M^n. The normalized scalar curvature ρ and the normal scalar curvature $\rho^\perp$ are defined by

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} \langle R(e_i, e_j)e_j, e_i \rangle, \tag{1}$$

and

$$\rho^\perp = \frac{2}{n(n-1)} \left(\sum_{i<j} \sum_{\alpha<\beta} \langle R^\perp(e_i, e_j)e_\beta, e_\alpha \rangle^2 \right)^{\frac{1}{2}}, \tag{2}$$

where R is the curvature tensor of the tangent bundle and $R^\perp$ is the normal curvature tensor of the normal bundle. In 1999, De Smet et al. [1] proposed the following well-known Normal Scalar Curvature Conjecture or DDVV Conjecture:

DDVV Conjecture: (c.f. [1]) *Let M^n be an $n(\geq 2)$-dimensional immersed submanifold in a real space form $\mathbb{M}^{n+p}(c)$. Then the inequality*

$$H^2 \geq \rho + \rho^\perp - c \tag{3}$$

holds at every point p of M^n. The formula (3) is called DDVV inequality.

Submanifolds achieving the equality everywhere in (3) are called Wintgen ideal submanifolds which carry interesting geometry and are not classified completely so far, see [2]. In 2007, Dillen et al. [3] transferred the conjecture into an algebraic version inequality:

Theorem 1. (c.f. [3]) *Let $B_1, B_2, \ldots, B_p$ be symmetric $(n \times n)$-matrices with trace zero. If*

$$\sum_{\alpha,\beta} \|[B_\alpha, B_\beta]\|^2 \leq \left(\sum_\alpha \|B_\alpha\|^2 \right)^2, \tag{4}$$

then DDVV Conjecture is true.

In 2008, DDVV Conjecture was solved completely by Ge-Tang [4] and Lu [5] independently through proving that the above algebraic inequality (4) holds true. Since then, DDVV type problems for submanifolds were studied in different ambient spaces, refer to [6–17].

Interestingly, Lu [5] simultaneously obtained an important rigidity result for compact minimal submanifolds immersed in $\mathbb{S}^{n+p}(c)$, which improved some classical rigidity results of Simons [18], Lawson [19], Chern et al. [20], Li-Li [21].

Theorem 2. (c.f. [5]) *Let M^n be an $n(\geq 2)$-dimensional compact minimal submanifold in $\mathbb{S}^{n+p}(c)$. If*

$$0 \leq \sigma + \lambda_2 \leq nc, \tag{5}$$

then M^n is a totally geodesic submanifold $S^n(c)$, or one of the Clifford torus $M_{r,n-r}$ $(1 \leq r \leq n-1)$, or the Veronese surface M^2. Here σ is the squared length of the second fundamental form, and λ_2 is the second largest eigenvalue of the fundamental matrix as stated in Definition 1. The Clifford torus $M_{r,n-r}$ is a Riemannian product of the form $S^r(\frac{nc}{r}) \times S^{n-r}(\frac{nc}{n-r})$, which is a minimal hypersurface immersed into $S^{n+1}(c)$.

Besides, it would be interesting and important to study the similar problems in a product space $\mathbb{M}^m(c) \times \mathbb{R}$. Now we use $\frac{\partial}{\partial t}$ to denote the unit $\mathbb{R}$ direction and write T for the projection of $\frac{\partial}{\partial t}$ on M. With using the tensor T, Chen and Cui [22] proved the corresponding interesting DDVV type inequality and obtained a pinching theorem in $\mathbb{M}^m(c) \times \mathbb{R}$. Precisely, the authors obtained the following two theorems.

Theorem 3. (c.f. [22]) *Let M^n be an n-dimensional immersed submanifold in $\mathbb{M}^m(c) \times \mathbb{R}$ $(m \geq n \geq 2)$. Then we have*

$$H^2 \geq \rho + \rho^{\perp} - c\left(1 - \frac{2}{n}|T|^2\right). \tag{6}$$

Theorem 4. (c.f. [22]) *Let M^n be an n-dimensional compact minimal submanifold in $\mathbb{S}^m(c) \times \mathbb{R}$ $(m \geq n \geq 2)$. Set $a = \max_{x \in M} |T|^2$. If*

$$0 \leq \sigma + \lambda_2 \leq c\left(n - (2n+1)a\right), \tag{7}$$

then $\sigma = 0$ or $\sigma + \lambda_2 = c\left(n - (2n+1)a\right)$.

Inspired by the above results, the first author [23] further generalized Chen-Cui's work to a product manifold of a space form and a Euclidean space of higher dimension. Recently, Roth [24] extended Theorem 3 to the case that the ambient space is a Riemannian warped product $I \times_f \mathbb{M}^m(c)$ by proving a new DDVV type inequality for submanifolds immersed in $I \times_f \mathbb{M}^m(c)$, which is similar to (3) and (6).

Theorem 5. (c.f. [24]) *Let M^n be an n-dimensional immersed submanifold in $I \times_f \mathbb{M}^m(c)$ $(m \geq n \geq 2)$. Then we have*

$$H^2 \geq \rho + \rho^{\perp} + \left(\frac{f'^2}{f^2} - \frac{c}{f^2}\right)\left(1 - \frac{2}{n}|T|^2\right) - \frac{2f''}{nf}|T|^2. \tag{8}$$

Hence, it seems natural and interesting to extend the classical pinching theorems (Theorems 2 and 4) obtained for submanifolds in real space forms, or in the product of a line with a real space form to warped product manifolds. In this article, we prove the following result:

Main Theorem: *Let M^n be an n-dimensional compact minimal submanifold in $I \times_f \mathbb{S}^m(c)$ $(m \geq n \geq 2)$ with warping function satisfying $c - (f'^2 - ff'') = c_1 f^4$ for some $c_1 > 0$ at every $t \in I$. If*

$$0 \leq \sigma + \lambda_2 \leq c_1 f^2 \left(n - (2n+1)|T|^2\right) - \frac{n|f''|}{f}, \tag{9}$$

then $\sigma = 0$ and M^n lies in a slice $\mathbb{S}^m(c)$, or $\sigma + \lambda_2 = c_1 f^2 \left(n - (2n+1)|T|^2\right) - \frac{n|f''|}{f}$.

Remark 1. *The assumption that $c - (f'^2 - ff'') = c_1 f^4$ in the Main Theorem is a second-order nonlinear ordinary differential equation, which can be rewritten as*

$$f'' - \frac{1}{f} f'^2 - (c_1 f^3 - \frac{c}{f}) = 0. \tag{10}$$

The substitution $\omega(f) = (f'(t))^2$ leads to a first-order linear differential equation $\omega'(f) = \frac{2}{f}\omega + 2(c_1 f^3 - \frac{c}{f})$. By the method of variation of parameters, then the solution of the above differential equation is given by

$$\omega(f) = \left[c_2 + \int e^{-F} \cdot 2(c_1 f^3 - \frac{c}{f})\,\mathrm{d}f\right] \cdot e^F,$$

where $F = \int \frac{2}{f}\,\mathrm{d}f = 2\ln f$ and c_2 is a undetermined constant. That is to say,

$$\begin{aligned}(f')^2 = \omega(f) &= f^2 \cdot \left[c_2 + \int 2(c_1 f - \frac{c}{f^3})\,\mathrm{d}f\right] \\ &= f^2 \cdot \left(c_2 + c_1 f^2 + \frac{c}{f^2}\right) \\ &= c_1 f^4 + c_2 f^2 + c.\end{aligned} \tag{11}$$

So, we can see that $f' = \pm\sqrt{c_1 f^4 + c_2 f^2 + c}$, which is a first-order separable equation.

A trivial example can be obtained by taking I as $\mathbb{R} = (-\infty, +\infty)$ and $f(t) = 1$ in the Main Theorem. Then we recover Theorem 4. Moreover, we can see easily that $f = \frac{e^{2t}-C}{e^{2t}+C}$ provides a particular solution of the Equation (11) for $c = c_1 = 1$ and $c_2 = -2$, where C is a real constant. Another non-trivial example is $\overline{M} = (0, \frac{\pi}{2}) \times_f \mathbb{S}^m(c)$, where $f : (0, \frac{\pi}{2}) \longrightarrow \mathbb{R}^+$, $f(t) = \tan t$. Needless to say, this function also satisfies (11) for $c = c_1 = 1$ and $c_2 = 2$. Hence our theorem can be view as a generalization of Theorem 4.

