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On d^*-Complete Topological Spaces and Related Fixed Point Results

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Abstract: In this paper, we introduce the concept of d^*-complete topological spaces, which include earlier defined classes of complete metric spaces and quasi b-metric spaces. Further, we prove some fixed point results for mappings defined on d^*-complete topological spaces, generalizing earlier results of Tasković, Ćirić and Prešić, Prešić, Bryant, Marjanović, Yen, Caccioppoli, Reich and Bianchini.

Keywords: fixed point; b-metric space; d^*-complete topological space

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1. Introduction

In 1920 in his PhD. dissertation, S. Banach formulated and proved a contraction mapping principle, which was published in 1922 [1]. It is one of the most important theorems in classical functional analysis because it gives:

(i) The existence of fixed point;
(ii) The uniqueness of such a fixed point;
(iii) Method for getting approximative fixed points;
(iv) Error estimates for approximative fixed point.

There are many (partial) generalizations considering only statements (i), (ii) and (iii) of the contraction mapping principle. Some of them are proved for non-metric spaces, in which the distance function need not be symmetric and need not satisfy triangle inequality. The notion of d-complete L spaces, or Kasahara spaces was introduced by S. Kasahara [2] (see also I. Rus [3]). In these spaces, the class of convergent sequences is axiomatic introduced, because these need not be topological spaces. The topological approach to Kasahara spaces was given in form of d-complete topological spaces by T. Hicks [4].

In this paper, we introduce the concept of d^*-complete topological spaces and prove that these include earlier defined classes of complete metric spaces and quasi b-metric spaces (M. H. Shah and N. Hussain [5]). Further, we prove some fixed point results for mappings defined on d^*-complete topological spaces which generalize earlier results of M. Tasković [6], Lj. Ćirić and S. B. Prešić [7],

S. B. Prešić [8,9], V. Bryant [10], M. Marjanović [11], C. L. Yen [12], R. Caccioppoli [13], S. Reich [14] and R. Bianchini [15].

2. Preliminaries

Let X be a nonempty set and $f : X \to X$ be an arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. For $\vartheta_0 \in X$, we say that a sequence (ϑ_n) defined by $\vartheta_n = f^n(\vartheta_0)$ is a sequence of Picard iterates of f at point ϑ_0 or (ϑ_n) is the orbit of f at point ϑ_0.

The next statement was presented in [16]. Its first part was discussed by D. Adamović [17].

Lemma 1. *(Aranđelović-Kečkić [16]) Let $X \neq \emptyset$ and a mapping $f : X \to X$. Let p be a natural number so that f^p possesses a unique fixed point, say u_*. Then*

(1) u_* is the unique fixed point of f;
(2) if X is a topological space and any sequence of Picard iterates defined by f^p is convergent to u_*, then the sequence of Picard iterates defined by f is convergent to u_*.

Let X be a Hausdorff topological space and $d : X \times X \to [0, \infty)$ be a given function. We define the following three properties:

(α) For any $\vartheta, \theta \in X$, $d(\vartheta, \theta) = 0$ if and only if $\vartheta = \theta$;
(β) For each sequence $(\vartheta_n) \subseteq X$, $\sum_{n=0}^{\infty} d(\vartheta_n, \vartheta_{n+1}) < \infty$ implies that (ϑ_n) is convergent;
(γ) For every sequence $(\vartheta_n) \subseteq X$, if there exist $L > 0$ and $\lambda \in [0, 1)$ such that $d(\vartheta_n, \vartheta_{n+1}) \leq L\lambda^n$ for $n = 0, 1, 2, \ldots$, then (ϑ_n) is a convergent sequence.

The pair (X, d) is a d-complete topological space if it satisfies (α) and (β).
The pair (X, d) is a d^*-complete topological space if it satisfies (α) and (γ).
It is obvious that complete metric spaces are examples of d^*-complete topological spaces, while the converse it is not true in general. The following example explains this fact.

Example 1. *Let $\mathbb{R}$ be the set of real numbers with the usual topology, $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational numbers with relative topology induced from real numbers $\mathbb{R}$ and $d : \mathbb{Q} \times \mathbb{Q} \to [0, \infty)$ be given as*

$$d(\vartheta, \theta) = \begin{cases} 0, \ \text{if } \vartheta = \theta; \\ \vartheta - \theta, \ \text{if there is } k \in \{1, 2, \ldots\} \text{ so that } \vartheta = 2^{-k} \text{ and } \theta = 2^{-k-1}; \\ 1, \qquad \text{otherwise}. \end{cases}$$

Clearly, the ordered pair (X, d) is a d^*-complete topological space. It is not a complete metric space because the symmetry does not hold.

