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Multiple Meixner Polynomials on a Non-Uniform Lattice

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Abstract: We consider two families of type II multiple orthogonal polynomials. Each family has orthogonality conditions with respect to a discrete vector measure. The r components of each vector measure are q-analogues of Meixner measures of the first and second kind, respectively. These polynomials have lowering and raising operators, which lead to the Rodrigues formula, difference equation of order $r+1$, and explicit expressions for the coefficients of recurrence relation of order $r+1$. Some limit relations are obtained.

Keywords: Hermite–Padé approximation; multiple orthogonal polynomials; discrete orthogonality; recurrence relations

MSC: 42C05; 33C47; 33E99

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1. Introduction

Hermite's proof [1] of the transcendence of the number e uses the notion of simultaneous approximation, which was subsequently studied in approximation theory and number theory [2–8]. Multiple orthogonal polynomials are polynomials that satisfy orthogonality conditions shared with respect to a set of measures [9–17]. They are related to the simultaneous rational approximation of a system of r analytic functions [18,19] and play an important role both in pure and applied mathematics (see for instance [20–22] as well as [23–27]). In this context, some families of continuous and discrete multiple orthogonal polynomials have been studied [3,28–30] as well as some multiple q-orthogonal polynomials [31–33]. The goal of the present paper is to study some multiple Meixner polynomials on a non-uniform lattice $x(s) = q^s - 1/q - 1, s = 0, 1, \dots$

The paper is structured as follows. Section 2 is devoted to introduce the necessary background material. In Section 3, we consider two families of multiple q-orthogonal polynomials, namely, multiple q-Meixner polynomials of the first and second kind, respectively. They are analogous to the discrete multiple Meixner polynomials studied in [28]. We obtain the raising and lowering q-difference operators as well as the Rodrigues-type formula, which lead to an explicit expression for the multiple q-Meixner polynomials. Then, the recurrence relations as well as the q-difference equations with respect to the independent variable $x(s)$ are obtained. In Section 4, some limit relations as the parameter q approaches 1 are studied. An appendix to the Section 3 is considered in Section 5, in which the AT-property of the involved system of q-discrete measures is addressed. We make concluding remarks in Section 6.

2. Background Material

Let $\vec{\mu} = (\mu_1, \dots, \mu_r)$ be a vector of r positive Borel measures supported on $\mathbb{R}$ with finite moments. By Ω_i we denote the smallest interval that contains $\operatorname{supp}(\mu_i)$. Define a multi-index $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, where $\mathbb{N}$ stands for the set of nonnegative integers. For the multi-index $\vec{n}$, a type II multiple orthogonal polynomial $P_{\vec{n}}$ is a polynomial of degree $\leq |\vec{n}| = n_1 + \cdots + n_r$, which satisfies the orthogonality conditions [34]

$$\int_{\Omega_i} P_{\vec{n}}(x)x^k d\mu_i(x) = 0, \qquad k = 0, \ldots, n_i - 1, \qquad i = 1, \ldots, r. \tag{1}$$

Special attention is paid to a unique solution of (1) (up to a multiplicative factor) with $\deg P_{\vec{n}}(x) = |\vec{n}|$ for every $\vec{n}$. In this situation the index is said to be normal [34]. In particular, if the above system of measures forms an AT system [34], then every multi-index is normal.

The polynomial $P_{\vec{n}}(z)$ is the common denominator of the simultaneous rational approximants $\frac{Q_{\vec{n},i}(z)}{P_{\vec{n}}(z)}$, to Cauchy transforms

$$\hat{\mu}_i(z) = \int_{\Omega_i} \frac{d\mu_i(x)}{z - x}, \quad z \notin \Omega_i \quad i = 1, \ldots, r, \tag{2}$$

of the vector components of $\vec{\mu} = (\mu_1, \ldots, \mu_r)$, i.e., for function (2) we have the following simultaneous rational approximation with prescribed order near infinity [34]

$$P_{\vec{n}}(z)\hat{\mu}_i(z) - Q_{\vec{n},i}(z) = \frac{\zeta_i}{z^{n_i+1}} + \cdots = \mathcal{O}(z^{-n_i-1}), \quad i = 1, \ldots, r.$$

If the measures in (1) are discrete

$$\mu_i = \sum_{k=0}^{N_i} \omega_{i,k}\delta_{x_{i,k}}, \qquad \omega_{i,k} > 0, \qquad x_{i,k} \in \mathbb{R}, \qquad N_i \in \mathbb{N}\cup\{+\infty\}, \qquad i = 1, 2, \ldots, r, \tag{3}$$

where $\delta_{x_{i,k}}$ denotes the Dirac delta function and $x_{i_1,k} \neq x_{i_2,k}$, $k = 0, \ldots, N_i$, whenever $i_1 \neq i_2$, the corresponding polynomial solution $P_{\vec{n}}(x)$ of the linear system of Equation (1) is called discrete multiple orthogonal polynomial (see [28] and the examples therein). In particular, the paper [28] considers discrete multiple orthogonal polynomial on the linear lattice $x(k) = k$, $k = 1, \ldots, N$, $N \in \mathbb{N}\cup\{+\infty\}$.

We will deal only with systems of discrete measures, for which $\Omega_i = \Omega \subset \mathbb{R}^+$ (the set of nonnegative reals) for each $i = 1, 2, \ldots, r$. Recall that the system of positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$, given in (3), forms an AT system if there exist r continuous functions $v_1, \ldots, v_r$ on Ω with $v_i(x_k) = \omega_{i,k}$, $k = 0, \ldots, N_i$, $i = 1, 2, \ldots, r$, such that the $|\vec{n}|$ functions

$$v_1(x), xv_1(x), \ldots, x^{n_1-1}v_1(x), \ldots, v_r(x), xv_r(x), \ldots, x^{n_r-1}v_r(x),$$

form a Chebyshev system on Ω for each multi-index $\vec{n}$ with $|\vec{n}| < N + 1$, i.e., every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x)v_i(x)$, where $Q_{n_i-1} \in \mathbb{P}_{n_i-1} \setminus \{0\}$, has at most $|\vec{n}| - 1$ zeros on Ω. Here $\mathbb{P}_m \subset \mathbb{P}$ denotes the linear subspace (of the space $\mathbb{P}$) of polynomials of degree at most $m \in \mathbb{Z}^+$.

In the sequel we will consider discrete multiple orthogonal polynomials on a non-uniform lattice $x(s) = q^s - 1/q - 1$ (see [35,36]).

Definition 1. *A polynomial $P_{\vec{n}}(x(s))$ on the lattice $x(s) = c_1q^s + c_3$, $q \in \mathbb{R}^+ \setminus \{1\}$, $c_1, c_3 \in \mathbb{R}$, is said to be a multiple q-orthogonal polynomial of a multi-index $\vec{n} \in \mathbb{N}^r$ with respect to positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$ (with finite moments) such that* $\operatorname{supp}(\mu_i) \subset \Omega_i \subset \mathbb{R}$, *$i = 1, 2, \ldots, r$, if the following conditions hold:*

$$\deg P_{\vec{n}}(x(s)) \leq |\vec{n}| = n_1 + n_2 + \cdots + n_r,$$

$$\sum_{s=0}^{N_i} P_{\vec{n}}(x(s))x(s)^k d\mu_i = 0, \qquad k = 0, \ldots, n_i - 1, \qquad N_i \in \mathbb{N}\cup\{+\infty\}. \tag{4}$$

In Section 3 we will deal with particular measures involving the q-Gamma function, which is defined as follows

$$\Gamma_q(s) = \begin{cases} f(s;q) = (1-q)^{1-s} \dfrac{\prod\limits_{k\geq 0}(1-q^{k+1})}{\prod\limits_{k\geq 0}(1-q^{s+k})}, & 0<q<1, \\ q^{\frac{(s-1)(s-2)}{2}} f(s;q^{-1}), & q>1. \end{cases} \tag{5}$$

See also [37,38] for the definition of the q-Gamma function. In addition, we use the q-analogue of the Stirling polynomials denoted by $[s]_q^{(k)}$, which is a polynomial of degree k in the variable $x(s) = (q^s-1)/(q-1)$, i.e.,

$$[s]_q^{(k)} = \prod_{j=0}^{k-1} \frac{q^{s-j}-1}{q-1} = x(s)x(s-1)\cdots x(s-k+1) \quad \text{for} \quad k>0, \quad \text{and} \quad [s]_q^{(0)} = 1. \tag{6}$$

Hereafter, confusion should be avoided between (6) and the notation for the q-analogue of a complex number $z \in \mathbb{C}$,

$$[z] = \frac{q^z - q^{-z}}{q-q^{-1}}. \tag{7}$$

The relation between (6) and (7) is as follows: $[z] = q^{1-z}[2z]_q^{(1)}/(q+1)$. The term q-analogue means that the expression $[z]$ tends to z, as q approaches 1. In general, we say that the function $f_q(s)$ is a q-analogue to the function $f(s)$ if for any sequence $(q_n)_{n\geq 0}$ approaching to 1, the corresponding sequence $\left(f_{q_n}(s)\right)_{n\geq 0}$ tends to $f(s)$ (see Section 4).

The following difference operators are used throughout this paper

$$\Delta \stackrel{\text{def}}{=} \frac{\triangle}{\triangle x(s-1/2)}, \qquad \nabla \stackrel{\text{def}}{=} \frac{\triangledown}{\triangledown x(s+1/2)}, \tag{8}$$

$$\nabla^{n_j} = \underbrace{\nabla\cdots\nabla}_{n_j \text{ times}}, \quad n_j \in \mathbb{N}, \tag{9}$$

where $\triangledown f(x) = f(x) - f(x-1)$ and $\triangle f(x) = \triangledown f(x+1)$ denote the backward and forward difference operators, respectively. When convenient, a less common notation taken from [38] will also be used: $\triangledown x_1(s) \stackrel{\text{def}}{=} \triangledown x(s+1/2) = \triangle x(s-1/2) = q^{s-1/2}$.