2. Preliminaries

Let t be an arc-length parameter of I and $\partial_t = \frac{\partial}{\partial t}$ be the unit vector field tangent to I. We consider the Riemannian warped product $\overline{M} = I \times_f \mathbb{M}^m(c)$ endowed with the Riemannian warped metric defined by

$$\langle\, , \rangle = dt^2 + f(t)^2 \langle\, , \rangle_{\mathbb{M}},$$

where $\langle\, , \rangle_{\mathbb{M}}$ denotes the standard Riemannian metric of $\mathbb{M}^m(c)$ and f is called the warping function of the warped product $I \times_f \mathbb{M}^m(c)$. Let M^n be an n-dimensional immersed connected submanifold in $I \times_f \mathbb{M}^m(c)$ with codimension $p = m + 1 - n (\geq 1)$. We denote by ∇ and $\overline{\nabla}$ the Riemannian connections of M^n and $\overline{M}$, respectively. Moreover, we use $\nabla^\perp$ for the normal connection of M^n.

Throughout this paper, we will agree on the following index ranges and use the Einstein summation convention unless otherwise stated:

$$\begin{aligned}&1 \leq A, B, C, \cdots \leq m+1; \\ &1 \leq i, j, k, \cdots \leq n; \\ &n+1 \leq \alpha, \beta, \gamma, \cdots \leq m+1.\end{aligned}$$

We choose $\{e_i\}_{i=1}^n$ and $\{e_\alpha\}_{\alpha=n+1}^{m+1}$ to be local orthonormal frames of the tangent bundle TM and the normal bundle $T^\perp M$, respectively. Let $\{\omega_A\}_{A=1}^{m+1}$ be the dual frame of $\{e_A\}_{A=1}^{m+1}$, and $\{\omega_{AB}\}_{A,B=1}^{m+1}$ be the Riemannian connection forms associated with $\{\omega_A\}_{A=1}^{m+1}$. In particular, $\{\omega_{ij}\}_{i,j=1}^n$ and $\{\omega_{\alpha\beta}\}_{\alpha,\beta=n+1}^{n+p}$ denote the Riemannian connection forms in TM and the normal connection forms in $T^\perp M$. From Cartan's lemma, we get

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

Denote by $h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha$ the second fundamental form and by $\sigma = \sum_{i,j,\alpha}(h_{ij}^\alpha)^2$ the squared length of h. The mean curvature vector is defined by $\vec{H} = \sum_\alpha H^\alpha e_\alpha$ with $H^\alpha = \frac{1}{n}\sum_i h_{ii}^\alpha$ and the mean curvature $H = |\vec{H}|$. Let A_α be the shape operator with respective to e_α. It is well known that h and A are related by

$$\langle h(e_i, e_j), e_\alpha \rangle = \langle A_{e_\alpha} e_i, e_j \rangle . \tag{12}$$

Definition 1. *The fundamental matrix F of M is a $p \times p$ matrix $F = \left(S_{\alpha\beta}\right)_{p\times p}$, where*

$$S_{\alpha\beta} = \langle A_\alpha, A_\beta \rangle = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta . \tag{13}$$

We can certainly assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ be all the eigenvalues of the fundamental matrix F. In particular, λ_1 and λ_2 are the largest and the second largest eigenvalue of F, respectively.

Obviously, by (13) and the definition of σ, it follows that

$$\mathrm{Tr}(F) = \sum_\alpha S_{\alpha\alpha} = \sum_{\mu=1}^p \lambda_\mu = \sum_{i,j,\alpha} \left(h_{ij}^\alpha\right)^2 = \sigma. \tag{14}$$

Recall that (c.f. [25], p. 74) the curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\overline{R}(X,Y) = \overline{\nabla}_{[X,Y]} - [\overline{\nabla}_X, \overline{\nabla}_Y] = \overline{\nabla}_{[X,Y]} - \overline{\nabla}_X\overline{\nabla}_Y + \overline{\nabla}_Y\overline{\nabla}_X, \text{ for any } X, Y \in T(\overline{M}).$$

We write

$$\overline{R}(e_A, e_B)e_C := \overline{R}_{CDAB} e_D$$

for any $e_A, e_B, e_C, e_D \in T(\overline{M})$.

From the properties of curvature tensor we find

$$\langle \overline{R}(e_A, e_B)e_C, e_D \rangle = \overline{R}_{CDAB} = \overline{R}_{ABCD}.$$

Similarly, it is convenient to write

$$R_{ijkl} := \langle R(e_i, e_j)e_k, e_l \rangle \text{ and } R^\perp_{\alpha\beta ij} := \langle R^\perp(e_\alpha, e_\beta)e_i, e_j \rangle.$$

The first and the second covariant derivatives of h_{ij}^α are respectively defined by

$$\nabla h_{ij}^\alpha = h_{ijk}^\alpha \omega_k = \mathrm{d}h_{ij}^\alpha - h_{mj}^\alpha \omega_{mi} - h_{im}^\alpha \omega_{mj} + h_{ij}^\beta \omega_{\beta\alpha}, \tag{15}$$

$$\nabla h_{ijk}^\alpha = h_{ijkl}^\alpha \omega_l = \mathrm{d}h_{ijk}^\alpha - h_{mjk}^\alpha \omega_{mi} - h_{imk}^\alpha \omega_{mj} - h_{ijm}^\alpha \omega_{mk} + h_{ijk}^\beta \omega_{\beta\alpha}. \tag{16}$$

We also denote

$$|\nabla h|^2 = \sum (h_{ijk}^\alpha)^2, \quad |\nabla^2 h|^2 = \sum (h_{ijkl}^\alpha)^2.$$

Then we have the well-known Codazzi equation and Ricci identity as below:

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = -\overline{R}_{\alpha ijk}, \tag{17}$$

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = h^{\alpha}_{mj} R_{mikl} + h^{\alpha}_{im} R_{mjkl} + h^{\beta}_{ij} R^{\perp}_{\beta\alpha kl}. \tag{18}$$

Decompose ∂_t into the tangential and normal parts as follows:

$$\partial_t = T + N = T^i e_i + N^{\alpha} e_{\alpha}. \tag{19}$$

Obviously, we have $|T|^2 + |N|^2 = |\partial_t|^2 = 1$.
By Gauss-Weingarten formulae one has

$$\begin{aligned}
\overline{\nabla}_{e_j} \partial_t &= \overline{\nabla}_{e_j} (T^i e_i + N^{\alpha} e_{\alpha}) \\
&= e_j(T^i) e_i + T^i \nabla_{e_j} e_i + T^i h^{\alpha}_{ij} e_{\alpha} \\
&\quad + e_j(N^{\alpha}) e_{\alpha} - N^{\alpha} h^{\alpha}_{jk} e_k + N^{\alpha} \nabla^{\perp}_{e_j} e_{\alpha} \\
&= T^i_{,j} e_i - N^{\alpha} h^{\alpha}_{ji} e_i + N^{\alpha}_{,j} e_{\alpha} + T^i h^{\alpha}_{ij} e_{\alpha} \\
&= (T^i_{,j} - N^{\alpha} h^{\alpha}_{ij}) e_i + (N^{\alpha}_{,j} + h^{\alpha}_{ij} T^i) e_{\alpha}.
\end{aligned} \tag{20}$$

We define $\pi : \overline{M} = I \times_f \mathbb{M}^m(c) \longrightarrow \mathbb{M}^m(c)$ to be the projection map, and $\pi(X) := X^* = X - \langle X, \partial_t \rangle \partial_t$ to be the orthogonal projection of X to the tangent space $\mathbb{TM}^m(c)$. Using Proposition 35 of Chapter 7 in [25], it follows that

$$\begin{aligned}
\overline{\nabla}_{e_j} \partial_t = \overline{\nabla}_{e_j^* + T^j \partial_t} \partial_t = \overline{\nabla}_{e_j^*} \partial_t = \frac{f'}{f} e_j^* &= \frac{f'}{f} (e_j - T^j \partial_t) \\
= \frac{f'}{f} (e_j - T^j \sum_i T^i e_i) - \frac{f'}{f} (T^j \sum_{\alpha} N^{\alpha} e_{\alpha}) \\
= \frac{f'}{f} (\delta_{ij} - T^j T^i) e_i - \frac{f'}{f} T^j N^{\alpha} e_{\alpha}.
\end{aligned} \tag{21}$$

Comparison of (20) and (21) shows that

$$T^i_{,j} = \sum_{\alpha} h^{\alpha}_{ij} N^{\alpha} + \frac{f'}{f} (\delta_{ij} - T^i T^j), \quad N^{\alpha}_{,j} = -\sum_i h^{\alpha}_{ij} T^i - \frac{f'}{f} T^j N^{\alpha}. \tag{22}$$

In [26], the authors deduced the structure equations for a semi-Riemannian submanifold immersed into a warped product $\pm I \times_f \mathbb{M}^m_k(c)$, where $I \subseteq \mathbb{R}$ and $\mathbb{M}^m_k(c)$ is a semi-Riemannian space form of constant nonzero sectional curvature c and index k. In the Riemannian case, we shall now derive the following structure equations by the moving frame method.