It is clearly also that any d^*-complete topological space (X, d) is d-complete, but the converse is not true.

Example 2. *Let $\mathbb{R}$ be the set of real numbers with the usual topology, $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational numbers with relative topology induced from real numbers $\mathbb{R}$ and $d : \mathbb{Q} \times \mathbb{Q} \to [0, \infty)$ be given as*

$$d(\vartheta, \theta) = \begin{cases} 0, \ \text{if } \vartheta = \theta; \\ \theta - \vartheta, \ \text{if there is } n \in \{1, 2, \ldots\} \text{ so that } \vartheta = \sum_{i=1}^{n} \frac{1}{i^2} \text{ and } \theta = \sum_{i=1}^{n+1} \frac{1}{i^2}; \\ \vartheta - \theta, \ \text{if there is } k \in \{1, 2, \ldots\} \text{ so that } \vartheta = 2^{-k} \text{ and } \theta = 2^{-k-1}; \\ 1, \qquad \text{otherwise}. \end{cases}$$

Clearly, the ordered pair (X, d) is a d-complete topological space. Furthermore, it is not a d^*-complete topological space. Indeed, there are no $L > 0$ and $\lambda \in [0, 1)$ such that

$$d(\vartheta_n, \vartheta_{n+1}) = \frac{1}{(n+1)^2} \leq L\lambda^n, \tag{1}$$

for all $n \in \mathbb{N}$, where the sequence (ϑ_n) is given as $\vartheta_n = \sum_{i=1}^{n} \frac{1}{i^2}$, $n \in \mathbb{N}$.

Remark 1. *Let $n_0 = \left\lfloor \frac{6L\lambda}{(1-\lambda)^3} \right\rfloor + 1$, then inequality*

$$\frac{1}{(n+1)^2} > L\lambda^n, \tag{2}$$

holds for all $n \geq n_0$. Namely, (2) follows from

$$(1+h)^{n+2} > \binom{n+2}{3} h^3, \tag{3}$$

where $h = \frac{1}{\lambda} - 1$.

Definition 1. *Let X and Y be topological spaces. A mapping $f : X \to Y$ is said to be sequentially continuous if for each sequence $(\vartheta_n) \subseteq X$ so that $\lim_{n\to\infty} \vartheta_n = p$, it follows that $\lim_{n\to\infty} f(\vartheta_n) = f(p)$.*

3. Quasi *b*-Metric Spaces

The concept of a quasi b-metric space was discussed by Shah and Hussain in [5]. In this section, we will show that each left complete quasi b-metric space is a d^*-complete topological space.

Definition 2. *Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a given function. (X, d) is said to be a quasi b-metric space if there is $s \in [1, \infty)$ such that for all $\rho, \varsigma, \tau \in X$:*

(a) $d(\rho, \varsigma) = 0$ *if and only if* $\rho = \varsigma$;
(b) $d(\rho, \tau) \leq s[d(\rho, \varsigma) + d(\varsigma, \tau)]$.

Every quasi b-metric space can be considered as a topological space, on which the topology is introduced by taking, for any $\vartheta \in X$, the collection $\{B_n(\vartheta) : n = 1, 2, \ldots\}$ as a base of neighborhood filter of the point ϑ. Here, the ball $\{B_n(\vartheta)\}$ is defined by

$$B_n(\vartheta) = \left\{\theta \in X : d(\vartheta, \theta) < \frac{1}{n}\right\}.$$

According to this definition for each $\vartheta \in X$ and $(\vartheta_n) \subseteq X$, from $\lim_{n\to\infty} d(\vartheta, \vartheta_n) = 0$, it follows that $\lim_{n\to\infty} \vartheta_n = \vartheta$.

Further, $(\vartheta_n) \subseteq X$ is said to be a left Cauchy sequence, if for every $\varepsilon > 0$, there is a positive integer n_0 so that $d(\vartheta_n, \vartheta_m) < \varepsilon$ for all $m > n \geq n_0$.

A quasi b-metric space (X, d) is said to be left complete if each its left Cauchy sequence is convergent.

Now, we need the following Lemma, which generalizes the result formulated and proved by R. Miculescu and A. Mihail [18] for b-metric spaces (for other related details, see [19–22]). Our proof is similar to [18], but for the convenience of the reader we shall give it.

Lemma 2. *Let (X,d) be a quasi b-metric space with constant s and $(\vartheta_n) \subseteq X$. Then*

$$d(\vartheta_0,\vartheta_k) \le s^n \sum_{i=0}^{k-1} d(\vartheta_i,\vartheta_{i+1}),$$

for any $n \in \mathbb{N}$ and every $k \in \{1,2,\ldots,2^n\}$.