Observe that

$$\triangledown^m \left(f(s)g(s)\right) = \sum_{k=0}^{m} \binom{m}{k} \left(\triangledown^k f(s)\right)\left(\triangledown^{m-k} g(s-k)\right), \quad m \in \mathbb{N}, \tag{10}$$

is a discrete analogue of the well-known Leibniz formula (product rule for derivatives). In particular,

$$\triangledown^m f(s) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(s-k). \tag{11}$$

Finally, we will make use of the following notations for multi-indices: The multi-index $\vec{e}_i$ denotes the standard r-dimensional unit vector with the i-th entry equals 1 and 0 otherwise, the multi-index $\vec{e}$ with all its r-entries equal 1. In addition, for any vector $\vec{\alpha} \in \mathbb{C}^r$ and number $p \in \mathbb{C}$,

$$\vec{\alpha}_{i,p} \stackrel{\text{def}}{=} \vec{\alpha} - \alpha_i(1-p)\vec{e}_i = (\alpha_1, \ldots, p\alpha_i, \ldots, \alpha_r). \tag{12}$$

Multiple Meixner Polynomials of the First and Second Kind

In [28], for multiple Meixner polynomials, it was considered two vector measures $\vec{\mu} = (\mu_1, \ldots, \mu_r)$ and $\vec{\nu} = (\nu_1, \ldots, \nu_r)$, where in both cases each component is a Pascal distribution (negative binomial distribution) with different parameters

$$\mu_i = \sum_{x=0}^{\infty} v^{\alpha_i,\beta}(x)\delta_x, \quad v^{\alpha_i,\beta}(x) = \begin{cases} \dfrac{\Gamma(\beta+x)}{\Gamma(\beta)} \dfrac{\alpha_i^x}{\Gamma(x+1)}, & x \in \mathbb{R} \setminus (\mathbb{Z}^- \cup \{-\beta, -\beta-1, \beta-2, \ldots\}), \\ 0, & \text{otherwise}, \end{cases}$$
$$\nu_i = \sum_{x=0}^{\infty} v^{\alpha,\beta_i}(x)\delta_x, \quad i = 1, \ldots, r.$$

Notice that $v^{\alpha,\beta_i}(x)$ is a C^∞-function on $\mathbb{R} \setminus \{-\beta_i, -\beta_i - 1, -\beta_i - 2, \ldots\}$ with simple poles at the points in $\{-\beta_i, -\beta_i - 1, -\beta_i - 2, \ldots\}$. For the above measures $0 < \alpha, \alpha_i < 1$, with all the α_i different, and $\beta, \beta_i > 0$ ($\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$). Under these conditions for both $\vec{\mu}$ and $\vec{\nu}$ the multi-index $\vec{n} \in \mathbb{N}^r$ is normal.

For the monic multiple Meixner polynomial of the first kind [28] corresponding to the multi-index $\vec{n} \in \mathbb{N}^r$ and the vector measure $\vec{\mu}$, define the monic polynomial $M_{\vec{n}}^{\vec{\alpha},\beta}(x)$ of degree $|\vec{n}|$ and different positive parameters $\alpha_1, \ldots, \alpha_r$ (indexed by $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$) and the same $\beta > 0$ which satisfies the orthogonality conditions

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{\vec{\alpha},\beta}(x)(-x)_j v^{\alpha_i,\beta}(x) = 0, \qquad j = 0, \ldots, n_i - 1, \qquad i = 1, \ldots, r,$$

where $(x)_j = (x)(x+1)\cdots(x+j-1)$, $(x)_0 = 1$, $j \geq 1$, denotes the Pochhammer symbol. This polynomial of degree j is used to deal more conveniently with the orthogonality conditions (1)–(3) on the linear lattice $\{x = 0, 1, \ldots\}$.

For the monic *multiple Meixner polynomial of the second kind* [28] corresponding to the multi-index $\vec{n} \in \mathbb{N}^r$ and the vector measure $\vec{\nu}$, define the monic polynomial $M_{\vec{n}}^{\alpha,\vec{\beta},}(x)$ of degree $|\vec{n}|$ and $\vec{\beta} = (\beta_1, \ldots, \beta_r)$, with different components, which satisfies the orthogonality conditions

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{\alpha,\vec{\beta}}(x)(-x)_j v^{\alpha,\beta_i}(x) = 0, \qquad j = 0, \ldots, n_i - 1, \qquad i = 1, \ldots, r.$$

For both families of multiple orthogonal polynomials the following r raising operators were found

$$\mathcal{L}^{\alpha_i,\beta}\left(M_{\vec{n}}^{\vec{\alpha},\beta}(x)\right) = -M_{\vec{n}+\vec{e}_i}^{\vec{\alpha},\beta-1}(x), \tag{13}$$
$$\mathcal{L}^{\alpha,\beta_i}\left(M_{\vec{n}}^{\alpha,\vec{\beta}}(x)\right) = -M_{\vec{n}+\vec{e}_i}^{\alpha,\vec{\beta}-\vec{e}_i}(x), \tag{14}$$

where

$$\mathcal{L}^{\sigma,\tau} \stackrel{\text{def}}{=} \frac{\sigma(\tau-1)}{(1-\sigma)\,v^{\sigma,\tau-1}(x)} \nabla v^{\sigma,\tau}(x), \quad (\sigma,\tau) \in \{(\alpha_i,\beta)\} \cup \{(\alpha,\beta_i)\}, \quad i = 1, \ldots, r.$$

As a consequence of (13) and (14), there holds the Rodrigues-type formulas

$$M_{\vec{n}}^{\vec{\alpha},\beta}(x) = (\beta)_{|\vec{n}|}\left(\prod_{i=1}^{r}\left(\frac{\alpha_i}{\alpha_i-1}\right)^{n_i}\right)\frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)}\mathcal{M}_{\vec{n}}^{\vec{\alpha}}\left(\frac{\Gamma(\beta+|\vec{n}|+x)}{\Gamma(\beta+|\vec{n}|)\Gamma(x+1)}\right), \tag{15}$$
$$M_{\vec{n}}^{\alpha,\vec{\beta}}(x) = \left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left(\prod_{i=1}^{r}(\beta_i)_{n_i}\right)\frac{\Gamma(x+1)}{\alpha^x}\mathcal{N}_{\vec{n}}^{\vec{\beta}}\left(\frac{\alpha^x}{\Gamma(x+1)}\right), \tag{16}$$

where $\mathcal{M}_{\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r} \left(\alpha_i^{-x} \bigtriangledown^{n_i} \alpha_i^{x}\right)$ and $\mathcal{N}_{\vec{n}}^{\vec{\beta}} = \prod_{i=1}^{r} \frac{\Gamma(\beta_i)}{\Gamma(\beta_i+x)} \bigtriangledown^{n_i} \frac{\Gamma(\beta_i+n_i+x)}{\Gamma(\beta_i+n_i)}$. Then, from (10) and (11) the above Formulas (15) and (16) provide an explicit expressions for the above polynomials $M_{\vec{n}}^{\vec{\alpha},\beta}(x)$ and $M_{\vec{n}}^{\alpha,\vec{\beta}}(x)$.

Two important algebraic properties are known for multiple Meixner polynomials [28], namely the $(r+1)$-order linear difference equations [39]

$$\prod_{i=1}^{r} \mathcal{L}^{\alpha_i,\beta+i+1-r} \left(\triangle M_{\vec{n}}^{\vec{\alpha},\beta}(x)\right) = -\sum_{i=1}^{r} n_i \prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{L}^{\alpha_j,\beta+j+1-r} \left(M_{\vec{n}}^{\vec{\alpha},\beta}(x)\right), \tag{17}$$

$$\prod_{i=1}^{r} \mathcal{L}^{\alpha,\beta_i+1} \left(\triangle M_{\vec{n}}^{\alpha,\vec{\beta}}(x)\right) = -\sum_{i=1}^{r} \frac{d_i \prod_{l=1}^{r} (n_l+\beta_l-\beta_i)}{\prod_{k=1,k\neq i}^{r-1} (\beta_i-\beta_k) \prod_{l=i+1}^{r} (\beta_l-\beta_i)} \prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{L}^{\alpha,\beta_j+1} \left(M_{\vec{n}}^{\alpha,\vec{\beta}}(x)\right), \tag{18}$$

where

$$d_i = \sum_{j=1}^{r} \frac{(-1)^{i+j} \prod_{k=1}^{r} (n_j+\beta_j-\beta_k)}{(n_j+\beta_j-\beta_i) \prod_{k=1,k\neq j}^{r-1} (n_k-n_j+\beta_k-\beta_j) \prod_{l=j+1}^{r} (n_j-n_l+\beta_j-\beta_l)},$$

and the recurrence relations [28]

$$\begin{aligned} xM_{\vec{n}}^{\vec{\alpha},\beta}(x) &= M_{\vec{n}+\vec{e}_k}^{\vec{\alpha},\beta}(x) + \left((\beta+|\vec{n}|)\left(\frac{\alpha_k}{1-\alpha_k}\right) + \sum_{i=1}^{r} \frac{n_i}{1-\alpha_i}\right) M_{\vec{n}}^{\vec{\alpha},\beta}(x) \\ &+ \sum_{i=1}^{r} \frac{\alpha_i n_i (\beta+|\vec{n}|-1)}{(\alpha_i-1)^2} M_{\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(x), \end{aligned} \tag{19}$$

$$\begin{aligned} xM_{\vec{n}}^{\alpha,\vec{\beta}}(x) &= M_{\vec{n}+\vec{e}_k}^{\alpha,\vec{\beta}}(x) + \left((n_k+\beta_k)\left(\frac{\alpha}{1-\alpha}\right) + \frac{|\vec{n}|}{1-\alpha}\right) M_{\vec{n}}^{\alpha,\vec{\beta}}(x) \\ &+ \alpha \sum_{i=1}^{r} \frac{n_i(\beta_i+n_i-1)}{(1-\alpha)^2} \prod_{j\neq i}^{r} \frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j} M_{\vec{n}-\vec{e}_i}^{\alpha,\vec{\beta}}(x). \end{aligned} \tag{20}$$

Note that each relation (19) and (20) involve r relations of nearest-neighbor polynomials. Moreover, each family of multiple Meixner polynomials $M_{\vec{n}}^{\vec{\alpha},\beta}(x)$ and $M_{\vec{n}}^{\alpha,\vec{\beta}}(x)$ forms common eigenfunctions of the above two linear difference operators of order $(r+1)$, namely (17)–(20), respectively.