Proposition 1. (c.f. [24,26]) *Let M^n be an n-dimensional immersed submanifold in $\overline{M} = I \times_f \mathbb{M}^m(c)$ $(m \geq n \geq 2)$. Then*

$$\begin{aligned}
R_{ijkl} =& \frac{c - (f'^2 - ff'')}{f^2} \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{il} T^j T^k + \delta_{jk} T^i T^l - \delta_{ik} T^j T^l - \delta_{jl} T^i T^k \right) \\
& - \frac{f''}{f} \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \right) + \sum_{\alpha} \left(h^{\alpha}_{ik} h^{\alpha}_{jl} - h^{\alpha}_{il} h^{\alpha}_{jk} \right),
\end{aligned} \tag{23}$$

$$-\overline{R}_{\alpha ijk} = h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = \frac{c - (f'^2 - ff'')}{f^2} N^{\alpha} \left(\delta_{ik} T^j - \delta_{ij} T^k \right), \tag{24}$$

$$R^{\perp}_{\alpha\beta ij} = \sum_k \left(h^{\alpha}_{ik} h^{\beta}_{jk} - h^{\alpha}_{jk} h^{\beta}_{ik} \right). \tag{25}$$

Proof. A direct computation gives

$$\begin{aligned} \overline{R}(e_i, e_j)e_k &= \overline{R}(e_i^* + T^i\partial_t, e_j^* + T^j\partial_t)(e_k^* + T^k\partial_t) \\ &= \overline{R}(e_i^*, e_j^*)e_k^* + T^k\overline{R}(e_i^*, e_j^*)\partial_t + T^j\overline{R}(e_i^*, \partial_t)e_k^* \\ &\quad + T^jT^k\overline{R}(e_i^*, \partial_t)\partial_t + T^i\overline{R}(\partial_t, e_j^*)e_k^* \\ &\quad + T^iT^k\overline{R}(\partial_t, e_j^*)\partial_t + T^iT^j\overline{R}(\partial_t, \partial_t)e_k^* + T^iT^jT^k\overline{R}(\partial_t, \partial_t)\partial_t. \end{aligned} \tag{26}$$

From the expression of the curvature tensor $\overline{R}$ (c.f. [25], p. 210) one has

$$\begin{aligned} \overline{R}(e_i^*, e_j^*)\partial_t &= \overline{R}(\partial_t, \partial_t)e_k^* = \overline{R}(\partial_t, \partial_t)\partial_t = 0, \\ \overline{R}(e_i^*, \partial_t)e_k^* &= -\frac{\langle e_i^*, e_k^* \rangle}{f} f''\partial_t, \\ \overline{R}(e_i^*, \partial_t)\partial_t &= \frac{f''}{f} e_i^*, \\ \overline{R}(\partial_t, e_j^*)e_k^* &= \frac{\langle e_j^*, e_k^* \rangle}{f} f''\partial_t, \\ \overline{R}(\partial_t, e_j^*)\partial_t &= -\frac{f''}{f} e_j^*. \end{aligned}$$

On substituting these into (26) we have

$$\begin{aligned} \langle \overline{R}(e_i, e_j)e_k, e_l \rangle &= \Big\langle \overline{R}(e_i^*, e_j^*)e_k^* + T^j\overline{R}(e_i^*, \partial_t)e_k^* + T^jT^k\overline{R}(e_i^*, \partial_t)\partial_t \\ &\qquad + T^i\overline{R}(\partial_t, e_j^*)e_k^* + T^iT^k\overline{R}(\partial_t, e_j^*)\partial_t \,,\, e_l^* + T^l\partial_t \Big\rangle \\ &= \Big\langle \overline{R}(e_i^*, e_j^*)e_k^* - T^j\frac{\langle e_i^*, e_k^* \rangle}{f} f''\partial_t + T^jT^k\frac{f''}{f} e_i^* \\ &\qquad + T^i\frac{\langle e_j^*, e_k^* \rangle}{f} f''\partial_t - T^iT^k\frac{f''}{f} e_j^* \,,\, e_l^* + T^l\partial_t \Big\rangle \\ &= \Big\langle \overline{R}(e_i^*, e_j^*)e_k^*, e_l^* \Big\rangle - T^jT^l\frac{\langle e_i^*, e_k^* \rangle}{f} f'' + T^jT^k\frac{f''}{f}\langle e_i^*, e_l^* \rangle \\ &\qquad + T^iT^l\frac{\langle e_j^*, e_k^* \rangle}{f} f'' - T^iT^k\frac{f''}{f}\langle e_j^*, e_l^* \rangle. \end{aligned} \tag{27}$$

We conclude similarly that

$$\begin{aligned} \langle \overline{R}(e_\alpha, e_i)e_j, e_k \rangle &= \Big\langle \overline{R}(e_\alpha^*, e_i^*)e_j^*, e_k^* \Big\rangle - T^iT^k\frac{\langle e_\alpha^*, e_j^* \rangle}{f} f'' + T^iT^j\frac{f''}{f}\langle e_\alpha^*, e_k^* \rangle \\ &\qquad + N^\alpha T^k\frac{\langle e_i^*, e_j^* \rangle}{f} f'' - N^\alpha T^j\frac{f''}{f}\langle e_i^*, e_k^* \rangle, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \langle \overline{R}(e_\alpha, e_\beta)e_i, e_j \rangle &= \Big\langle \overline{R}(e_\alpha^*, e_\beta^*)e_i^*, e_j^* \Big\rangle - N^\beta T^j\frac{\langle e_\alpha^*, e_i^* \rangle}{f} f'' + N^\beta T^i\frac{f''}{f}\langle e_\alpha^*, e_j^* \rangle \\ &\qquad + N^\alpha T^j\frac{\langle e_\beta^*, e_i^* \rangle}{f} f'' - N^\alpha T^i\frac{f''}{f}\langle e_\beta^*, e_j^* \rangle. \end{aligned} \tag{29}$$

Observe that

$$\langle e_i^*, e_j^* \rangle = \langle e_i - T^i\partial_t, e_j - T^j\partial_t \rangle = \delta_{ij} - T^iT^j, \quad \langle e_\alpha^*, e_i^* \rangle = \langle e_\alpha - N^\alpha\partial_t, e_i - T^i\partial_t \rangle = -N^\alpha T^i.$$

Now (27) becomes

$$\begin{aligned}
\left\langle \overline{R}(e_i, e_j)e_k, e_l \right\rangle &= \frac{c - f'^2}{f^2}\Big[(\delta_{ik} - T^iT^k)(\delta_{jl} - T^jT^l) - (\delta_{il} - T^iT^l)(\delta_{jk} - T^jT^k)\Big] \\
&\quad - \frac{\delta_{ik} - T^iT^k}{f} f''T^jT^l + \frac{\delta_{il} - T^iT^l}{f} f''T^jT^k \\
&\quad + \frac{\delta_{jk} - T^jT^k}{f} f''T^iT^l - \frac{\delta_{jl} - T^jT^l}{f} f''T^iT^k \\
&= \frac{c - f'^2}{f^2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{il}T^jT^k + \delta_{jk}T^iT^l - \delta_{ik}T^jT^l - \delta_{jl}T^iT^k\right) \\
&\quad + \frac{f''}{f}\left(\delta_{il}T^jT^k + \delta_{jk}T^iT^l - \delta_{ik}T^jT^l - \delta_{jl}T^iT^k\right) \\
&= \frac{c - (f'^2 - ff'')}{f^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \delta_{il}T^jT^k + \delta_{jk}T^iT^l \\
&\quad - \delta_{ik}T^jT^l - \delta_{jl}T^iT^k) - \frac{f''}{f}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right).
\end{aligned} \tag{30}$$