Proof. We use the method of mathematical induction in the proof. Denote by $P(n)$ the statement

$$d(\vartheta_0,\vartheta_k) \le s^n \sum_{i=0}^{k-1} d(\vartheta_i,\vartheta_{i+1}), k \in \{1,2,\ldots,2^n-1,2^n\}.$$

Obviously, $P(0)$ is true. Now, we prove that $P(n) \Rightarrow P(n+1), \forall n \ge 0$.
Let $P(n)$ be true for some positive integer n.
Then for any $k \in \{1,2,\ldots,2^n-1,2^n\}$, by $P(n)$ we obtain

$$d(\vartheta_0,\vartheta_k) \le s^n \sum_{i=0}^{k-1} d(\vartheta_i,\vartheta_{i+1}),$$

which implies

$$d(\vartheta_0,\vartheta_k) \le s^{n+1} \sum_{i=0}^{k-1} d(\vartheta_i,\vartheta_{i+1}),$$

because $s \ge 1$. For every $k \in \{2^n+1,2^n+2,\ldots,2^{n+1}-1,2^{n+1}\}$, we have

$$\begin{aligned} d(\vartheta_0,\vartheta_k) &\le s[d(\vartheta_0,\vartheta_{2^n}) + d(\vartheta_{2^n},\vartheta_k] \\ &\le s[s^n \sum_{i=0}^{2^n-1} d(\vartheta_i,\vartheta_{i+1}) + s^n \sum_{i=2^n}^{k} d(\vartheta_i,\vartheta_{i+1})] \\ &= s^{n+1} \sum_{i=0}^{k-1} d(\vartheta_i,\vartheta_{i+1}). \end{aligned}$$

So, by induction, $P(n)$ holds for every $n \ge 0$. □

The following theorem is a generalization of recent results of R. Miculescu and A. Mihail [18] for b-metric spaces. The proof is similar to [18]. Again, for the convenience of the reader, we present it.

Theorem 1. *Every left complete quasi b-metric space is a d^*-complete topological space.*

Proof. Let (X,d) be a left complete quasi b-metric space with constant s, $\lambda \in [0,1)$ and $(\vartheta_n) \subseteq X$ such that

$$d(\vartheta_{n+1},\vartheta_{n+2}) \le \lambda d(\vartheta_n,\vartheta_{n+1}), n = 0,1,2,\ldots.$$

We shall prove that (ϑ_n) is a left Cauchy sequence.

Let m, k be arbitrary positive integers and $j = [\log_2(k)]$. Then

$$\begin{aligned}
d(\vartheta_{m+1}, \vartheta_{m+k}) &\leq sd(\vartheta_{m+1}, \vartheta_{m+2}) + sd(\vartheta_{m+2}, \vartheta_{m+k}) \\
&\leq sd(\vartheta_{m+1}, \vartheta_{m+2}) + s^2 d(\vartheta_{m+2}, \vartheta_{m+2^2}) + s^2 d(\vartheta_{m+2^2}, \vartheta_{m+k}) \\
&\leq sd(\vartheta_{m+1}, \vartheta_{m+2}) + s^2 d(\vartheta_{m+2}, \vartheta_{m+2^2}) + s^3 d(\vartheta_{m+2^2}, \vartheta_{m+2^3}) \\
&+ s^3 d(\vartheta_{m+2^3}, \vartheta_{m+k}) \\
&\vdots \\
&\leq \sum_{n=1}^{j} s^n d(\vartheta_{m+2^{n-1}}, \vartheta_{m+2^n}) + s^{j+1} d(\vartheta_{m+2^j}, \vartheta_{m+k}).
\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}
d(\vartheta_{m+1}, \vartheta_{m+k}) &\leq \sum_{n=1}^{j} s^n d(\vartheta_{m+2^{n-1}}, \vartheta_{m+2^n}) + s^{j+1} d(\vartheta_{m+2^j}, \vartheta_{m+k}) \\
&\leq \sum_{n=1}^{j} s^{2n} \Big(\sum_{i=m}^{m+2^{n-1}-1} d(\vartheta_{2^{n-1}+i}, \vartheta_{2^{n-1}+i+1}) \Big) \\
&+ s^{2j+2} \sum_{i=m}^{m+k-2^j-1} d(\vartheta_{i+2^j}, \vartheta_{1+i+2^j}) \\
&\leq \sum_{n=1}^{j+1} s^{2n} \sum_{i=m}^{m+2^{n-1}-1} d(\vartheta_{2^{n-1}+i}, \vartheta_{2^{n-1}+i+1}) \\
&\leq d(\vartheta_0, \vartheta_1) \sum_{n=1}^{j+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} \lambda^{m+2^{n-1}+i} \\
&\leq \frac{d(\vartheta_0, \vartheta_1)\lambda^m}{1-\lambda} \sum_{n=0}^{j+1} s^{2n} \lambda^{2^{n-1}} \\
&= \frac{d(\vartheta_0, \vartheta_1)\lambda^m}{1-\lambda} \sum_{n=0}^{j+1} \lambda^{2n\log_\lambda(s)+2^{n-1}}.
\end{aligned}$$