3. Multiple Meixner Polynomials on a Non-Uniform Lattice

Some algebraic properties will be studied in this section: The Rodrigues-type formula, some recurrence relations and the difference equations with respect to the independent discrete variable $x(s)$. For the q-difference equation (of order $r+1$) we will proceed as follows. First, we define an r-dimensional subspace $\mathbb{V}$ of polynomials of degree at most $|\vec{n}|-1$ in the variable $x(s)$ by using some interpolation conditions. Then, we find the lowering operator and express its action on the polynomials as a linear combination of the basis vectors of $\mathbb{V}$. This operator depends on the specific family of multiple orthogonal polynomials, therefore some 'ad hoc' computations are needed. Finally, we combine the lowering and the raising operators to derive the q-difference equation. A similar procedure is given in [31,32,36,39–41]. Finally, the recurrence relations will be derived from some specific difference operators used in Theorems 2 and 4.

3.1. On Some q-Analogues of Multiple Meixner Polynomials of the First Kind

Consider the following vector measure $\vec{\mu}_q$ with positive q-discrete components on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k)\delta(k-s), \qquad \omega_i > 0, \qquad i = 1, 2, \dots, r. \tag{21}$$

Here $\omega_i(s) = v_q^{\alpha_i,\beta}(s) \triangle x(s-1/2)$, and

$$v_q^{\alpha_i,\beta}(s) = \begin{cases} \dfrac{\alpha_i^s \Gamma_q(\beta+s)}{\Gamma_q(s+1)}, & \text{if} \quad s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

where $0 < \alpha_i < 1$, $\beta > 0$, $i = 1, 2, \dots, r$, and with all the α_i different.

The system of measures $\mu_1, \mu_2, \dots, \mu_r$ given in (21) forms an AT system on $\mathbb{R}^+$ (see Lemma 9).

Definition 2. *A polynomial $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$, that verifies the orthogonality conditions*

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(k)} v_q^{\alpha_i,\beta}(s) \triangle x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \dots, r, \tag{23}$$

is said to be the q-Meixner multiple orthogonal polynomial of the first kind. See also (4) *with respect to measure* (21).

Notice that for $r = 1$ we recover the scalar q-Meixner polynomials given in [35] and that the orthogonality conditions (4) have been written more conveniently as (23), in which the monomials $x(s)^k$ were replaced by $[s]_q^{(k)}$. In addition, because we have an AT-system of positive discrete measures the q-Meixner multiple orthogonal polynomial of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [28], theorem 2.1, pp. 26–27). Finally, in Section 4 we will recover the multiple Meixner polynomials of the first kind given in [28] as a limiting case of $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$.

Let us replace $[s]_q^{(k)}$ in (23) by

$$[s]_q^{(k)} = \frac{q^{k-1/2}}{[k+1]_q^{(1)}} \nabla [s+1]_q^{(k+1)}, \tag{24}$$

then, we have

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \nabla [s+1]_q^{(k+1)} v_q^{\alpha_i,\beta}(s) \triangle x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \dots, r.$$

Using summation by parts and condition $v_q^{\alpha_i}(-1) = v_q^{\alpha_i}(\infty) = 0$, we have that for any two polynomials ϕ and ψ in the variable $x(s)$,

$$\sum_{s=0}^{\infty} \Delta\phi(s)\psi(s) v_q^{\alpha_i,\beta}(s) \bigtriangledown x_1(s) = -\sum_{s=0}^{\infty} \phi(s) \nabla \left(\psi(s) v_q^{\alpha_i,\beta}(s) \right) \triangle x(s-1/2). \tag{25}$$

Thus, the following relation

$$\sum_{s=0}^{\infty} \nabla \left(M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \right) [s]_q^{(k+1)} \triangle x(s-1/2) = -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \Delta [s]_q^{(k+1)} \triangle x(s-1/2)$$
$$= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \nabla [s+1]_q^{(k+1)} \triangle x(s-1/2),$$

holds. Equivalently,

$$\sum_{s=0}^{\infty} \nabla \left(M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \right) [s]_q^{(k+1)} \triangle x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \dots, r.$$

Observe that

$$\nabla \left(M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \right) = q^{-|\vec{n}|+1/2} \frac{c_{q,\vec{n}}^{\alpha_i,\beta-1}}{\alpha_i x(\beta-1)} v_q^{\alpha_i/q,\beta-1}(s) \mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s),$$

where

$$c_{q,\vec{n}}^{\alpha_i,\beta} = \left(\alpha_i q^{|\vec{n}|+\beta} - 1 \right). \tag{26}$$

This coefficient will be extensively used throughout the paper and $\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s)$ represents a monic polynomial $x^{|\vec{n}|+1}$ + lower degree terms. Consequently,

$$\sum_{s=0}^{\infty} \mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s) v_q^{\alpha_i/q,\beta-1}(s) [s]_q^{(k+1)} \triangle x(s-1/2) = \sum_{s=0}^{\infty} \nabla \left(M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \right) [s]_q^{(k+1)} \triangle x(s-1/2) = 0. \tag{27}$$

From the next Lemma 1 we will conclude that $\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s) = M_{q,\vec{n}+\vec{e}_i}^{\alpha_1,\dots,\alpha_i/q,\dots,\alpha_r,\beta-1}(s)$.

Lemma 1. *Let the vector subspace $\mathbb{W} \subset \mathbb{P}$ of polynomials $W(s)$ of degree at most $|\vec{n}|+1$ in the variable $x(s)$ be defined by conditions*

$$\sum_{s=0}^{\infty} W(s) [s]_q^{(k)} v_q^{\alpha_j/q,\beta-1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j, \qquad j = 1, \dots, r,$$
$$W(-1) \neq 0.$$

Then, the spanning set of the system $\left\{ M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s) \right\}_{j=1}^{r}$ coincides with $\mathbb{W}$ (see notation (12) for the index $\vec{\alpha}_{i,1/q}$).

Proof. The polynomials $M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(-1) \neq 0$, $j = 1, \dots, r$, because they have exactly $|\vec{n}|+1$ different zeros on $\mathbb{R}^+$. Moreover, from orthogonality relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s) [s]_q^{(k)} v_q^{\alpha_j/q,\beta-1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j, \qquad j = 1, \dots, r,$$

we have that the system of polynomials $M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s)$, $j = 1, \dots, r$, belongs to $\mathbb{W}$.

Assume that there exist numbers λ_j, $j = 1, \dots, r$, such that

$$\sum_{j=1}^{r} \lambda_j M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s) = 0, \qquad \text{where} \qquad \sum_{j=1}^{r} |\lambda_j| > 0. \tag{28}$$

Multiplying the previous equation by $[s]_q^{(n_k-1)} v_q^{\alpha_k,\beta-1}(s) \bigtriangledown x_1(s)$ and then summing from $s=0$ to ∞, one gets

$$\sum_{j=1}^{r} \lambda_j \sum_{s=0}^{\infty} M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s)[s]_q^{(n_k-1)} v_q^{\alpha_k,\beta-1}(s) \bigtriangledown x_1(s) = 0.$$

Thus, from relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s)[s]_q^{(n_k-1)} v_q^{\alpha_k,\beta-1}(s) \bigtriangledown x_1(s) = c\delta_{j,k}, \qquad c \in \mathbb{R}\setminus\{0\}, \tag{29}$$

one concludes that $\lambda_k = 0$ for $k = 1,\ldots,r$. Here $\delta_{j,k}$ denotes the Kronecker delta symbol. Thus, the assumption (28) is false, so the system $\left\{ M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}_{j,1/q},\beta-1}(s) \right\}_{j=1}^{r}$ is linearly independent in $\mathbb{W}$. Moreover, we know that any polynomial from vector subspace $\mathbb{W}$ is determined by its $|\vec{n}|+2$ coefficients while $(|\vec{n}|+2+r)$ conditions are imposed on $\mathbb{W}$. Consequently the dimension of $\mathbb{W}$ is at most r. Therefore, span $\left\{ M_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q},\beta-1}(s) \right\}_{i=1}^{r} = \mathbb{W}$. $\square$

From Equation (27) and Lemma 1 we have

$$\nabla\left(M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) v_q^{\alpha_i,\beta}(s) \right) = q^{-|\vec{n}|+1/2} \frac{c_{q,\vec{n}}^{\alpha_i,\beta-1}}{\alpha_i x(\beta-1)} v_q^{\alpha_i/q,\beta-1}(s) M_{q,\vec{n}+\vec{e}_i}^{\alpha_1,\ldots,\alpha_i/q,\ldots,\alpha_r,\beta-1}(s).$$

Then, for monic q-Meixner multiple orthogonal polynomials of the first kind we have r raising operators

$$\mathcal{D}_q^{\alpha_i,\beta} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = -q^{1/2} M_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q},\beta-1}(s), \qquad i = 1,\ldots,r, \tag{30}$$

where

$$\mathcal{D}_q^{\alpha_i,\beta} \stackrel{\text{def}}{=} -\frac{\alpha_i x(\beta-1)}{q^{-|\vec{n}|} c_{q,\vec{n}}^{\alpha_i,\beta-1}} \left(\frac{1}{v_q^{\alpha_i/q,\beta-1}(s)} \nabla v_q^{\alpha_i,\beta}(s) \right).$$

Furthermore,

$$\mathcal{D}_q^{\alpha_i,\beta} f(s) = \frac{q^{|\vec{n}|+1/2}}{c_{q,\vec{n}}^{\alpha_i,\beta-1}} \Big(\Big(\alpha_i q^{\beta-1}\left(x(1-\beta) - x(s)\right) + x(s) \Big) \mathcal{I} - x(s) \bigtriangledown \Big) f(s),$$

for any function $f(s)$ defined on the discrete variable s. Here $\mathcal{I}$ denotes the identity operator. We call $\mathcal{D}_q^{\alpha_i,\beta}$ a raising operator since the i-th component of the multi-index $\vec{n}$ in (30) is increased by 1.

In the sequel we will only consider monic q-Meixner multiple orthogonal polynomials of the first kind.

Proposition 1. *The following q-analogue of Rodrigues-type formula holds:*

$$M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \mathcal{G}_q^{\vec{n},\vec{\alpha},\beta} \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \left(\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \right), \tag{31}$$

where

$$\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r} \mathcal{M}_{q,n_i}^{\alpha_i}, \quad \mathcal{M}_{q,n_i}^{\alpha_i} = (\alpha_i)^{-s} \nabla^{n_i} (\alpha_i q^{n_i})^s, \tag{32}$$

and

$$\mathcal{G}_q^{\vec{n},\vec{\alpha},\beta} = (-1)^{|\vec{n}|} [-\beta]_q^{(|\vec{n}|)} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \frac{\alpha_i^{n_i} \prod_{j=1}^{n_i} q^{|\vec{n}|_i+\beta+j-1}}{\prod_{j=1}^{n_i} \left(\alpha_i q^{|\vec{n}|+\beta+j-1} - 1\right)} \right) \left(\prod_{i=1}^{r} q^{n_i \sum_{j=i}^{r} n_j} \right), \tag{33}$$

with $|\vec{n}|_i = n_1 + \cdots + n_{i-1}$, $|\vec{n}|_1 = 0$.