Likewise, we can deduce that

$$\begin{aligned}
\langle \overline{R}(e_\alpha, e_i)e_j, e_k \rangle &= \frac{c - f'^2}{f^2}\Big[(-N^\alpha T^j)(\delta_{ik} - T^iT^k) - (-N^\alpha T^k)(\delta_{ij} - T^iT^j)\Big] \\
&\quad - \frac{-N^\alpha T^j}{f} f''T^iT^k + \frac{-N^\alpha T^k}{f} f''T^iT^j \\
&\quad + \frac{\delta_{ij} - T^iT^j}{f} f''N^\alpha T^k - \frac{\delta_{ik} - T^iT^k}{f} f''N^\alpha T^j \\
&= \frac{c - f'^2}{f^2}\left(\delta_{ij}N^\alpha T^k - \delta_{ik}N^\alpha T^j\right) + \frac{f''}{f}\left(\delta_{ij}N^\alpha T^k - \delta_{ik}N^\alpha T^j\right) \\
&= \frac{c - (f'^2 - ff'')}{f^2} N^\alpha \left(\delta_{ij}T^k - \delta_{ik}T^j\right),
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
\langle \overline{R}(e_\alpha, e_\beta)e_i, e_j \rangle &= \frac{c - f'^2}{f^2}\Big[(-N^\alpha T^i)(-N^\beta T^j) - (-N^\alpha T^j)(-N^\beta T^i)\Big] \\
&\quad + \frac{N^\alpha T^i}{f} f''N^\beta T^j - \frac{N^\alpha T^j}{f} f''N^\beta T^i \\
&\quad - \frac{N^\beta T^i}{f} f''N^\alpha T^j + \frac{N^\beta T^j}{f} f''N^\alpha T^i = 0.
\end{aligned} \tag{32}$$

Combining (30)–(32) with the standard Gauss, Codazzi and Ricci equations gives the proof of Proposition 1. □

By Proposition 1, we can obtain a lower bound of $|\nabla h|^2$, which extends Proposition 1 in [27].

Proposition 2. *Let M^n be an n-dimensional immersed connected submanifold in $\overline{M} = I \times_f \mathbb{M}^m(c)$ $(m \geq n \geq 2)$. Let*

$$\eta_{ijk}^\alpha = \frac{1}{3}(h_{ijk}^\alpha + h_{jki}^\alpha + h_{kij}^\alpha).$$

Then we have

$$\sum_{i,j,k}(h^{\alpha}_{ijk})^2=\sum_{i,j,k}(\eta^{\alpha}_{ijk})^2+\frac{2(n-1)\big[c-(f'^2-ff'')\big]^2}{3f^4}(N^{\alpha})^2|T|^2. \tag{33}$$

Proof. It follows from the definition of η^{α}_{ijk} and (24) that

$$\begin{aligned}
h^{\alpha}_{ijk}&=\eta^{\alpha}_{ijk}+\frac{1}{3}(h^{\alpha}_{ijk}-h^{\alpha}_{ikj})+\frac{1}{3}(h^{\alpha}_{jik}-h^{\alpha}_{jki})\\
&=\eta^{\alpha}_{ijk}-\frac{1}{3}(\overline{R}_{\alpha ijk}+\overline{R}_{\alpha jik})\\
&=\eta^{\alpha}_{ijk}+\frac{c-(f'^2-ff'')}{3f^2}N^{\alpha}\big(T^i\delta_{jk}+T^j\delta_{ik}-2T^k\delta_{ij}\big).
\end{aligned}$$

Squaring the both sides of the above equation, and summing over i,j,k, it turns out that

$$\begin{aligned}
\sum_{i,j,k}(h^{\alpha}_{ijk})^2&=\sum_{i,j,k}\Big\{(\eta^{\alpha}_{ijk})^2+\frac{2\big[c-(f'^2-ff'')\big]}{3f^2}N^{\alpha}\eta^{\alpha}_{ijk}(T^i\delta_{jk}+T^j\delta_{ik}-2T^k\delta_{ij})\\
&\quad+\frac{\big[c-(f'^2-ff'')\big]^2}{9f^4}(N^{\alpha})^2(T^i\delta_{jk}+T^j\delta_{ik}-2T^k\delta_{ij})^2\Big\}\\
&=\sum_{i,j,k}(\eta^{\alpha}_{ijk})^2+\frac{\big[c-(f'^2-ff'')\big]^2}{9f^4}(N^{\alpha})^2\sum_{i,j,k}\big(T^i\delta_{jk}+T^j\delta_{ik}-2T^k\delta_{ij}\big)^2\\
&=\sum_{i,j,k}(\eta^{\alpha}_{ijk})^2+\frac{2(n-1)\big[c-(f'^2-ff'')\big]^2}{3f^4}(N^{\alpha})^2|T|^2.
\end{aligned}$$

The proof is completed. □

Remark 2. *In Proposition 2, suppose that* $f'^2-ff''\neq c$ *and* $h^{\alpha}_{ijk}=0$ *for any* i,j,k,α, *then* $|T|=0$ *or* $|N|=0$, i.e., M^n *is either contained in a slice* $\mathbb{M}^m(c)$, *or* ∂_t *is everywhere tangent to* M^n. *In the latter case, it is of the form* $M^n=J\times_{\tilde{f}}P$, *where* J *is an open subinterval of* I, P *is an* $(n-1)$*-dimentional submanifold of* $\mathbb{M}^m(c)$ and $\tilde{f}$ *is the restriction of* f *on* I.

3. Proof of Main Theorem

In this section, we will give the proof of our Main Theorem. We shall adopt the similar procedure as in the proof of [22]. Firstly, we proceed to calculate $\frac{1}{2}\Delta\|A_{\alpha}\|^2$. For arbitrary fixed α, we conclude from (17) and (18) that

$$\begin{aligned}
\frac{1}{2}\Delta\|A_{\alpha}\|^2=\frac{1}{2}\Delta\Big(\sum_{i,j}(h^{\alpha}_{ij})^2\Big)&=(h^{\alpha}_{ijk})^2+h^{\alpha}_{ij}h^{\alpha}_{ijkk}\\
&=(h^{\alpha}_{ijk})^2+nH^{\alpha}_{,ij}h^{\alpha}_{ij}-h^{\alpha}_{ij}\overline{R}_{\alpha kik,j}-h^{\alpha}_{ij}\overline{R}_{\alpha ijk,k}\\
&\quad+h^{\alpha}_{ij}h^{\alpha}_{mi}R_{mkjk}+h^{\alpha}_{ij}h^{\alpha}_{km}R_{mijk}+h^{\alpha}_{ij}h^{\beta}_{ki}R^{\perp}_{\beta\alpha jk}.
\end{aligned} \tag{34}$$

From (24) we obtain

$$\begin{aligned}
-h^{\alpha}_{ij}\overline{R}_{\alpha kik,j}&=h^{\alpha}_{ij}\left(\frac{c-(f'^2-ff'')}{f^2}N^{\alpha}(T^i\delta_{kk}-T^k\delta_{ki})\right)_{,j}\\
&=\left(\frac{c-(f'^2-ff'')}{f^2}\right)'h^{\alpha}_{ij}T^jN^{\alpha}(T^i\delta_{kk}-T^k\delta_{ki})
\end{aligned}$$

$$+\frac{c-(f'^2-ff'')}{f^2}h^\alpha_{ij}N^\alpha_{,j}(T^i\delta_{kk}-T^k\delta_{ki})$$
$$+\frac{c-(f'^2-ff'')}{f^2}h^\alpha_{ij}N^\alpha(T^i_{,j}\delta_{kk}-T^k_{,j}\delta_{ki}). \quad (35)$$

Note that

$$h^\alpha_{ij}T^jN^\alpha(T^i\delta_{kk}-T^k\delta_{ki})=(n-1)h^\alpha_{ij}T^jN^\alpha T^i=(n-1)N^\alpha\langle A_\alpha T,T\rangle.$$

By (22) one has

$$\begin{aligned}h^\alpha_{ij}N^\alpha_{,j}(T^i\delta_{kk}-T^k\delta_{ki})&=-h^\alpha_{ij}(h^\alpha_{lj}T^l+\frac{f'}{f}T^jN^\alpha)(T^i\delta_{kk}-T^k\delta_{ki})\\&=-(n-1)h^\alpha_{ij}(h^\alpha_{lj}T^l+\frac{f'}{f}T^jN^\alpha)T^i\\&=-(n-1)|A_\alpha T|^2-\frac{(n-1)f'}{f}N^\alpha\langle A_\alpha T,T\rangle,\end{aligned}$$

and

$$\begin{aligned}h^\alpha_{ij}N^\alpha(T^i_{,j}\delta_{kk}-T^k_{,j}\delta_{ki})&=h^\alpha_{ij}N^\alpha\left(h^\beta_{ij}N^\beta+\frac{f'}{f}(\delta_{ij}-T^iT^j)\right)\delta_{kk}\\&\quad-h^\alpha_{ij}N^\alpha\left(h^\beta_{kj}N^\beta+\frac{f'}{f}(\delta_{kj}-T^kT^j)\right)\delta_{ki}\\&=(n-1)h^\alpha_{ij}N^\alpha\left(h^\beta_{ij}N^\beta+\frac{f'}{f}(\delta_{ij}-T^iT^j)\right)\\&=(n-1)N^\alpha N^\beta\mathrm{Tr}(A_\alpha A_\beta)+\frac{n(n-1)f'}{f}\langle\overrightarrow{H},N\rangle\\&\quad-\frac{(n-1)f'}{f}N^\alpha\langle A_\alpha T,T\rangle.\end{aligned}$$