For each $M > 0$, there is a positive integer n_0 such that

$$2n\log_\lambda(s) + 2^{n-1} - n \geq M$$

for any $n_0 \in \{n_0+1, n_0+2, \ldots\}$ because

$$\lim_{n\to\infty} 2n\log_\lambda(s) + 2^{n-1} - n = \infty. \tag{4}$$

From (4), we get that

$$\lambda^{2n\log_\lambda(s)+2^{n-1}} \leq \lambda^{M+n},$$

for each $n \in \{n_0+1, n_0+2, \ldots\}$. So, there is a real number $S > 0$ such that

$$\sum_{n=1}^{\infty} \lambda^{2n\log_\lambda(s)+2^{n-1}} = S.$$

This implies that

$$d(\vartheta_{m+1}, \vartheta_{m+k}) \leq \frac{d(\vartheta_0, \vartheta_1)\lambda^m}{1-\lambda} S,$$

for all $m, k \in \mathbb{N}$. So, (ϑ_n) is a left Cauchy sequence. □

Remark 2. *Let $\mathbb{Q}$ and d be defined as in Example 1. Suppose that $(\mathbb{Q}, d)$ is a quasi b-metric space. Then there exists $s \geq 1$ such that $d(\rho, \tau) \leq s[d(\rho, \varsigma) + d(\varsigma, \tau)]$ for all $\rho, \varsigma, \tau \in X$. Let l be a positive integer such that $s < 2^l$, $\rho = 2^{-l}$, $\varsigma = 2^{-l-1}$ and $\tau = 2^{-l-2}$. Hence $d(\rho, \tau) = 1$, $d(\rho, \varsigma) = 2^{-l-1}$ and $d(\varsigma, \tau) = 2^{-l-2}$. We get that*

$$1 = d(\rho, \tau) \leq s[d(\rho, \varsigma) + d(\varsigma, \tau)] < 2^l(2^{-l-1} + 2^{-l-2}) = \frac{3}{4},$$

which is contradiction.

So we obtain that class of d^*-complete topological spaces is more general then class of left complete quasi b-metric spaces.

4. Main Results

Now, we shall prove that the product of d^*-complete topological spaces is a d^*-complete topological space.

Theorem 2. *Let $(X_1, d_1), \ldots, (X_n, d_n)$ be d^*-complete topological spaces, $X = X_1 \times \cdots \times X_n$ be the product space and $d : X \times X \to [0, \infty)$ be defined by*

$$d((p_1, \ldots, p_n), (q_1, \ldots, q_n)) = \max\{d_1(p_1, q_1), \ldots, d_n(p_n, q_n)\},$$

where $p_i, q_i \in X_i$ for any $i = 1, 2, \ldots, n$. Then $(X_1 \times \cdots \times X_n, d)$ is a d^*-complete topological space.

Proof. Let $(y_n) \subseteq X$ be a sequence defined by

$$y_k = (\vartheta_{k1}, \ldots, \vartheta_{kn}), \quad k = 0, 1, 2 \ldots.$$

If there are $L > 0$ and $\lambda \in [0, 1)$ such that $d(y_k, y_{k+1}) \leq \lambda^k L$ for $k = 0, 1, 2, \ldots$, then for every $1 \leq i \leq n$, we get that $d(\vartheta_{ki}, \vartheta_{(k+1)i}) \leq \lambda^k L$, which implies that the sequence $(\vartheta_{ki}) \subseteq X_i$ is convergent for each $i = 1, \ldots, n$, because (X_i, d_i) is a d^*-complete topological space. So (y_k) is a convergent sequence in X, because all its coordinate sequences (ϑ_{ki}) for $1 \leq i \leq n$, are convergent. □

Lemma 3. *Let $X_1, \ldots, X_n$ be Hausdorff topological spaces, $X = X_1 \times \cdots \times X_n$, $d : X \times X \to [0, \infty)$ be a mapping defined by*

$$d((\vartheta_1, \ldots, \vartheta_n), (y_1, \ldots, y_n)) = \max\{d_1(\vartheta_1, y_1), \ldots, d_n(\vartheta_n, y_n)\},$$

and $f_i : X \to X_i$ be sequentially continuous functions and $F : X \to X$ be defined by

$$F(\vartheta) = (f_1(\vartheta_1), ..., f_n(\vartheta_n)),$$

where $\vartheta = (\vartheta_1, ..., \vartheta_n) \in X$. Then F is a sequentially continuous function.