Proof. For $i = 1, \ldots, r$, applying k_i-times the raising operators (30) in a recursive way one obtains

$$\prod_{i=1}^{r} \left(\frac{\alpha_i}{q^{k_i}} \right)^{-s} \nabla^{k_i} \alpha_i^s \frac{\Gamma_q(\beta+s)}{\Gamma_q(\beta)\Gamma_q(s+1)} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = [\beta-1]_q^{(|\vec{k}|)} q^{|\vec{k}|/2} \left(\prod_{i=1}^{r} \frac{\prod_{j=1}^{k_i} \left(\alpha_i q^{|\vec{n}|+\beta-j} - 1\right)}{\alpha_i^{k_i}} \right)$$
$$\times \prod_{i=1}^{r} q^{-n_i \sum_{j=i}^{r} k_j} \prod_{i=1}^{r-1} q^{-k_i \sum_{j=i+1}^{r} k_j} M_{q,\vec{n}+\vec{k}}^{\alpha_1/q^{k_1},\ldots,\alpha_r/q^{k_r},\beta-|\vec{k}|}(s) \frac{\Gamma_q(\beta-|\vec{k}|+s)}{\Gamma_q(\beta-|\vec{k}|)\Gamma_q(s+1)}.$$

Taking $n_1 = n_2 = \cdots = n_r = 0$ and replacing β by $\beta + |\vec{k}|$, α_i by $\alpha_i q^{k_i}$, and k_i by n_i, for $i = 1, \ldots, r$, yields the Formula (31). □

3.2. *q-Difference Equation for the q-Analogue of Multiple Meixner Polynomials of the First Kind*

We will find a lowering operator for the q-Meixner multiple orthogonal polynomials of the first kind. We will follow a similar strategy used in [32].

Lemma 2. *Let $\mathbb{V}$ be the linear subspace of polynomials $Q(s)$ on the lattice $x(s)$ of degree at most $|\vec{n}| - 1$ defined by the following conditions*

$$\sum_{s=0}^{\infty} Q(s) [s]_q^{(k)} v_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2 \qquad \textit{and} \qquad j = 1, \ldots, r.$$

Then, the system $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^{r}$, where $\vec{\alpha}_{i,q} = (\alpha_1, \ldots, q\alpha_i, \ldots, \alpha_r)$, is a basis for $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_j}^{\vec{\alpha}_{j,q},\beta+1}(s) [s]_q^{(k)} v_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2, \qquad j = 1, \ldots, r,$$

we have that polynomials $M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)$, $i = 1, \ldots, r$, belong to $\mathbb{V}$.

Now, aimed to get a contradiction, let us assume that there exist constants λ_i, $i = 1, \ldots, r$, such that

$$\sum_{i=1}^{r} \lambda_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s) = 0, \qquad \text{where} \qquad \sum_{i=1}^{r} |\lambda_i| > 0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)} v_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s)$ and then taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(n_k-1)} v_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s) = 0.$$

Thus, from relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(n_k-1)} v_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s) = c\delta_{i,k}, \qquad c \in \mathbb{R} \setminus \{0\}, \tag{34}$$

we deduce that $\lambda_k = 0$ for $k = 1, \ldots, r$. Here $\delta_{i,k}$ represents the Kronecker delta symbol. Therefore, the vectors $\left\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\right\}_{i=1}^{r}$ are linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial of $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}| - r)$ linear conditions are imposed on $\mathbb{V}$. Consequently the dimension of $\mathbb{V}$ is at most r. Hence, the system $\left\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\right\}_{i=1}^{r}$ spans $\mathbb{V}$, which completes the proof. □

Now we will prove that the operator (8) is indeed a lowering operator for the sequence of q-Meixner multiple orthogonal polynomials of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$.

Lemma 3. *The following relation holds:*

$$\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2} \frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}} [n_i]_q^{(1)} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s). \tag{35}$$

Proof. Using summation by parts we have

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(k)} v_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \nabla\left([s]_q^{(k)} v_q^{q\alpha_j,\beta+1}(s)\right) \bigtriangledown x_1(s) \\ &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \varphi_{j,k}(s) v_q^{\alpha_j,\beta}(s) \bigtriangledown x_1(s),\end{aligned} \tag{36}$$

where

$$\varphi_{j,k}(s) = q^{1/2}\left(\frac{q^{\beta}x(s)}{x(\beta)} + 1\right)[s]_q^{(k)} - q^{-1/2}\frac{x(s)}{\alpha_j x(\beta)}[s-1]_q^{(k)},$$

is a polynomial of degree $\le k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (23) we get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(k)} v_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2, \qquad j = 1, \ldots, r.$$

Hence, from Lemma 2, $\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \in \mathbb{V}$. Moreover, $\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ can be expressed as a linear combination of polynomials $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^{r}$, i.e.,

$$\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^{r} \xi_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s), \qquad \sum_{i=1}^{r} |\xi_i| > 0. \tag{37}$$

Multiplying both sides of the Equation (37) by $[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s)$ and using relations (34) one has

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s) = \sum_{i=1}^{r} \xi_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s)$$
$$= \xi_k \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s). \quad (38)$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of Equation (36), then Equation (38) transforms into

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s) = -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\varphi_{k,n_k-1}(s) v_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s)$$
$$= \frac{q^{-1/2}\left(1-\alpha_k q^{n_k+\beta}\right)}{\alpha_k x(\beta)} \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k)} v_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s). \quad (39)$$

For this transformation we have used that $x(s)[s-1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get

$$\varphi_{k,n_k-1}(s) = -\frac{q^{-1/2}\left(1-\alpha_k q^{n_k+\beta}\right)}{\alpha_k x(\beta)}[s]_q^{(n_k)} + \text{lower degree terms.}$$

On the other hand, from (30) one has that

$$\frac{q^{-1/2}\left(1-\alpha_k q^{|\vec{n}|+\beta}\right)}{\alpha_k x(\beta)} v_q^{\alpha_k,\beta}(s) M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = -q^{|\vec{n}|-1/2} \nabla\left(v_q^{q\alpha_k,\beta+1}(s) M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)\right). \quad (40)$$

Considering (40) and using once more summation by parts on the right-hand side of Equation (39) we obtain

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s)$$
$$= -q^{|\vec{n}|-1} \frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}} \sum_{s=0}^{\infty} [s]_q^{(n_k)} \nabla\left(v_q^{q\alpha_k,\beta+1}(s) M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)\right) \bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-1} \frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}} \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)\left(\Delta[s]_q^{(n_k)}\right) v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s).$$

Since $\Delta[s]_q^{(n_k)} = q^{3/2-n_k}[n_k]_q^{(1)}[s]_q^{(n_k-1)}$, we have

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-n_k+1/2} \frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}} [n_k]_q^{(1)} \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)[s]_q^{(n_k-1)} v_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s).$$

Comparing this equation with (38), we obtain the coefficients in the expansion (37), i.e.,

$$\xi_k = q^{|\vec{n}|-n_k+1/2} \frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}} [n_k]_q^{(1)}.$$

Therefore, relation (35) holds. □

Theorem 1. *The q-Meixner multiple orthogonal polynomial of the first kind $M^{\vec{\alpha},\beta}_{q,\vec{n}}(s)$ satisfies the following $(r+1)$-order q-difference equation*

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1}\Delta M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) = -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1}M^{\vec{\alpha},\beta}_{q,\vec{n}}(s). \tag{41}$$

Proof. Since the operators (30) commute, we write

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1} = \left(\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1}\right)\mathcal{D}_q^{q\alpha_i,\beta+1}. \tag{42}$$

Using (30) when acting on Equation (35) with the product of operators (42), we obtain (41), i.e.,

$$\begin{aligned}\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1}\Delta M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) &= \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1}\left(\mathcal{D}_q^{q\alpha_i,\beta+1}M^{\vec{\alpha}_{i,q},\beta+1}_{q,\vec{n}-\vec{e}_i}(s)\right)\\ &= -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1}M^{\vec{\alpha},\beta}_{q,\vec{n}}(s).\end{aligned}$$

This completes the proof of the theorem. □

3.3. Recurrence Relation for q-Meixner Multiple Orthogonal Polynomials of the First Kind

In this section we will study the nearest neighbor recurrence relation for any multi-index $\vec{n}$. The approach presented here differs from those used in [28,42]. We begin by defining the following linear difference operator

$$\mathcal{F}_{q,n_i} := g_{q,i}^{-1}(s)\nabla^{n_i}g_{q,k}(s), \tag{43}$$

where n_i is the i-th entry of the vector index $\vec{n}$ and $g_{q,k}$ is defined in the variable s and depends on the i-th component of the vector orthogonality measure $\vec{\mu}$. In the case that $g_{q,k}$ depends also on the i-th component of $\vec{n}$, then the index $k = n_i$; otherwise $k = i$.