Putting these expressions into (35) gives

$$\begin{aligned}-h^\alpha_{ij}\overline{R}_{\alpha kik,j}&=(n-1)\left(\frac{c-(f'^2-ff'')}{f^2}\right)'N^\alpha\langle A_\alpha T,T\rangle\\&\quad+\frac{(n-1)\left[c-(f'^2-ff'')\right]}{f^2}\left(N^\alpha N^\beta\mathrm{Tr}(A_\alpha A_\beta)-|A_\alpha T|^2\right)\\&\quad+\frac{(n-1)f'\left[c-(f'^2-ff'')\right]}{f^3}\left(n\langle\overrightarrow{H},N\rangle-2N^\alpha\langle A_\alpha T,T\rangle\right).\end{aligned} \quad (36)$$

Using Codazzi Equation (24) again, we have

$$\begin{aligned}-h^\alpha_{ij}\overline{R}_{\alpha ijk,k}&=h^\alpha_{ij}\left(\frac{c-(f'^2-ff'')}{f^2}N^\alpha(T^j\delta_{ik}-T^k\delta_{ij})\right)_{,k}\\&=\left(\frac{c-(f'^2-ff'')}{f^2}\right)'h^\alpha_{ij}T^kN^\alpha(T^j\delta_{ik}-T^k\delta_{ij})\\&\quad+\frac{c-(f'^2-ff'')}{f^2}h^\alpha_{ij}N^\alpha_{,k}(T^j\delta_{ik}-T^k\delta_{ij})\\&\quad+\frac{c-(f'^2-ff'')}{f^2}h^\alpha_{ij}N^\alpha(T^j_{,k}\delta_{ik}-T^k_{,k}\delta_{ij}).\end{aligned} \quad (37)$$

First observe that

$$h_{ij}^{\alpha}T^kN^{\alpha}(T^j\delta_{ik}-T^k\delta_{ij})=N^{\alpha}\langle A_{\alpha}T,T\rangle-n\langle\overrightarrow{H},N\rangle|T|^2.$$

Then by (22) again, we have

$$\begin{aligned}h_{ij}^{\alpha}N_{,k}^{\alpha}(T^j\delta_{ik}-T^k\delta_{ij})=&-h_{ij}^{\alpha}\left(h_{lk}^{\alpha}T^l+\frac{f'}{f}T^kN^{\alpha}\right)\left(T^j\delta_{ik}-T^k\delta_{ij}\right)\\=&-|A_{\alpha}T|^2+nH^{\alpha}\langle A_{\alpha}T,T\rangle-\frac{f'}{f}N^{\alpha}\langle A_{\alpha}T,T\rangle+\frac{nf'}{f}\langle\overrightarrow{H},N\rangle|T|^2,\end{aligned}$$

and

$$\begin{aligned}h_{ij}^{\alpha}N^{\alpha}(T_{,k}^j\delta_{ik}-T_{,k}^k\delta_{ij})=&\,h_{ij}^{\alpha}N^{\alpha}\left(h_{jk}^{\beta}N^{\beta}+\frac{f'}{f}(\delta_{jk}-T^jT^k)\right)\delta_{ik}\\&-h_{ij}^{\alpha}N^{\alpha}\left(h_{kk}^{\beta}N^{\beta}+\frac{f'}{f}(\delta_{kk}-T^kT^k)\right)\delta_{ij}\\=&\,N^{\alpha}N^{\beta}\mathrm{Tr}(A_{\alpha}A_{\beta})+\frac{nf'}{f}H^{\alpha}N^{\alpha}-\frac{f'}{f}N^{\alpha}\langle A_{\alpha}T,T\rangle\\&-n^2H^{\alpha}N^{\alpha}\langle\overrightarrow{H},N\rangle-\frac{n^2f'}{f}H^{\alpha}N^{\alpha}+\frac{nf'}{f}H^{\alpha}N^{\alpha}|T|^2\\=&\,N^{\alpha}N^{\beta}\mathrm{Tr}(A_{\alpha}A_{\beta})-n^2\langle\overrightarrow{H},N\rangle^2+\frac{nf'}{f}\langle\overrightarrow{H},N\rangle|T|^2\\&-\frac{n(n-1)f'}{f}\langle\overrightarrow{H},N\rangle-\frac{f'}{f}N^{\alpha}\langle A_{\alpha}T,T\rangle.\end{aligned}$$

Substituting the above expressions into (37) we have

$$\begin{aligned}-h_{ij}^{\alpha}\overline{R}_{\alpha ijk,k}=&\left(\frac{c-(f'^2-ff'')}{f^2}\right)'\left(N^{\alpha}\langle A_{\alpha}T,T\rangle-n\langle\overrightarrow{H},N\rangle|T|^2\right)\\&+\frac{c-(f'^2-ff'')}{f^2}\left(N^{\alpha}N^{\beta}\mathrm{Tr}(A_{\alpha}A_{\beta})+nH^{\alpha}\langle A_{\alpha}T,T\rangle-|A_{\alpha}T|^2-n^2\langle\overrightarrow{H},N\rangle^2\right)\\&+\frac{f'[c-(f'^2-ff'')]}{f^3}\left(2n\langle\overrightarrow{H},N\rangle|T|^2-2N^{\alpha}\langle A_{\alpha}T,T\rangle-n(n-1)\langle\overrightarrow{H},N\rangle\right).\end{aligned}\tag{38}$$

Summing (36) and (38) leads to

$$\begin{aligned}&-h_{ij}^{\alpha}\overline{R}_{\alpha kik,j}-h_{ij}^{\alpha}\overline{R}_{\alpha ijk,k}\\=&\,n\left\{\left(\frac{c-(f'^2-ff'')}{f^2}\right)'-\frac{2f'[c-(f'^2-ff'')]}{f^3}\right\}(N^{\alpha}\langle A_{\alpha}T,T\rangle-\langle\overrightarrow{H},N\rangle|T|^2)\\&+\frac{n[c-(f'^2-ff'')]}{f^2}\left(N^{\alpha}N^{\beta}\mathrm{Tr}(A_{\alpha}A_{\beta})+H^{\alpha}\langle A_{\alpha}T,T\rangle-|A_{\alpha}T|^2-n\langle\overrightarrow{H},N\rangle^2\right).\end{aligned}\tag{39}$$

Using Gauss Equation (23) we get

$$\begin{aligned}&h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mkjk}+h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk}\\=&\,h_{ij}^{\alpha}h_{mi}^{\alpha}\Bigg[\frac{c-(f'^2-ff'')}{f^2}\Big(\delta_{mj}\delta_{kk}-\delta_{mk}\delta_{kj}\\&+\delta_{mk}T^kT^j+\delta_{kj}T^mT^k-\delta_{mj}T^kT^k-\delta_{kk}T^mT^j\Big)\\&-\frac{f''}{f}(\delta_{mj}\delta_{kk}-\delta_{mk}\delta_{kj})+(h_{mj}^{\beta}h_{kk}^{\beta}-h_{mk}^{\beta}h_{kj}^{\beta})\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+h_{ij}^{\alpha}h_{km}^{\alpha}\Big[\frac{c-(f'^2-ff'')}{f^2}\Big(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij}\\
&+\delta_{mk}T^iT^j+\delta_{ij}T^mT^k-\delta_{mj}T^iT^k-\delta_{ik}T^mT^j\Big)\\
&-\frac{f''}{f}(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})+(h_{mj}^{\beta}h_{ik}^{\beta}-h_{mk}^{\beta}h_{ij}^{\beta})\Big]\\
=&\frac{c-(f'^2-ff'')}{f^2}\Big((n-1)\|A_\alpha\|^2-|T|^2\|A_\alpha\|^2-(n-2)|A_\alpha T|^2\Big)\\
&-\frac{(n-1)f''}{f}\|A_\alpha\|^2+nH^{\beta}\mathrm{Tr}(A_\alpha^2A_\beta)-\mathrm{Tr}(A_\alpha^2A_\beta^2)\\
&+\frac{c-(f'^2-ff'')}{f^2}\Big(\|A_\alpha\|^2-n^2(H^\alpha)^2+2nH^\alpha\langle A_\alpha T,T\rangle-2|A_\alpha T|^2\Big)\\
&-\frac{f''}{f}\Big(\|A_\alpha\|^2-n^2(H^\alpha)^2\Big)+\mathrm{Tr}\big((A_\alpha A_\beta)^2\big)-\big(\mathrm{Tr}(A_\alpha A_\beta)\big)^2\\
=&\frac{c-(f'^2-ff'')}{f^2}\Big[\big(n-|T|^2\big)\|A_\alpha\|^2-n^2H^2\\
&-n|A_\alpha T|^2+2nH^\alpha\langle A_\alpha T,T\rangle\Big]-\frac{nf''}{f}\Big(\|A_\alpha\|^2-nH^2\Big)\\
&+\mathrm{Tr}\big((A_\alpha A_\beta)^2\big)-\mathrm{Tr}(A_\alpha^2A_\beta^2)+nH^\beta\mathrm{Tr}(A_\alpha^2A_\beta)-S_{\alpha\beta}^2.
\end{aligned}
\tag{40}
$$