Proof. Let $(y_k) \subseteq X$ be a sequence defined by

$$y_k = (\vartheta_{k1}, \ldots, \vartheta_{kn}), \quad k = 0, 1, 2 \ldots,$$

such that $\lim_{k\to\infty} \vartheta_{ki} = \vartheta_i$ for each $i = 1, \ldots, n$. Let $y = (\vartheta_1, \ldots, \vartheta_n)$. That is, (y_k) is convergent to y. Then

$$\begin{aligned} \lim_{k\to\infty} F(y_k) = \lim_{k\to\infty} (f_1(\vartheta_{k1}, \ldots, \vartheta_{kn}), \ldots, f_n(\vartheta_{k1}, \ldots, \vartheta_{kn})) = \\ (\lim_{k\to\infty} f_1(\vartheta_{k1}, \ldots, \vartheta_{kn}), \ldots, \lim_{k\to\infty} f_n(\vartheta_{k1}, \ldots, \vartheta_{kn})) = \\ (f_1(\lim_{k\to\infty} \vartheta_{k1}, \ldots, \lim_{k\to\infty} \vartheta_{kn}), \ldots, f_n(\lim_{k\to\infty} \vartheta_{k1}, \ldots, \lim_{k\to\infty} \vartheta_{kn})) = \\ (f_1(\vartheta_1, \ldots, \vartheta_n), \ldots, f_n(\vartheta_1, \ldots, \vartheta_n))) = F(y). \end{aligned}$$

□

Next theorem generalizes earlier results presented by M. Tasković [6] on complete metric spaces (case $n = 2$) to d^*-complete topological spaces.

Theorem 3. *Let $(X_1, d_1),\ldots,(X_n, d_n)$ be d^*-complete topological spaces, $X = X_1 \times \cdots \times X_n$, $d : X \times X \to [0, \infty)$ be a function defined by*

$$d((\vartheta_1, \ldots, \vartheta_n), (y_1, \ldots, y_n)) = \max\{d_1(\vartheta_1, y_1), \ldots, d_n(\vartheta_n, y_n)\},$$

$f_i : X \to X_i$ be sequentially continuous functions and $F : X \to X$ be defined by

$$F(\vartheta) = (f_1(\vartheta), ..., f_n(\vartheta)),$$

where $\vartheta = (\vartheta_1, ..., \vartheta_n)$. If there is $0 \leq \lambda < 1$ such that

$$d(F(\vartheta), F(\theta)) \leq \lambda d(\vartheta, \theta),$$

for all $\vartheta, \theta \in X$, then

(1) F admits a unique fixed point, say $p \in X$;
(2) for every $\vartheta_0 \in X$, the sequence of Picard iterates (ϑ_n) defined by F at ϑ_0 converges to p.

Proof. By Lemma 3, we get that F is sequentially continuous. Let $\vartheta_0 \in X$ be arbitrary, and (ϑ_n) be a sequence of Picard iterations defined by F at ϑ_0. We have

$$d(\vartheta_{n+1}, \vartheta_{n+2}) \leq \lambda d(\vartheta_n, \vartheta_{n+1}), \quad n = 0, 1, 2, \ldots,$$

which implies that (ϑ_n) is a convergent sequence because (X, d) is a d^*-complete topological space. Let $p = \lim_{n\to\infty} \vartheta_n (\in X)$. Then

$$p = \lim_{n\to\infty} \vartheta_n = \lim_{n\to\infty} \vartheta_{n+1} = \lim_{n\to\infty} F(\vartheta_n) = F(p),$$

because X is a Hausdorff topological space and F is a sequentially continuous mapping.

Let $q = F(q)$. Then from $d(F(p), F(q)) \leq \lambda d(p, q)$, we obtain $d(p, q) \leq \lambda d(p, q)$. We get easily the uniqueness. □

The next theorem extends earlier result proved by Lj. Ćirić and S. B. Prešić, [7] for complete metric spaces to d^*-complete topological spaces.