Lemma 4. *Let n_i be a positive integer and let $f(s)$ be a function defined on the discrete variable s. The following relation is valid*

$$\mathcal{F}_{q,n_i}x(s)f_q(s) = q^{-n_i+1/2}x(n_i)g_{q,i}^{-1}(s)\nabla^{n_i-1}g_{q,k}(s)f_q(s) + q^{-n_i}\big(x(s)-x(n_i)\big)\mathcal{F}_{q,n_i}f_q(s). \tag{44}$$

Proof. Let us act n_i-times with backward difference operators (9) on the product of functions $x(s)f(s)$. Assume that $n_i \geq N > 1$,

$$\begin{aligned}\nabla^{n_i}x(s)f(s) &= \nabla^{n_i-1}\big(\nabla x(s)f(s)\big) = \nabla^{n_i-1}\big(q^{-1/2}f(s)+x(s-1)\nabla f(s)\big)\\ &= q^{-1/2}\nabla^{n_i-1}f(s)+\nabla^{n_i-1}\big(x(s-1)\nabla f(s)\big)\\ &= q^{-1/2}\nabla^{n_i-1}f(s)+\nabla^{n_i-2}\big(\nabla x(s-1)\nabla f(s)\big).\end{aligned} \tag{45}$$

Repeating this process, but on the second term of the right-hand side of Equation (45)

$$\begin{aligned}\nabla^{n_i}x(s)f(s) &= \big(q^{1/2-n_i}+\cdots+q^{-5/2}+q^{-3/2}+q^{-1/2}\big)\nabla^{n_i-1}f(s)+x(s-n_i)\nabla^{n_i}f(s)\\ &= q^{1/2-n_i}x(n_i)\nabla^{n_i-1}f(s)+x(s-n_i)\nabla^{n_i}f(s).\end{aligned}$$

Thus,

$$\nabla^{n_i} x(s) f(s) = q^{-n_i+1/2} x(n_i) \nabla^{n_i-1} f(s) + q^{-n_i}\big(x(s) - x(n_i)\big)\nabla^{n_i} f(s), \qquad n_i \geq 1. \tag{46}$$

Now, to involve the difference operator $\mathcal{F}_{q,n_i}$ in the above equation, we multiply the Equation (46) from the left by $g_{q,i}(s)^{-1}$ and replace $f(s)$ by $g_{q,k}(s)f(s)$. Therefore, the Equation (46) transforms into (44). □

Theorem 2. *The q-Meixner multiple orthogonal polynomials of the first kind satisfy the following* $(r+2)$*-term recurrence relation*

$$x(s)M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \;=\; M_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha},\beta}(s) + b_{\vec{n},k} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) + \sum_{i=1}^{r} \frac{x(n_i)\alpha_i q^{|\vec{n}|+n_i-1} x(\beta+|\vec{n}|-1)}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta-1} c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta-2}} B_{\vec{n},i} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(s), \tag{47}$$

where

$$\begin{aligned} b_{\vec{n},k} = {} & -\alpha_k q^{|\vec{n}|+n_k+1} \frac{x(\beta+|\vec{n}|)}{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}} + (q-1)\prod_{i=1}^{r} \frac{x(n_i)}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}} \left(q^{|\vec{n}|+\beta}\prod_{i=1}^{r}\alpha_i q^{n_i} - 1\right) \\ & + \sum_{i=1}^{r} \frac{x(n_i)}{q^{-|\vec{n}|}} \left(\frac{\alpha_i q^{n_i}-1}{c_{q,\vec{n}}^{\alpha_i,\beta}} \prod_{i=1}^{r} \frac{c_{q,\vec{n}}^{\alpha_i,\beta}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}} - \frac{\alpha_i q^{|\vec{n}|+\beta+n_i-1}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta-1}} \frac{\alpha_i q^{n_i}-1}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}} \prod_{j\neq i}^{r} \frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}} \right) \end{aligned}$$

and

$$B_{\vec{n},i} = \frac{\alpha_i q^{n_i}-1}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}} \prod_{j\neq i}^{r} \frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}} \prod_{i=1}^{r} \frac{c_{q,\vec{n}}^{\alpha_i,\beta-1}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}.$$

Proof. Let

$$f_{\mathbf{n}}(s;\beta) = \frac{\Gamma_q(\beta+\mathbf{n}+s)}{\Gamma_q(\beta+\mathbf{n})\Gamma_q(s+1)}, \quad \text{where} \quad \mathbf{n} = |\vec{n}|.$$

We will use Lemma 4 involving this function $f_{\mathbf{n}}(s;\beta)$ as well as difference operator (32). Consider equation

$$\begin{aligned} (\alpha_k)^{-s}\nabla^{n_k+1}(\alpha_k q^{n_k+1})^s f_{\mathbf{n}+1}(s;\beta) &= (\alpha_k)^{-s}\nabla^{n_k}\left(q^{-s+1/2}\,\nabla\left((\alpha_k q^{n_k+1})^s f_{\mathbf{n}+1}(s;\beta)\right)\right) \\ &= q^{1/2}(\alpha_k)^{-s}\nabla^{n_k}\left((\alpha_k q^{n_k})^s\left(1+\frac{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}x(s)\right)f_{\mathbf{n}}(s;\beta)\right), \end{aligned}$$

which can be rewritten in terms of difference operators (32) as follows

$$q^{-1/2}\mathcal{M}_{q,n_k+1}^{\alpha_k} f_{\mathbf{n}+1}(s;\beta) = \mathcal{M}_{q,n_k}^{\alpha_k} f_{\mathbf{n}}(s;\beta) + \frac{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)} \mathcal{M}_{q,n_k}^{\alpha_k} x(s) f_{\mathbf{n}}(s;\beta). \tag{48}$$

Since operators (32) commute, the multiplication of Equation (48) from the left-hand side by the product $\prod_{\substack{i=1 \\ i\neq k}}^{r} \mathcal{M}_{q,n_i}^{\alpha_i}$ yields the following relation

$$\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} x(s) f_{\mathbf{n}}(s;\beta) = \frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}} \left(q^{-1/2}\mathcal{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}} f_{\mathbf{n}+1}(s;\beta) - \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} f_{\mathbf{n}}(s;\beta)\right). \tag{49}$$

Let us recursively use Lemma 4 involving the product of r difference operators acting on the function $f_{\mathbf{n}}(s;\beta)$, which in this case is the operator $\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}$ (see expression (32)). Thus,

$$\left(q^{|\vec{n}|}\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}x(s) - q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\frac{x(n_i)c_{q,\vec{n}+n_j\vec{e}_j}^{\alpha_j,\beta}}{(\alpha_i q^{n_i}-1)^{-1}\prod\limits_{\nu=1}^{r}c_{q,\vec{n}}^{\alpha_\nu,\beta}}\prod_{l=1}^{r}\mathcal{M}_{q,n_l-\delta_{l,i}}^{\alpha_l}\right) f_{\mathbf{n}}(s;\beta)$$
$$= \left(x(s)\prod_{i=1}^{r}\frac{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}{c_{q,\vec{n}}^{\alpha_i,\beta}} - \sum_{i=1}^{r}\frac{q^{|\vec{n}|}x(n_i)}{c_{q,\vec{n}}^{\alpha_i,\beta}(\alpha_i q^{n_j}-1)^{-1}} + \frac{1-q^{2|\vec{n}|+\beta}\prod\limits_{i=1}^{r}\alpha_i}{(q-1)^{-1}}\prod_{i=1}^{r}\frac{x(n_i)}{c_{q,\vec{n}}^{\alpha_i,\beta}}\right)\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}f_{\mathbf{n}}(s;\beta). \tag{50}$$

Using the expressions (49) and (50) one gets

$$x(s)\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}f_{\mathbf{n}}(s;\beta) = q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{c_{q,\vec{n}}^{\alpha_i,\beta}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}\frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}}\mathcal{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}}f_{\mathbf{n}+1}(s;\beta)$$
$$+\prod_{i=1}^{r}\frac{c_{q,\vec{n}}^{\alpha_i,\beta}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}\left(\sum_{i=1}^{r}\frac{q^{|\vec{n}|}x(n_i)}{(\alpha_i q^{n_j}-1)^{-1}c_{q,\vec{n}}^{\alpha_i,\beta}} - \frac{1-q^{2|\vec{n}|+\beta}\prod\limits_{i=1}^{r}\alpha_i}{(q-1)^{-1}}\prod_{i=1}^{r}\frac{x(n_i)}{c_{q,\vec{n}}^{\alpha_i,\beta}} - \frac{q^{|\vec{n}|}\alpha_k x(\beta+|\vec{n}|)}{q^{-n_k-1}c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}}\right)\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}f_{\mathbf{n}}(s;\beta)$$
$$-q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\frac{x(n_i)\left(\alpha_i q^{n_i}-1\right)}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}\prod_{l=1}^{r}\mathcal{M}_{q,n_l-\delta_{l,i}}^{\alpha_l}f_{\mathbf{n}}(s;\beta).$$

Observe that when $l=i$ in the above expression we have

$$\mathcal{M}_{q,n_i-1}^{\alpha_i}f_{\mathbf{n}}(s;\beta) = q^{-1/2}\frac{\alpha_i q^{\beta+|\vec{n}|+n_i-1}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta-1}}\mathcal{M}_{q,n_i}^{\alpha_i}f_{\mathbf{n}}(s;\beta) - \frac{1}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta-1}}\mathcal{M}_{q,n_i-1}^{\alpha_i}f_{\mathbf{n}-1}(s;\beta).$$

Therefore,

$$x(s)\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}f_{\mathbf{n}}(s;\beta)$$
$$= q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{c_{q,\vec{n}}^{\alpha_i,\beta}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}\frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha_k,\beta+1}}\mathcal{M}_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}}f_{\mathbf{n}+1}(s;\beta) + b_{\vec{n},k}\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}}f_{\mathbf{n}}(s;\beta)$$
$$-q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\frac{x(n_i)\left(\alpha_i q^{n_i}-1\right)}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta}}\frac{1}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha_i,\beta-1}}\prod_{l=1}^{r}\mathcal{M}_{q,n_l-\delta_{l,i}}^{\alpha_l}f_{\mathbf{n}-1}(s;\beta).$$

Finally, multiplying from the left both sides of the previous expression by $\mathcal{G}_q^{\vec{n},\vec{\alpha},\beta}\frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)}$ and using Rodrigues-type Formula (31) we obtain (47). This completes the proof of the theorem. □

3.4. On Some q-Analogue of Multiple Meixner Polynomials of the Second Kind

Consider the following vector measure $\vec{v}_q$ with positive q-discrete components

$$v_i = \sum_{s=0}^{\infty} v_q^{\alpha,\beta_i}(k)\,\triangle\, x\,(k-1/2)\,\delta\,(k-s)\,, \quad i=1,2,\ldots,r, \tag{51}$$

where $v_q^{\alpha,\beta_i}(s)$ is defined in (22), but here the domain for its non-identically zero part is $s \in \Omega = \mathbb{R} \setminus \{\mathbb{Z}^- \cup \{-\beta_i, -\beta_i - 1, -\beta_i - 2, \dots\}\}$, $\beta_i > 0$, $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$, and $0 < \alpha < 1$. Indeed,

$$v_q^{\alpha,\beta_i}(s) = \begin{cases} \dfrac{\alpha^x \Gamma_q(\beta_i + s)}{\Gamma_q(s+1)}, & \text{if } s \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. *A polynomial $M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions*

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(k)} v_q^{\alpha,\beta_i}(s) \triangle x(s-1/2) = 0, \qquad 0 \leq k \leq n_i - 1, \qquad i = 1, \dots, r, \tag{52}$$

is said to be the q-Meixner multiple orthogonal polynomial of the second kind.