By Ricci Equation (25) we have

$$
h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}^{\perp}=h_{ij}^{\alpha}h_{ki}^{\beta}(h_{jl}^{\beta}h_{kl}^{\alpha}-h_{kl}^{\beta}h_{jl}^{\alpha})=\mathrm{Tr}\big((A_\alpha A_\beta)^2\big)-\mathrm{Tr}(A_\alpha^2A_\beta^2).
\tag{41}
$$

Substituting (39)–(41) into (34) gives

$$
\begin{aligned}
\frac{1}{2}\Delta\|A_\alpha\|^2=&\frac{1}{2}\Delta\Big(\sum_{i,j}(h_{ij}^{\alpha})^2\Big)=(h_{ijk}^{\alpha})^2+h_{ij}^{\alpha}h_{ijkk}^{\alpha}\\
=&(h_{ijk}^{\alpha})^2+nH_{,ij}^{\alpha}h_{ij}^{\alpha}+n\Big\{\Big(\frac{c-(f'^2-ff'')}{f^2}\Big)'\\
&-\frac{2f'[c-(f'^2-ff'')]}{f^3}\Big\}\big(N^\alpha\langle A_\alpha T,T\rangle-\langle\overrightarrow{H},N\rangle|T|^2\big)\\
&+\frac{c-(f'^2-ff'')}{f^2}\Big((n-|T|^2)\|A_\alpha\|^2-n^2H^2-2n|A_\alpha T|^2\\
&+3nH^\alpha\langle A_\alpha T,T\rangle+nN^\alpha N^\beta\mathrm{Tr}(A_\alpha A_\beta)-n^2\langle\overrightarrow{H},N\rangle^2\Big)\\
&-\frac{nf''}{f}\Big(\|A_\alpha\|^2-nH^2\Big)+nH^\beta\mathrm{Tr}(A_\alpha^2A_\beta)-\|[A_\alpha,A_\beta]\|^2-S_{\alpha\beta}^2.
\end{aligned}
\tag{42}
$$

Assume that M^n is a minimal submanifold satisfying $c-(f'^2-ff'')=c_1f^4$ at each $t\in I$, where c_1 is a positive constant. We thus obtain

$$
\Big(\frac{c-(f'^2-ff'')}{f^2}\Big)'-\frac{2f'[c-(f'^2-ff'')]}{f^3}=0,
$$

and then (42) becomes

$$
\begin{aligned}
\frac{1}{2}\Delta\|A_\alpha\|^2=&(h_{ijk}^{\alpha})^2+c_1f^2\Big((n-|T|^2)\|A_\alpha\|^2\\
&-2n|A_\alpha T|^2+nN^\alpha N^\beta\mathrm{Tr}(A_\alpha A_\beta)\Big)
\end{aligned}
$$

$$-\frac{nf''}{f}\|A_\alpha\|^2 - \|[A_\alpha, A_\beta]\|^2 - S^2_{\alpha\beta}. \tag{43}$$

Proof of Main Theorem: For fixed $x \in M^n$, we can take a local coordinate system $\{U; (x_1, x_2, \ldots, x_n)\}$ and a suitable local orthonormal normal frame around x such that $F(x) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 = \cdots = \lambda_r > \lambda_{r+1} \geq \cdots \geq \lambda_p$. For simplicity, we denote $A_{n+\alpha}$ briefly by A_α for $1 \leq \alpha \leq p$, and define $A_{r+1} = 0$ if $r = p$. Furthermore, for arbitrary integer $q \geq 2$, we define

$$f_q := \mathrm{Tr}(F^q) = \sum_{\alpha_1,\ldots,\alpha_q} S_{\alpha_1\alpha_2} \cdot S_{\alpha_2\alpha_3} \cdot \cdots \cdot S_{\alpha_q\alpha_1}$$

to be a smooth function on M^n. A straightforward calculation gives rise to, at $x \in M$,

$$|\nabla f_q|^2 = q^2 \sum_k \Big(\sum_\alpha \lambda_\alpha^{q-1} \nabla_{\frac{\partial}{\partial x_k}} S_{\alpha\alpha} \Big)^2, \tag{44}$$

where the covariant derivative of $S_{\alpha\beta}$ is given by

$$\nabla_{\frac{\partial}{\partial x_k}} S_{\alpha\beta} = \frac{\partial S_{\alpha\beta}}{\partial x_k} + \omega_{\alpha\gamma}\left(\frac{\partial}{\partial x_k}\right) S_{\gamma\beta} + \omega_{\beta\gamma}\left(\frac{\partial}{\partial x_k}\right) S_{\gamma\alpha}.$$

Here and subsequently, we use λ_α instead of $\lambda_{\alpha-n}$ for $n+1 \leq \alpha \leq n+p$.
Applying the Cauchy-Schwarz inequality to (44) gives

$$|\nabla f_q|^2 = q^2 \sum_k \Big(\sum_\alpha \lambda_\alpha^{\frac{q}{2}} (\lambda_\alpha^{\frac{q-2}{2}} \nabla_{\frac{\partial}{\partial x_k}} S_{\alpha\alpha}) \Big)^2 \leq q^2 f_q \sum_{k,\alpha} \lambda_\alpha^{q-2} (\nabla_{\frac{\partial}{\partial x_k}} S_{\alpha\alpha})^2. \tag{45}$$

Combining the definition of f_q and (43), we obtain

$$\begin{aligned}
\frac{1}{2q}\Delta f_q =& \frac{1}{2} \sum_{s+t=q-2} \sum_{\alpha,\beta,k} \lambda_\alpha^s \lambda_\beta^t (\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\beta})^2 + \frac{1}{2}\sum_\alpha (\lambda_\alpha^{q-1}\Delta\|A_\alpha\|^2) \\
=& \frac{1}{2} \sum_{s+t=q-2} \sum_{\alpha,\beta,k} \lambda_\alpha^s \lambda_\beta^t (\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\beta})^2 + \sum_\alpha \Big(\lambda_\alpha^{q-1} \sum_{i,j,k} (h^\alpha_{ijk})^2 \Big) \\
& - \sum_{\alpha\neq\beta} \|[A_\alpha, A_\beta]\|^2 \lambda_\alpha^{q-1} - f_{q+1} - \frac{nf''}{f} f_q \\
& + c_1 f^2 \left\{ (n - |T|^2) f_q + n \sum_\alpha \lambda_\alpha^q (N^\alpha)^2 - 2n \sum_\alpha \lambda_\alpha^{q-1} |A_\alpha T|^2 \right\}.
\end{aligned} \tag{46}$$

Observe that

$$\begin{aligned}
& \sum_{s+t=q-2} \sum_{\alpha,\beta,k} \lambda_\alpha^s \lambda_\beta^t (\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\beta})^2 \geq \sum_{s+t=q-2} \sum_{\alpha,k} \lambda_\alpha^{q-2} (\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\alpha})^2 \\
& = (q-1) \sum_{\alpha,k} \lambda_\alpha^{q-2} (\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\alpha})^2, \\
& \sum_\alpha \lambda_\alpha^q (N^\alpha)^2 \geq 0.
\end{aligned}$$

Using the Cauchy-Schwarz inequality we get

$$\begin{aligned}
& \sum_\alpha \lambda_\alpha^{q-1} |A_\alpha T|^2 = \sum_{\alpha,i,j,k} \lambda_\alpha^{q-1} h^\alpha_{ij} h^\alpha_{ik} T^j T^k \\
& \leq \sum_\alpha \lambda_\alpha^{q-1} \sqrt{\sum_{ij} (h^\alpha_{ij})^2} \sqrt{\sum_{ik} (h^\alpha_{ik})^2} \sqrt{\sum_j (T^j)^2} \sqrt{\sum_k (T^k)^2} = f_q |T|^2.
\end{aligned}$$