Theorem 4. *Let (X, d) be a d^*-complete topological space and $f : X^n \to X$ be a sequentially continuous mapping. If there is $0 \leq \lambda < 1$ so that*

$$d(f(\vartheta_1, \ldots, \vartheta_n), f(\vartheta_2, \ldots, \vartheta_{n+1})) \leq \lambda \max\{d(\vartheta_1, \vartheta_2), \ldots, d(\vartheta_n, \vartheta_{n+1})\},$$

holds for every $\vartheta_1, \ldots, \vartheta_n, \vartheta_{n+1} \in X$. Then

(I) there is $p \in X$ so that

$$p = f(\underbrace{p, \ldots, p}_{n\ times});$$

(II) for arbitrary $\vartheta_1, \ldots, \vartheta_n \in X$, the sequence $(\vartheta_n) \subseteq X$ defined by

$$\vartheta_{k+n} = f(\vartheta_k, \ldots, \vartheta_{k+n-1}), \qquad k = 1, 2, \ldots,$$

is convergent and

$$\lim_{k\to\infty}\vartheta_k = f(\underbrace{\lim_{k\to\infty}\vartheta_k,\ldots,\lim_{k\to\infty}\vartheta_k}_{n\ times});$$

(III) if

$$d(f(\underbrace{\tau,\ldots,\tau}_{n\ times}), f(\underbrace{v,\ldots,v}_{n\ times})) < d(\tau, v)$$

for all $\tau, v \in X$*, then the point* p *is unique.*

Proof. Assertions (I) and (II). Let $d : X^n \times X^n \to [0,\infty)$ be defined by

$$d((\vartheta_1,\ldots,\vartheta_n),(y_1,\ldots,y_n)) = \max\{d_1(\vartheta_1,y_1),\ldots,d_n(\vartheta_n,y_n)\}.$$

From Theorem 2, it follows that (X,d) is a d^*-complete topological space. Let $F : X^n \to X^n$ be defined by

$$F(\vartheta_1,\ldots,\vartheta_n) = (\vartheta_2,\ldots,\vartheta_{n-1},\vartheta_n, f(\vartheta_1,\ldots,\vartheta_n)).$$

We have that F is a sequentially continuous mapping on X^n, because f is a sequentially continuous mapping on X.

Let $(y_k) \subseteq X^n$ be a sequence defined by

$$y_k = (\vartheta_k,\vartheta_{k+1},\ldots,\vartheta_{k+n-1}),$$

for arbitrary $\vartheta_1,\ldots,\vartheta_n \in X$ and $(\vartheta_n) \subseteq X$ be defined by

$$\vartheta_{k+n} = f(\vartheta_k,\ldots,\vartheta_{k+n-1}), \qquad k = 1,2,\ldots.$$

We get that

$$d(y_{k+1},y_{k+2}) \le \lambda d(y_k,y_{k+1}) \qquad k = 1,2,\ldots,$$

which implies that (y_k) is a convergent sequence because (X^n,d) is a d^*-complete topological space. Let $z = \lim\limits_{k\to\infty} y_k$. Since X^n is a Hausdorff topological space and F is a sequentially continuous mapping on X^n, one writes

$$z = \lim_{k\to\infty} y_k = \lim_{k\to\infty} y_{k+1} = \lim_{k\to\infty} F(y_k) = F(z).$$

From $z = F(z)$, it follows

$$\lim_{k\to\infty} y_k = (\lim_{k\to\infty}\vartheta_{k+1},\lim_{k\to\infty}\vartheta_{k+2},\ldots,\lim_{k\to\infty}\vartheta_{k+n}) = (\lim_{k\to\infty}\vartheta_k,\lim_{k\to\infty}\vartheta_k,\ldots,\lim_{k\to\infty}\vartheta_k),$$

which implies

$$\lim_{k\to\infty}\vartheta_k = f(\underbrace{\lim_{k\to\infty}\vartheta_k,\ldots,\lim_{k\to\infty}\vartheta_k}_{n\ times}).$$

Let $p = \lim\limits_{k\to\infty}\vartheta_k$. Hence, $p = f(\underbrace{p,\ldots,p}_{n\ times})$.

Assertion (III). Suppose there is $q \in X$ (with $q \neq p$) so that $q = f(\underbrace{q,\ldots,q}_{n\ times})$. In view of the assumption that

$$d(f(\underbrace{u,\ldots,u}_{n\ times}), f(\underbrace{v,\ldots,v}_{n\ times})) < d(u,v)$$

for each $u, v \in X$, then

$$d(f(\underbrace{p,\ldots,p}_{n\ times}), f(\underbrace{q,\ldots,q}_{n\ times}) < d(p,q).$$

That is, $0 < d(p,q) < d(p,q)$, which is a contradiction. Consequently, $p = q$, and so the uniqueness is ensured. □

The following corollary corresponds to the result proved by S. B. Prešić [8,9] in the setting of d^*-complete topological spaces.