The general orthogonality relations (4) have been conveniently written involving the q-analogue of the Stirling polynomials (6) as in relations (52). In Section 5 we will address the AT-property of the system of positive discrete measures (51). This fact guarantees that the q-Meixner multiple orthogonal polynomial of the second kind $M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [28], theorem 2.1, pp. 26–27). In Section 4, the multiple Meixner polynomials of the second kind (16) given in [28] will be recovered as q approaches 1.

To find a raising operator we substitute $[s]_q^{(k)}$ in (52) for the finite-difference expression (24) and then we use summation by parts along with conditions $v_q^{\alpha,\beta_i}(-1) = v_q^{\alpha,\beta_i}(\infty) = 0$. Thus,

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) \nabla [s+1]_q^{(k+1)} v_q^{\alpha,\beta_i}(s) \triangle x(s-1/2) = 0, \qquad 0 \leq k \leq n_i - 1, \qquad i = 1, \dots, r.$$

Using (25), one gets

$$\begin{aligned} \sum_{s=0}^{\infty} \nabla \left(M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) v_q^{\alpha,\beta_i}(s) \right) [s]_q^{(k+1)} \triangle x(s-1/2) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) v_q^{\alpha,\beta_i}(s) \Delta [s]_q^{(k+1)} \triangle x(s-1/2) \\ &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) v_q^{\alpha,\beta_i}(s) \nabla [s+1]_q^{(k+1)} \triangle x(s-1/2). \end{aligned}$$

Hence

$$\sum_{s=0}^{\infty} \nabla \left(M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) v_q^{\alpha,\beta_i}(s) \right) [s]_q^{(k+1)} \triangle x(s-1/2) = 0, \qquad 0 \leq k \leq n_i - 1, \qquad i = 1, \dots, r,$$

where

$$\nabla \left(M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) v_q^{\alpha,\beta_i}(s) \right) = \frac{q^{-|\vec{n}|+1/2} c_{q,\vec{n}}^{\alpha,\beta_i - 1}}{\alpha x(\beta_i - 1)} v_q^{\alpha/q,\beta_i - 1}(s) \mathcal{P}_{q,\vec{n}+\vec{e}_i}(s).$$

$\mathcal{P}_{q,\vec{n}+\vec{e}_i}(s)$ denotes a monic polynomial of degree $|\vec{n}| + 1$. Therefore, from (52) the relation

$$\sum_{s=0}^{\infty} \mathcal{P}_{q,\vec{n}+\vec{e}_i}(s) v_q^{\alpha/q,\beta_i - 1}(s) [s]_q^{(k+1)} \triangle x(s-1/2) = \sum_{s=0}^{\infty} \nabla \left(M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) v_q^{\alpha,\beta_i}(s) \right) [s]_q^{(k+1)} \triangle x(s-1/2) = 0,$$

implies that $\mathcal{P}_{q,\vec{n}+\vec{e}_i}(s) = M^{\alpha/q,\vec{\beta}-\vec{e}_i}_{q,\vec{n}+\vec{e}_i}(s)$. Therefore

$$\nabla\left(M^{\alpha,\vec{\beta}}_{q,\vec{n}}(s)v^{\alpha,\beta_i}_q(s)\right) = \frac{q^{-|\vec{n}|+1/2}c^{\alpha,\beta_i-1}_{q,\vec{n}}}{\alpha x(\beta_i-1)}v^{\alpha/q,\beta_i-1}_q(s)M^{\alpha/q,\vec{\beta}-\vec{e}_i}_{q,\vec{n}+\vec{e}_i}(s),$$

which leads to the following r raising operators for the monic q-Meixner multiple orthogonal polynomials of the second kind

$$\mathcal{D}^{\alpha,\beta_i}_q M^{\alpha,\vec{\beta}}_{q,\vec{n}}(s) = -q^{1/2}M^{\alpha/q,\vec{\beta}-\vec{e}_i}_{q,\vec{n}+\vec{e}_i}(s)\,. \tag{53}$$

The operator $\mathcal{D}^{\alpha,\beta_i}_q$ is given in (30) with the replacements: α_i by α and β by β_i, respectively. Indeed,

$$\mathcal{D}^{\alpha,\beta_i}_q f(s) = \frac{q^{|\vec{n}|+1/2}}{c^{\alpha,\beta_i-1}_{q,\vec{n}}}\left(\left(\alpha q^{\beta_i-1}\left(x(1-\beta_i)-x(s)\right)+x(s)\right)\mathcal{I}-x(s)\,\nabla\right)f(s), \tag{54}$$

holds for any function $f(s)$ defined on the discrete variable s.

Proposition 2. *The following finite-difference analogue of the Rodrigues-type formula holds:*

$$M^{\alpha,\vec{\beta}}_{q,\vec{n}}(s) = \mathcal{G}^{\vec{n},\vec{\beta},\alpha}_q\frac{\Gamma_q(s+1)}{\alpha^s}\mathcal{N}^{\vec{\beta}}_{q,\vec{n}}\left(\frac{\left(\alpha q^{|\vec{n}|}\right)^s}{\Gamma_q(s+1)}\right), \tag{55}$$

where

$$\mathcal{N}^{\vec{\beta}}_{q,\vec{n}} = \prod_{i=1}^{r}\mathcal{N}^{\beta_i}_{q,n_i}, \quad \mathcal{N}^{\beta_i}_{q,n_i} = \frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i+s)}\nabla^{n_i}\frac{\Gamma_q(\beta_i+n_i+s)}{\Gamma_q(\beta_i+n_i)}, \tag{56}$$

and

$$\mathcal{G}^{\vec{n},\vec{\beta},\alpha}_q = (-1)^{|\vec{n}|}\left(\alpha q^{|\vec{n}|}\right)^{|\vec{n}|}q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{\prod_{j=1}^{n_i}q^{\beta_i+j-1}}{\prod_{j=1}^{n_i}c^{\alpha,\beta_i+j-1}_{q,\vec{n}}}\right)\left(\prod_{i=1}^{r}[-\beta_i]^{(n_i)}_q\right). \tag{57}$$

Proof. We follow the same pattern given in Proposition 1 adapted to the operator $\mathcal{N}^{\vec{\beta}}_{q,\vec{n}}$. For $i=1,\ldots,r$, by applying k_i-times the raising operators (53) in a recursive way, the following expression holds

$$\prod_{i=1}^{r}\frac{\Gamma(\beta_i-k_i)}{\Gamma(\beta_i-k_i+s)}\nabla^{k_i}\frac{\Gamma_q(\beta_i+s)}{\Gamma_q(\beta_i)}\frac{(\alpha)^s}{\Gamma_q(s+1)}M^{\alpha,\vec{\beta}}_{q,\vec{n}}(s) = \prod_{i=1}^{r}[\beta_i-1]^{(k_i)}_q q^{|\vec{k}|/2}q^{-(|\vec{k}|)|\vec{n}|}$$
$$\times\left(\prod_{i=1}^{r}\alpha^{-k_i}\prod_{j=1}^{k_i}c^{\alpha,\beta_i-j}_{q,\vec{n}}\right)M^{\alpha/q^{|\vec{k}|},\beta_1-k_1,\ldots,\beta_r-k_r}_{q,\vec{n}+\vec{k}}(s)\frac{(\alpha/q^{|\vec{k}|})^s}{\Gamma_q(s+1)}.$$

Let $n_1=n_2=\cdots=n_r=0$ and replace β_i by β_i+k_i and α by $\alpha q^{|\vec{k}|}$. Finally, if we rename the new index component k_i with the old index component n_i, for $i=1,\ldots,r$, the expression (55) holds. □

3.5. q-Difference Equation for the q-Analogue of Multiple Meixner Polynomials of the Second Kind

In this section we will find the lowering operator for the q-Meixner multiple orthogonal polynomials of the second kind.

Lemma 5. *The q-Meixner multiple orthogonal polynomials of the second kind satisfy the following property*

$$\sum_{s=0}^{\infty}M^{q\alpha,\vec{\beta}+\vec{e}_i}_{q,\vec{n}-\vec{e}_i}(s)\,[s]^{(n_k-1)}_q\,v^{q\alpha,\beta_k+1}_q(s)\,\nabla x_1(s) = m_{k,i}\sum_{s=0}^{\infty}M^{q^r\alpha,\vec{\beta}+\vec{e}}_{q,\vec{n}-\vec{e}}(s)\,[s]^{(n_k-1)}_q\,v^{q^r\alpha,\beta_k+1}_q(s)\,\nabla x_1(s),$$

where

$$m_{k,i}=\frac{1-\alpha q^{|\vec{n}|+\beta_i}}{\alpha q^{|\vec{n}|+\beta_i}}\frac{1}{x(n_k+\beta_k-\beta_i)}\prod_{j=1}^{r}\frac{\alpha q^{|\vec{n}|+\beta_j}}{1-\alpha q^{|\vec{n}|+\beta_j}}x(n_k+\beta_k-\beta_j),\quad k,i=1,2,\ldots,r, \tag{58}$$

and $\vec{e}=\sum_{i=1}^{r}\vec{e}_i$.