Applying the above estimates to (46), it follows that

$$\begin{aligned}\frac{1}{2q}\Delta f_q \geq&\frac{q-1}{2}\sum_{\alpha,k}\lambda_\alpha^{q-2}(\nabla_{\frac{\partial}{\partial x^k}}S_{\alpha\alpha})^2+\sum_\alpha\Big(\lambda_\alpha^{q-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)\\&-\sum_{\alpha\neq\beta}\|[A_\alpha,A_\beta]\|^2\lambda_\alpha^{q-1}-f_{q+1}-\frac{nf''}{f}f_q\\&+c_1f^2\Big(n-(2n+1)|T|^2\Big)f_q.\end{aligned}\tag{47}$$

It is straightforward to show that

$$\begin{aligned}&f_q=r\lambda_1^q+\sum_{\alpha=r+1}^p\lambda_\alpha^q\geq r\lambda_1^q,\\&f_q=r\lambda_1^q+\sum_{\alpha=r+1}^p\lambda_\alpha^q\leq r\lambda_1^q+(p-r)\lambda_{r+1}^q\leq r\lambda_1^q+(p-r)\lambda_{r+1}^q,\\&f_{q+1}=r\lambda_1^{q+1}+\sum_{\alpha=r+1}^p\lambda_\alpha^{q+1}\leq r\lambda_1^{q+1}+(p-r)\lambda_{r+1}^{q+1}\leq r\lambda_1^{q+1}+(p-r)\sigma\lambda_{r+1}^q.\end{aligned}$$

Following Lu's paper ([5], Lemma 2), we see that

$$\sum_{\beta=2}^p\|[A_1,A_\beta]\|^2\leq\|A_1\|^2\Big(\sum_{\beta=2}^p\|A_\beta\|^2+\|A_2\|^2\Big).\tag{48}$$

From Lemma 1 in [20] we have

$$\|[A_\alpha,A_\beta]\|^2\leq 2\|A_\alpha\|^2\|A_\beta\|^2\tag{49}$$

for any $1\leq\alpha,\beta\leq p$.

By applying (48) and (49) we conclude

$$\begin{aligned}\sum_{\alpha\neq\beta}\|[A_\alpha,A_\beta]\|^2\lambda_\alpha^{q-1}=&\sum_{\alpha=1}^r\sum_{\alpha\neq\beta}\|[A_\alpha,A_\beta]\|^2\lambda_\alpha^{q-1}+\sum_{\alpha=r+1}^p\sum_{\alpha\neq\beta}\|[A_\alpha,A_\beta]\|^2\lambda_\alpha^{q-1}\\\leq& r\|A_1\|^2\Big(\sum_{\beta=2}^p\|A_\beta\|^2+\|A_2\|^2\Big)\lambda_1^{q-1}+2\sum_{\alpha=r+1}^p\sum_{\alpha\neq\beta}\|A_\alpha\|^2\|A_\beta\|^2\lambda_\alpha^{q-1}\\\leq& r\Big(\sum_{\beta=2}^p\|A_\beta\|^2+\|A_2\|^2\Big)\|A_1\|^{2q}+2(p-r)\sigma\lambda_{r+1}^q,\end{aligned}$$

where the last step is based on

$$\sum_{\alpha=r+1}^p\sum_{\alpha\neq\beta}\|A_\alpha\|^2\|A_\beta\|^2\lambda_\alpha^{q-1}\leq\sum_{\alpha=r+1}^p\sum_{\alpha\neq\beta}\|A_\beta\|^2\lambda_{r+1}^q\leq\sum_{\alpha=r+1}^p\sigma\lambda_{r+1}^q=(p-r)\sigma\lambda_{r+1}^q.$$

Substituting the above estimates into (47), we thus obtain

$$\begin{aligned}\frac{1}{q}\Delta f_q\geq&(q-1)\sum_{\alpha,k}\lambda_\alpha^{q-2}(\nabla_{\frac{\partial}{\partial x^k}}S_{\alpha\alpha})^2+2\sum_\alpha\Big(\lambda_\alpha^{q-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\Big)\\&-2r\Big(\sum_{\beta=2}^p\|A_\beta\|^2+\lambda_2\Big)\lambda_1^q-4(p-r)\sigma\lambda_{r+1}^q\end{aligned}$$

$$
\begin{aligned}
& - 2r\lambda_1^{q+1} - 2(p-r)\sigma\lambda_{r+1}^q - \frac{2n|f''|}{f}\left(r\lambda_1^q + (p-r)\lambda_{r+1}^q\right) \\
& + 2c_1 f^2\left(n - (2n+1)|T|^2\right) r\lambda_1^q \\
=& (q-1)\sum_{\alpha,k}\lambda_\alpha^{q-2}(\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\alpha})^2 + 2\sum_\alpha\left(\lambda_\alpha^{q-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\right) \\
& - 6(p-r)\sigma\lambda_{r+1}^q - \frac{2n(p-r)|f''|}{f}\lambda_{r+1}^q \\
& + 2r\lambda_1^q\Big[c_1 f^2(n - (2n+1)|T|^2) - \sigma - \lambda_2 - \frac{n|f''|}{f}\Big].
\end{aligned}
\tag{50}
$$

Letting $g_q = (f_q)^{\frac{1}{q}}$ and by (44), we get

$$
|\nabla g_q|^2 = \frac{1}{q^2} f_q^{\frac{2}{q}-2}|\nabla f_q|^2 \le \sum_k\Big(\sum_\alpha(\frac{\lambda_\alpha^q}{f_q})^{\frac{q-1}{q}}\cdot(\nabla_{\frac{\partial}{\partial x_k}} S_{\alpha\alpha})\Big)^2 \le C\sigma \tag{51}
$$

for some constant C. It follows that $\int_M \Delta g_q = 0$.

By (50) and (45) one has

$$
\begin{aligned}
\Delta g_q =& \frac{1}{q} f_q^{\frac{1}{q}-1}\Delta f_q + \frac{1}{q}(\frac{1}{q}-1) f_q^{\frac{1}{q}-2}|\nabla f_q|^2 \\
\ge& (q-1) f_q^{\frac{1}{q}-1}\sum_{\alpha,k}\lambda_\alpha^{q-2}(\nabla_{\frac{\partial}{\partial x^k}} S_{\alpha\alpha})^2 + 2 f_q^{\frac{1}{q}-1}\sum_\alpha\left(\lambda_\alpha^{q-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\right) \\
& - 6(p-r)\sigma f_q^{\frac{1}{q}-1}\lambda_{r+1}^q - \frac{2n(p-r)|f''|}{f} f_q^{\frac{1}{q}-1}\lambda_{r+1}^q \\
& + 2r\lambda_1^q f_q^{\frac{1}{q}-1}\Big[c_1 f^2(n - (2n+1)|T|^2) - \sigma - \lambda_2 - \frac{n|f''|}{f}\Big] \\
& + \frac{1}{q}(\frac{1}{q}-1) f_q^{\frac{1}{q}-2}|\nabla f_q|^2 \\
\ge& 2 f_q^{\frac{1}{q}-1}\sum_\alpha\left(\lambda_\alpha^{q-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\right) - 6(p-r)\sigma f_q^{\frac{1}{q}-1}\lambda_{r+1}^q \\
& - \frac{2n(p-r)|f''|}{f} f_q^{\frac{1}{q}-1}\lambda_{r+1}^q + 2r\lambda_1^q f_q^{\frac{1}{q}-1}\Big[c_1 f^2(n - (2n+1)|T|^2) \\
& - \sigma - \lambda_2 - \frac{n|f''|}{f}\Big].
\end{aligned}
\tag{52}
$$

Integrating the both sides of the formula (52) and using $\int_M \Delta g_q = 0$, it follows that

$$
\begin{aligned}
0 \ge& \int_M f_q^{\frac{1}{q}-1}\sum_\alpha\left(\lambda_\alpha^{q-1}\sum_{i,j,k}(h_{ijk}^\alpha)^2\right) - 3(p-r)\int_M \sigma f_q^{\frac{1}{q}-1}\lambda_{r+1}^q \\
& - n(p-r)\int_M \frac{|f''|}{f} f_q^{\frac{1}{q}-1}\lambda_{r+1}^q + \int_M r\lambda_1^q f_q^{\frac{1}{q}-1}\Big[c_1 f^2(n - (2n+1)|T|^2) \\
& - \sigma - \lambda_2 - \frac{n|f''|}{f}\Big].
\end{aligned}
\tag{53}
$$