Corollary 1. *Let (X,d) be a d^*-complete topological space and $f : X^n \to X$ be a sequentially continuous mapping. If there are $0 \leq \lambda_1, \ldots, \lambda_n < 1$ such that $\sum_{i=1}^{n} \lambda_i < 1$ and*

$$d(f(\vartheta_1,\ldots,\vartheta_n), f(\vartheta_2,\ldots,\vartheta_{n+1})) \leq \lambda_1 d(\vartheta_1,\vartheta_2) + \cdots + \lambda_n d(\vartheta_n,\vartheta_{n+1}),$$

holds for every $\vartheta_1,\ldots,\vartheta_n,\vartheta_{n+1} \in X$, then

(i) there is $p \in X$ so that

$$p = f(\underbrace{p,\ldots,p}_{n\ times});$$

(ii) for arbitrary $\vartheta_1,\ldots,\vartheta_n \in X$, the sequence $(\vartheta_n) \subseteq X$ defined by

$$\vartheta_{k+n} = f(\vartheta_k,\ldots,\vartheta_{k+n-1}), \qquad k = 1,2,\ldots,$$

is convergent and

$$\lim_{k\to\infty} \vartheta_k = f(\underbrace{\lim \vartheta_k,\ldots,\lim \vartheta_k}_{n\ times});$$

(iii) if

$$d(f(\underbrace{u,\ldots,u}_{n\ times}), f(\underbrace{v,\ldots,v}_{n\ times})) < d(u,v),$$

for each $u, v \in X$, then the fixed point p is unique.

Proof. It follows from Theorem 4 and relation

$$\lambda_1 d(\vartheta_1,\vartheta_2) + \cdots + \lambda_n d(\vartheta_n,\vartheta_{n+1}) \leq \sum_{i=1}^{n} \lambda_i \max_{1\leq i\leq n} d(\vartheta_i,\vartheta_{i+1}).$$

□

The next theorem extends earlier results presented by V. Bryant [10], M. Marjanović [11], C. L. Yen [12], R. Caccioppoli [13], S. Reich [14] and R. Bianchini [15] for complete metric spaces to d^*-complete topological spaces.

Theorem 5. *Let (X,d) be a d^*-complete topological space, $\lambda \in [0,1)$ and $f : X \to X$ be a sequentially continuous mapping such that*

$$d(f^{n+1}(\vartheta), f^{n+1}(\theta)) \leq \lambda \max_{0\leq i\leq n} \{d(f^i(\vartheta), f^i(\theta)), d(f^i(\vartheta), f^{i+1}(\vartheta)), d(f^i(\theta), f^{i+1}(\theta))\},$$

for each $\vartheta, \theta \in X$. Then

(1) f admits a unique fixed point $p \in X$;

(2) for each $\vartheta \in X$, the sequence of Picard iterates $(f^n(\vartheta))$ converges to p.

Proof. Let $\vartheta_1 \in X$ be arbitrary, (ϑ_n) be a sequence defined by f at ϑ_1. We get that

$$d(\vartheta_{n+1}, \vartheta_{n+2}) \leq \lambda \max\{d(\vartheta_1, \vartheta_2), \ldots, d(\vartheta_n, \vartheta_{n+1})\}, \tag{5}$$

holds for every n. By Theorem 4, there is $p \in X$ so that $p = \lim_{n\to\infty} \vartheta_n$ and

$$p = f^{n+1}(p).$$

Suppose $q = f^{n+1}(q)$. Then from

$$d(f^{n+1}(p), f^{n+1}(q)) \leq \lambda \max_{0\leq i\leq n} \{d(f^i(p), f^i(q)), d(f^j(p), f^{i+1}(p)), d(f^i(q), f^{i+1}(q))\},$$

we obtain

$$d(p, q) \leq \lambda d(p, q).$$

Hence, $p = q$, i.e., f^n has a unique fixed point.

From (5), it follows that

$$d(f(\vartheta), f(\theta)) \leq \lambda d(\vartheta, \theta),$$

which implies that

$$d(f^k(\vartheta), f^k(\theta)) \leq \lambda^k d(\vartheta, \theta).$$

Let $\vartheta_0 \in X$ be arbitrary, and (ϑ_k) be a sequence of Picard iteration defined by f at ϑ_0. So,

$$d(\vartheta_{k+1}, \vartheta_{k+2}) \leq d(\vartheta_{k+1}, \vartheta_{k+2}) \leq \lambda d(\vartheta_k, \vartheta_{k+1}) \leq \lambda^{k+1} d(\vartheta_0, \vartheta_1).$$

Hence (ϑ_k) is a convergent sequence. Let $\lim_{k\to\infty} \vartheta_k = p$. So

$$p = \lim_{k\to\infty} \vartheta_k = \lim_{k\to\infty} \vartheta_{k+1} = \lim_{k\to\infty} f(\vartheta_k) = f(p),$$

because f is sequentially continuous. Hence, $p = f(p)$.