Proof. By shifting conveniently the parameters involved in (53) and (54), respectively, one has

$$\begin{aligned}M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)&=-q^{-1/2}\mathcal{D}_q^{q\alpha,\beta_i+1}\left(M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\right)\\&=-\frac{q^{|\vec{n}|-1}}{1-\alpha q^{|\vec{n}|+\beta_i}}\left\{\left(\alpha q^{\beta_i+1}\left(x(s)-x\left(-\beta_i\right)\right)-x(s)\right)M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)+x(s)\bigtriangledown M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{s=0}^{\infty}M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)\,[s]_q^{(n_k-1)}\,v_q^{\alpha,\beta_k+1}(s)\bigtriangledown x_1(s)&=-\frac{q^{|\vec{n}|-1}}{1-\alpha q^{|\vec{n}|+\beta_i}}\sum_{s=0}^{\infty}[s]_q^{(n_k-1)}\,v_q^{\alpha,\beta_k+1}(s)\bigtriangledown x_1(s)\\&\times\left\{\left(\alpha q^{\beta_i+1}\left(x(s)-x\left(-\beta_i\right)\right)-x(s)\right)M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)+x(s)\bigtriangledown M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\right\}.\end{aligned}$$

Using summation by parts in the above expression we have

$$\begin{aligned}&\sum_{s=0}^{\infty}M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)\,[s]_q^{(n_k-1)}\,v_q^{\alpha,\beta_k+1}(s)\bigtriangledown x_1(s)\\&=\frac{\alpha q^{|\vec{n}|+\beta_i}}{1-\alpha q^{|\vec{n}|+\beta_i}}x(n_k+\beta_k-\beta_i)\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\,[s]_q^{(n_k-1)}\,v_q^{q\alpha,\beta_k+1}(s)\bigtriangledown x_1(s)\\&\quad+\frac{\alpha q^{|\vec{n}|-n_k+2}}{1-\alpha q^{|\vec{n}|+\beta_i}}x(\beta_k-1)x(n_k+\beta_k-1)\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\,[s]_q^{(n_k-2)}\,v_q^{q\alpha,\beta_k+1}(s)\bigtriangledown x_1(s).\end{aligned}$$

From the orthogonality conditions the following relation holds:

$$\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\,[s]_q^{(n_k-2)}\,v_q^{q\alpha,\beta_k+1}(s)\bigtriangledown x_1(s)=0.$$

Therefore,

$$\begin{aligned}\sum_{s=0}^{\infty}M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)\,[s]_q^{(n_k-1)}\,v_q^{\alpha,\beta_k+1}(s)\bigtriangledown x_1(s)&=\frac{\alpha q^{|\vec{n}|+\beta_i}}{1-\alpha q^{|\vec{n}|+\beta_i}}x(n_k+\beta_k-\beta_i)\\&\times\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\,[s]_q^{(n_k-1)}\,v_q^{q\alpha,\beta_k+1}(s)\bigtriangledown x_1(s).\end{aligned} \tag{59}$$

Then, by iterating recursively (59), the relation (58) holds. This completes the proof of the lemma. □

Lemma 6. *Let* $M=(m_{k,i})_{k,i=1}^{r}$ *be the matrix with entries given in (58). Then, M is non-singular.*

Proof. Let us rewrite the entries in M as $m_{k,i} = c_k d_i / [n_k + \beta_k - \beta_i]_q$, where

$$c_k = q^{(1-n_k-\beta_k)/2} \prod_{j=1}^{r} \frac{\alpha q^{|\vec{n}|+\beta_j}}{1-\alpha q^{|\vec{n}|+\beta_j}} x(n_k+\beta_k-\beta_j),$$

$$d_i = q^{\beta_i/2}\left(\frac{1-\alpha q^{|\vec{n}|+\beta_i}}{\alpha q^{|\vec{n}|+\beta_i}}\right),$$

$$[n_k+\beta_k-\beta_i]_q = q^{(1-n_k-\beta_k+\beta_i)/2} x(n_k+\beta_k-\beta_i).$$

The matrix M is the product of three matrices; that is $M = C \cdot A \cdot D$, where $A = \left(1/[n_k+\beta_k-\beta_i]_q\right)_{k,i=1}^{r}$ and matrices C, D are the diagonal matrices $C = \operatorname{diag}(c_1, c_2, \ldots, c_r)$, $D = \operatorname{diag}(d_1, d_2, \ldots, d_r)$, respectively. □

In ([31], lemma 3.2, p. 7) it was proved that A is nonsingular. Therefore, M is also a nonsingular matrix. Indeed,

$$\det M = q^{(r-|\vec{n}|)/2}\left(\prod_{j=1}^{r} c_j d_j\right)\det A,$$

$$= \frac{\prod_{k=1}^{r-1}\prod_{l=k+1}^{r} x(\beta_l-\beta_k) q^{n_l} x(n_k-n_l+\beta_k-\beta_l)}{\prod_{k=1}^{r}\prod_{l=1}^{r} x(n_l+\beta_l-\beta_k)}. \tag{60}$$

Lemma 7. *Let $\mathbb{V}$ be the subspace of polynomials ϑ on the discrete variable $x(s)$, such that* $\deg \vartheta \le |\vec{n}| - 1$ *and*

$$\sum_{s=0}^{\infty} \vartheta(s)\, [s]_q^{(k)}\, v_q^{q\alpha,\beta_j+1}(s) \bigtriangledown x_1(s) = 0, \quad 0 \le k \le n_j - 2, \quad j = 1, 2, \ldots, r.$$

Then, the system $\left\{M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\right\}_{i=1}^{r}$ *is linearly independent in* $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_j}^{q\alpha,\vec{\beta}+\vec{e}_j}(s)\, [s]_q^{(k)}\, v_q^{q\alpha,\beta_j+1}(s) \bigtriangledown x_1(s) = 0, \quad 0 \le k \le n_j - 2, \quad j = 1, 2, \ldots, r,$$

we have that polynomials $M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s) \in \mathbb{V}$, for $i = 1, 2, \ldots, r$.

Suppose that there exist constants λ_i, $i = 1, \ldots, r$, such that

$$\sum_{i=1}^{r} \lambda_i M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s) = 0, \qquad \text{where} \qquad \sum_{i=1}^{r} |\lambda_i| > 0. \tag{61}$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s)$ and then taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s) [s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s) = 0.$$

Using Lemma 5 and relation $\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)[s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s) \neq 0$, we obtain the following homogeneous linear system of equations

$$\sum_{i=1}^{r} m_{k,i}\lambda_i = 0, \qquad k = 1, \dots, r,$$

or equivalently, in matrix form $M\lambda = 0$, where $\lambda = (\lambda_1, \dots, \lambda_r)^T$. From Lemma 6, we have that M is nonsingular, which implies $\lambda_i = 0$ for $i = 1, \dots, r$; that is, the previous assumption (61) is false. Therefore, $\left\{ M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s) \right\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial from subspace $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}| - r)$ conditions are imposed on $\mathbb{V}$, consequently the dimension of $\mathbb{V}$ is at most r. Therefore, the system $\{M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\}_{i=1}^{r}$ spans $\mathbb{V}$. This completes the proof of the lemma. □

Now we will prove that operator (8) is indeed a lowering operator for the sequence of q-Meixner multiple orthogonal polynomials of the second kind $M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)$.

Lemma 8. *The following relation holds:*

$$\Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) = \sum_{i=1}^{r} \xi_i M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s), \tag{62}$$

where

$$\xi_i = \frac{\prod_{l=1}^{r} x(n_l + \beta_l - \beta_i)}{\prod_{k=1,k\neq i}^{r} x(\beta_i - \beta_k) \prod_{l=i+1}^{r} x(\beta_l - \beta_i)} \sum_{j=1}^{r} \frac{(1 - \alpha q^{n_j+\beta_j}) q^{|\vec{n}|-n_j+1/2}}{(1 - \alpha q^{|\vec{n}|+\beta_j}) x(n_j + \beta_j - \beta_i)}$$
$$\times \frac{(-1)^{i+j} \prod_{k=1}^{r} x(n_j + \beta_j - \beta_k)}{\prod_{k=1,k\neq j}^{r-1} q^{n_j} x(n_k - n_j + \beta_k - \beta_j) \prod_{l=j+1}^{r} q^{n_l} x(n_j - n_l + \beta_j - \beta_l)}. \tag{63}$$

Proof. Using summation by parts we have

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(k)} v_q^{q\alpha,\beta_j+1}(s) \bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) \nabla\left([s]_q^{(k)} v_q^{q\alpha,\beta_j+1}(s)\right) \bigtriangledown x_1(s) \\ &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) \varphi_{j,k}(s) v_q^{\alpha,\beta_j}(s) \bigtriangledown x_1(s),\end{aligned} \tag{64}$$

where

$$\varphi_{j,k}(s) = q^{1/2}\left(\frac{q^{\beta_j} x(s)}{x(\beta_j)} + 1\right)[s]_q^{(k)} - q^{-1/2}\frac{x(s)}{\alpha x(\beta_j)}[s-1]_q^{(k)},$$

is a polynomial of degree $\leq k+1$ in the variable $x(s)$. Then, from the orthogonality conditions (52) we get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(k)} v_q^{q\alpha,\beta_j+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \leq k \leq n_j - 2, \qquad j = 1, \dots, r.$$

From Lemma 7, $\Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) \in \mathbb{V}$. Moreover, $\Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)$ can be expressed as a linear combination of polynomials $\{M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s)\}_{i=1}^{r}$, i.e.,

$$\Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) = \sum_{i=1}^{r} \xi_i M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s), \qquad \sum_{i=1}^{r} |\xi_i| > 0. \tag{65}$$

Thus, for finding explicity $\xi_1, \dots, \xi_r$ one takes into account Lemma 5 and (65) to get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s) = \left(\sum_{i=1}^{r} \xi_i m_{k,i}\right) \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}}^{q^r\alpha,\vec{\beta}+\vec{e}}(s)[s]_q^{(n_k-1)} \times v_q^{q^r\alpha,\beta_k+1}(s) \bigtriangledown x_1(s). \tag{66}$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of Equation (64), then left-hand side of Equation (66) transforms into relation

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)\varphi_{k,n_k-1}(s) v_q^{\alpha,\beta_k}(s) \bigtriangledown x_1(s) \\ &= \frac{q^{1/2}\left(1-\alpha_k q^{n_k+\beta_k}\right)}{\alpha q^{n_k+\beta_k}} \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(n_k)} v_q^{\alpha,\beta_k+1}(s) \bigtriangledown x_1(s).\end{aligned}$$

We have used that $x(s)[s-1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get

$$\varphi_{k,n_k-1}(s) = -\frac{q^{-1/2}\left(1-\alpha q^{n_k+\beta_k}\right)}{\alpha x(\beta_k)}[s]_q^{(n_k)} + \text{lower degree terms}.$$

Using Lemma 5, we have that

$$\begin{aligned}&\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)[s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s) \\ &\quad= \frac{(1-\alpha q^{n_k+\beta_k})q^{|\vec{n}|-n_k+1/2}}{1-\alpha q^{|\vec{n}|+\beta_k}} x(n_k) \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{q\alpha,\vec{\beta}+\vec{e}_k}(s)[s]_q^{(n_k-1)} v_q^{q\alpha,\beta_k+1}(s) \bigtriangledown x_1(s) \\ &\quad= \tilde{b}_k \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}}^{q^r\alpha,\vec{\beta}+\vec{e}}(s)[s]_q^{(n_k-1)} v_q^{q^r\alpha,\beta_k+1}(s) \bigtriangledown x_1(s),\end{aligned} \tag{67}$$

where

$$\tilde{b}_k = \frac{q^{1/2}(1-\alpha q^{n_k+\beta_k})}{\alpha q^{n_k+\beta_k}} \prod_{i=1}^{r} \frac{\alpha q^{|\vec{n}|+\beta_i}}{1-\alpha q^{|\vec{n}|+\beta_i}} x(n_k+\beta_k-\beta_i).$$