For fixed $x \in M$, we see that

$$
\lim_{q\to\infty}\frac{\lambda_{r+1}^q}{f_q} = \lim_{q\to\infty}\frac{\lambda_{r+1}^q}{r\lambda_1^q + \sum\limits_{\alpha=r+1}^p \lambda_\alpha^q} \le \lim_{q\to\infty}\frac{1}{r}\left(\frac{\lambda_{r+1}}{\lambda_1}\right)^q = 0,
$$

$$\lim_{q\to\infty}\frac{\lambda_\alpha^{q-1}}{f_q^{1-\frac{1}{q}}}=\lim_{q\to\infty}\left(\frac{\lambda_\alpha^q}{f_q}\right)^{\frac{q-1}{q}}\leq\lim_{q\to\infty}\left(\frac{\lambda_{r+1}^q}{f_q}\right)^{\frac{q-1}{q}}=0 \text{ for } \forall\ \alpha\geq r+1,$$

$$\lim_{q\to\infty}\frac{\lambda_1^{q-1}}{f_q^{1-\frac{1}{q}}}=\lim_{q\to\infty}\left(\frac{\lambda_1^q}{f_q}\right)^{\frac{q-1}{q}}=\frac{1}{r}.$$

As $q\to\infty$, applying the above estimates to (53), we have

$$0\geq\int_M\frac{1}{r}\sum_{i,j,k}\sum_{\alpha\leq r}(h_{ijk}^\alpha)^2+\int_M\|A_1\|^2\Big[c_1f^2\big(n-(2n+1)|T|^2\big)-\sigma-\lambda_2-\frac{n|f''|}{f}\Big]. \tag{54}$$

It follows from the hypothesis (9) that

$$\sum_{i,j,k}\sum_{\alpha\leq r}(h_{ijk}^\alpha)^2=0,\quad \|A_1\|^2\Big[c_1f^2\big(n-(2n+1)|T|^2\big)-\sigma-\lambda_2-\frac{n|f''|}{f}\Big]=0.$$

One thus obtains $\|A_1\|^2=0$, or $\sigma+\lambda_2=c_1f^2\left(n-(2n+1)|T|^2\right)-\frac{n|f''|}{f}$.

The first case shows that M is a totally geodesic submanifold in $I\times_f\mathbb{S}^m(c)$. Since $\sigma=0$, we infer that $h_{ijk}^\alpha=0$ for any i,j,k,α. It follows immediately from Remark 2 and the compactness of M that M lies in a slice $\mathbb{S}^m(c)$. This is the desired conclusion. □

Author Contributions: writing—review and editing, X.Z.; supervision, Z.H. Both authors equally contributed to this manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors is greatly indebted to Fu, Y. for many stimulating conversations. The authors would also like to express their sincere thanks to anonymous reviewers for helpful comments and suggestions on the original manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. De Smet, P.J.; Dillen, F.; Verstraelen, L.; Vrancken, L. A pointwise inequality in submanifold theory. *Arch. Math. (Brno)* **1999**, *35*, 115–128.
2. Chen, B.Y. Classification of Wintgen ideal surfaces in Euclidean 4-space with equal Gauss and normal curvature. *Ann. Global Anal. Geom.* **2010**, *38*, 145–160. [CrossRef]
3. Dillen, F.; Fastenakels, J.; van der Veken, J. Remarks on an inequality involving the normal scalar curvature. In *Pure and Applied Differential Geometry*; Dillen, F., Van de Woestyne, I., Eds.; Shaker: Aachen, Germany, 2007; pp. 83–92.
4. Ge, J.Q.; Tang, Z.Z. A proof of the DDVV conjecture and its equality case. *Pac. J. Math.* **2008**, *237*, 87–95. [CrossRef]
5. Lu, Z.Q. Normal scalar curvature conjecture and its applications. *J. Funct. Anal.* **2011**, *261*, 1284–1308. [CrossRef]
6. Alodan, H.; Chen, B.Y.; Deshmukh, S.; Vîlcu, G.E. A generalized Wintgen inequality for quaternionic CR-submanifolds. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2020**, *114*, 129. [CrossRef]
7. Aydin, M.E.; Mihai, A.; Mihai, I. Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature. *Bull. Math. Sci.* **2017**, *7*, 155–166. [CrossRef]
8. Aydin, M.E.; Mihai, I. Wintgen inequality for statistical surfaces. *Math. Inequal. Appl.* **2019**, *22*, 123–132. [CrossRef]
9. Aytimur, H.; Özgür, C. Cihan Inequalities for submanifolds in statistical manifolds of quasi-constant curvature. *Ann. Polon. Math.* **2018**, *121*, 197–215. [CrossRef]
10. Bansal, P.; Uddin, S.; Shahid, M.H. On the normal scalar curvature conjecture in Kenmotsu statistical manifolds. *J. Geom. Phys.* **2019**, *142*, 37–46. [CrossRef]

11. Boyom, M.N.; Aquib, M.; Shahid, M.H.; Jamali, M. Generalized Wintegen type inequality for Lagrangian submanifolds in holomorphic statistical space forms. In *International Conference on Geometric Science of Information*; Springer: Cham, Switzerland, 2017; pp. 162–169.
12. Boyom, M.N.; Jabeen, Z.; Lone, M.A.; Lone, M.S.; Shahid, M.H. Generalized Wintgen inequality for Legendrian submanifolds in Sasakian statistical manifolds. In *International Conference on Geometric Science of Information*; Springer: Cham, Switzerland, 2019; pp. 407–412.
13. Görünüş, R.; Erken, İ.K.; Yazla, A.; Murathan, C. A generalized Wintgen inequality for Legendrian submanifolds in almost Kenmotsu statistical manifolds. *Int. Electron. J. Geom.* **2019**, *12*, 43–56.
14. Macsim, G.; Ghişoiu, V. Generalized Wintgen inequality for Lagrangian submanifolds in quaternionic space forms. *Math. Inequal. Appl.* **2019**, *22*, 803–813. [CrossRef]
15. Mihai, I. On the generalized Wintgen inequality for Lagrangian submanifolds in complex space forms. *Nonlinear Anal.* **2014**, *95*, 714–720. [CrossRef]
16. Mihai, I. On the generalized Wintgen inequality for Legendrian submanifolds in Sasakian space forms. *Tohoku Math. J.* **2017**, *69*, 43–53. [CrossRef]
17. Murathan, C.; Şahin, B. A study of Wintgen like inequality for submanifolds in statistical warped product manifolds. *J. Geom.* **2018**, *109*, 30. [CrossRef]
18. Simons, J. Minimal varieties in Riemannian manifolds. *Ann. Math.* **1968**, *88*, 62–105. [CrossRef]
19. Lawson, B. Local rigidity theorems for minimal hypersurfaces. *Ann. Math.* **1969**, *89*, 187–197. [CrossRef]
20. Chern, S.S.; Carmo, M.D.; Kobayashi, S. Minimal submanifolds of a sphere with second fundamental form of constant length. In *Functional Analysis and Related Fields*; Browder, F.E., Ed.; Springer: New York, NY, USA, 1970; pp. 59–75.
21. Li, A.M.; Li, J.M. An intrinsic rigidity theorem for minimal submanifolds in a sphere. *Arch. Math. (Basel)* **1992**, *58*, 582–594.
22. Chen, Q.; Cui, Q. Normal scalar curvature and a pinching theorem in $\mathbb{S}^m \times \mathbb{R}$ and $\mathbb{H}^m \times \mathbb{R}$. *Sci. China Math.* **2011**, *54*, 1977–1984. [CrossRef]
23. Zhan, X. A DDVV type inequality and a pinching theorem for compact minimal submanifolds in a generalized cylinder $\mathbb{S}^{n_1}(c) \times \mathbb{R}^{n_2}$. *Results Math.* **2019**, *74*, 102. [CrossRef]
24. Roth, J. A DDVV inequality for submanifolds of warped products. *Bull. Aust. Math. Soc.* **2017**, *95*, 495–499. [CrossRef]
25. O'Neill, B. *Semi-Riemannian Geometry, with Application to Relativity*; Academic Press: New York, NY, USA, 1983.
26. Lawn, M.A.; Ortega, M. A fundamental theorem for hypersurfaces in semi-Riemannian warped products. *J. Geom. Phys.* **2015**, *90*, 55–70. [CrossRef]
27. Hou, Z.H.; Qiu, W.H. Submanifolds with parallel mean curvature vector field in product spaces. *Vietnam J. Math.* **2015**, *43*, 705–723. [CrossRef]