We prove that f^{n+1} has a unique fixed point p, which is the limit of all sequences of Picard iterations defined by $f^{n+1}\vartheta$. By Lemma 1, it follows that f has a unique fixed point $p \in X$ and for each $\vartheta \in X$, the sequence of Picard iterates defined by f at ϑ converges to p. □

The next corollary extends the known results presented by S. Reich [14] and R. Bianchini [15] from complete metric spaces to d^*-complete topological spaces.

Corollary 2. *Let (X, d) be a d^*-complete topological space, $\lambda \in [0, 1)$ and $f : X \to X$ be a self-mapping on X. Suppose that f is a sequentially continuous mapping. If*

$$d(f(\vartheta), f(\theta)) \leq \lambda \max\{d(\vartheta, \theta), d(\vartheta, f(\vartheta)), d(\theta, f(\theta))\}, \tag{6}$$

for each $\vartheta, \theta \in X$, then

(I) f has a unique fixed point $p \in X$;
(II) for each $\vartheta \in X$, the sequence of Picard iterates $(f^n(\vartheta))$ converges to p.

In theorem of R. Bianchini [15], the inequality

$$d(f(x), f(y)) \leq \lambda \max\{d(x, f(x)), d(y, f(y))\},$$

was used instead of inequality (6). In theorem of S. Reich [14], the inequality

$$d(f(\vartheta), f(\theta)) \leq \alpha d(\vartheta, \theta) + \beta d(\vartheta, f(\theta)) + \gamma d(\theta, f(\theta)),$$

where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$, was used instead of inequality (6).

From Theorem 5, the next corollary extends famous results presented by V. Bryant [10] for complete metric spaces to d^*-complete topological spaces.

Corollary 3. *Let (X, d) be a d^*-complete topological spaces, $\lambda \in [0, 1)$ and $f : X \to X$. Suppose that f is a sequentially continuous mapping. If there is a positive integer n so that*

$$d(f^n(\vartheta), f^n(\theta)) \leq \lambda d(\vartheta, \theta),$$

for each $\vartheta, \theta \in X$, then f has a unique fixed point, which is the limit of the sequence of Picard iterates of f at an arbitrary point $\vartheta \in X$.

By Corollary 3, we obtain the following result which extends the theorem of C. L. Yen [12] from complete metric spaces to d^*-complete topological spaces.

Corollary 4. *Let (X, d) be a d^*-complete topological spaces, $\lambda \in [0, 1)$ and $f : X \to X$. Suppose that f is a sequentially continuous mapping. If there exist positive integers m, n such that*

$$d(f^m(\vartheta), f^n(\vartheta)) \leq \lambda d(\vartheta, \theta),$$

for each $\vartheta, \theta \in X$, then f has a unique fixed point, which is the limit of the sequence of Picard iterates of f at an arbitrary point $\vartheta \in X$.

Proof. Put $\vartheta = f^n(z)$ and $\theta = f^m(z)$. We get that f^{m+n} satisfies all conditions of Corollary 2. □

By Corollary 2, the next result extends the known theorem of R. Caccioppoli from complete metric spaces to d^*-complete topological spaces.

Corollary 5. *Let (X, d) be a d^*-complete topological space, $\lambda \in [0, 1)$ and $f : X \to X$. Suppose that f is a sequentially continuous mapping. If there is a sequence of nonnegative reals (c_n) so that $\sum_{n=1}^{+\infty} c_n < \infty$ and*

$$d(f^n(\vartheta), f^n(\theta)) \leq c_n d(\vartheta, \theta),$$

for each $\vartheta, \theta \in X$, then f has a unique fixed point, which is the limit of the sequence of Picard iterates of f at an arbitrary point $\vartheta \in X$.

Proof. For some positive integer n, we have $c_n < 1$. Now, the statement follows from Corollary 2. □

5. Conclusions

In this paper, we introduce the concept of d^*-complete topological spaces. We give an example of a d^*-complete topological space, which is not a complete metric space. We show that the product of d^*-complete topological spaces is also a d^*-complete topological space. We also establish that every left complete quasi b-metric space is a d^*-complete topological space. Moreover, we prove some fixed point results for contraction mappings in the setting of d^*-complete topological spaces. These obtained results are generalizations of many known ones in the literature.

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