From Equations (66) and (67) we get the following linear system of equations for the unknown coefficients $\xi_1, \dots, \xi_r$,

$$b_j = \sum_{i=1}^{r} \xi_i s_{j,i}, \quad k = 1, \dots, r, \quad \Longleftrightarrow \quad S\xi = b, \quad \xi = (\xi_1, \dots, \xi_r), \tag{68}$$

where the entries of the vector b and matrix S are as follows

$$b_j = \frac{(1-\alpha q^{n_j+\beta_j})q^{|\vec{n}|-n_j+1/2}}{(1-\alpha q^{|\vec{n}|+\beta_j})}, \quad s_{j,i} = m_{j,i}.$$

The above system (68) has a unique solution if and only if the matrix S is nonsingular. From Lemma 6, Formula (60), this condition is fulfilled. Accordingly, if $C_{j,i}$ stands for the cofactor of the entry $s_{j,i}$ and $S_i(b)$ denotes the matrix obtained from S replacing its ith column by b, then

$$\xi_i = \frac{\det S_i(b)}{\det S}, \quad i = 1, \ldots, r.$$

From Lemma 6,

$$\begin{aligned} \det S_i(b) &= \sum_{j=1}^{r} b_j C_{j,i} \\ &= \sum_{j=1}^{r} b_j (-1)^{i+j} \prod_{k=1,k\neq i}^{r-1} \prod_{l=k+1,l\neq i}^{r} x(\beta_l - \beta_k) \frac{\prod_{k=1,k\neq j}^{r-1} \prod_{l=k+1,l\neq j}^{r} q^{n_l} x(n_k - n_l + \beta_k - \beta_l)}{\prod_{k=1,k\neq i}^{r} \prod_{l=1,l\neq j}^{r} x(n_k + \beta_k - \beta_l)}. \end{aligned}$$

Therefore, relation (62) holds. □

Theorem 3. *The q-Meixner multiple orthogonal polynomial of the second kind* $M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s)$ *satisfies the following* $(r+1)$*-order q-difference equation*

$$\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_i+1} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) = -\sum_{i=1}^{r} q^{1/2} \xi_i \prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_j+1} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s), \tag{69}$$

where ξ_i*s are the constants in* (63).

Proof. Since the operators (53) commute, we write

$$\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_i+1} = \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_j+1} \right) \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_i+1}. \tag{70}$$

□

Using Formula (53) in Equation (62) by acting with the product of operators (70), we obtain the desired relation (69); that is,

$$\begin{aligned} \prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_i+1} \Delta M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) &= \sum_{i=1}^{r} \xi_i \prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_j+1} \left(\mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_i+1} M_{q,\vec{n}-\vec{e}_i}^{q\alpha,\vec{\beta}+\vec{e}_i}(s) \right) \\ &= -\sum_{i=1}^{r} q^{1/2} \xi_i \prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{q\alpha,\beta_j+1} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s). \end{aligned}$$

3.6. Recurrence Relation for q-Meixner Multiple Orthogonal Polynomials of the Second Kind

Theorem 4. *The q-Meixner multiple orthogonal polynomials of the second kind satisfy the following* $(r+2)$*-term recurrence relation*

$$\begin{aligned} x(s) M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) &= M_{q,\vec{n}+\vec{e}_k}^{\alpha,\vec{\beta}}(s) + b_{\vec{n},k} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) \\ &+ \alpha q^{2|\vec{n}|-1} \sum_{i=1}^{r} \frac{x(n_i) x(\beta_i + n_i - 1)}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i-1} c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i-2}} \prod_{j\neq i}^{r} \frac{x(n_i + \beta_i - \beta_j)}{x(n_i + \beta_i - n_j - \beta_j)} B_{\vec{n},i} M_{q,\vec{n}-\vec{e}_i}^{\alpha,\vec{\beta}}(s), \end{aligned} \tag{71}$$

where

$$b_{\vec{n},k}=\prod_{i=1}^{r}\frac{c_{q,\vec{n}}^{\alpha,\beta_i}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i}}\left(\sum_{i=1}^{r}\frac{-q^{|\vec{n}|}x(n_i)}{q^{n_i}c_{q,\vec{n}}^{\alpha,\beta_i}}-\frac{\alpha q^{2|\vec{n}|+1}x(\beta_k+n_k)}{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha,\beta_k+1}}\right)$$
$$+(q-1)\left(\alpha q^{|\vec{n}|}\sum_{i=1}^{r}\frac{x(n_i)x(n_i+\beta_i-1)}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i}c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i-1}}\prod_{j\neq i}^{r}\frac{x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}+\prod_{i=1}^{r}\frac{-x(n_i)}{c_{q,\vec{n}}^{\alpha,\beta_i}}\prod_{i=1}^{r}\frac{c_{q,\vec{n}}^{\alpha,\beta_i}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i}}\right)$$

and

$$B_{\vec{n},i}=\frac{\alpha q^{|\vec{n}|}-1}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i}}\prod_{i=1}^{r}\frac{c_{q,\vec{n}}^{\alpha,\beta_i-1}}{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i}}.$$

Proof. Let

$$g_{\mathbf{n}}(s;\alpha)=\frac{(\alpha q^{\mathbf{n}})^s}{\Gamma_q(s+1)},\quad \text{where}\quad \mathbf{n}=|\vec{n}|.$$

We will use Lemma 4 involving this function $g_{\mathbf{n}}(s;\alpha)$ as well as difference operator (56). Consider the following equation

$$\frac{\Gamma_q(\beta_k)}{\Gamma_q(\beta_k+s)}\nabla^{n_k+1}\frac{\Gamma_q(\beta_k+n_k+1+s)}{\Gamma_q(\beta_k+n_k+1)}\frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)}$$
$$=\frac{\Gamma_q(\beta_k)}{\Gamma_q(\beta_k+s)}\nabla^{n_k}\left(q^{-s+1/2}\nabla\left(\frac{\Gamma_q(\beta_k+n_k+1+s)}{\Gamma_q(\beta_k+n_k+1)}\frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)}\right)\right)$$
$$=q^{1/2}\frac{\Gamma_q(\beta_k)}{\Gamma_q(\beta_k+s)}\nabla^{n_k}\left(\frac{\Gamma_q(\beta_k+n_k+s)}{\Gamma_q(\beta_k+n_k)}\left(1+\frac{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha,\beta_k+1}}{(\alpha_k q^{|\vec{n}|+1})x(\beta_k+n_k)}x(s)\right)\frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}\right),$$

which can be rewritten as follows

$$\mathcal{N}_{q,n_k+1}^{\beta_k}\frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)}=q^{1/2}\mathcal{N}_{q,n_k}^{\beta_k}\frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}+q^{1/2}\frac{c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha,\beta_k+1}}{(\alpha q^{|\vec{n}|+1})x(\beta_k+n_k)}\mathcal{N}_{q,n_k}^{\beta_k}x(s)\frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}. \tag{72}$$

Since operators (56) commute, the multiplication of Equation (72) from the left-hand side by the product $\prod_{\substack{i=1\\i\neq k}}^{r}\mathcal{N}_{q,n_i}^{\beta_i}$ yields

$$\mathcal{N}_{q,\vec{n}}^{\vec{\beta}}x(s)\frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}=\frac{\alpha x(\beta_k+n_k)}{q^{-|\vec{n}|-1/2}c_{q,\vec{n}+n_k\vec{e}_k}^{\alpha,\beta_k+1}}\left(\mathcal{N}_{q,\vec{n}+\vec{e}_k}^{\vec{\beta}}\frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)}-q^{1/2}\mathcal{N}_{q,\vec{n}}^{\vec{\beta}}\frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}\right). \tag{73}$$

Let us recursively use Lemma 4 involving the product of r difference operators $\prod_{i=1}^{r}\mathcal{F}_{q,n_i}$ acting on the function $g_{\mathbf{n}}(s;\alpha)$, that is, the operator $\mathcal{N}_{q,\vec{n}}^{\vec{\beta}}$ (see expression (56)). Thus,

$$q^{|\vec{n}|}\mathcal{N}_{q,\vec{n}}^{\vec{\beta}}x(s)g_{\mathbf{n}}(s;\alpha)=\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{q^{1/2}x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}\frac{x(n_i)c_{q,\vec{n}+n_j\vec{e}_j}^{\alpha,\beta_j}}{\prod_{\nu=1}^{r}c_{q,\vec{n}}^{\alpha,\beta_\nu}}\prod_{l=1}^{r}\mathcal{N}_{q,n_l-\delta_{l,i}}^{\beta_l}g_{\mathbf{n}}(s;\alpha)$$
$$+\left(q^{|\vec{n}|}\sum_{i=1}^{r}\frac{x(n_i)}{q^{n_i}c_{q,\vec{n}}^{\alpha,\beta_i}}+(q-1)\prod_{i=1}^{r}\frac{x(n_i)}{c_{q,\vec{n}}^{\alpha,\beta_i}}\right)\mathcal{N}_{q,\vec{n}}^{\vec{\beta}}g_{\mathbf{n}}(s;\alpha)+\prod_{i=1}^{r}\frac{c_{q,\vec{n}+n_i\vec{e}_i}^{\alpha,\beta_i}}{c_{q,\vec{n}}^{\alpha,\beta_i}}x(s)\mathcal{N}_{q,\vec{n}}^{\vec{\beta}}g_{\mathbf{n}}(s;\alpha). \tag{74}$$

Hence, using expressions (73) and (74) one gets

$$
\begin{aligned}
x(s)\mathcal{N}^{\vec{\beta}}_{q,\vec{n}}g_{\mathbf{n}}(s;\alpha) &= q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{c^{\alpha,\beta_i}_{q,\vec{n}}}{c^{\alpha,\beta_i}_{q,\vec{n}+n_i\vec{e}_i}}\frac{\alpha q^{|\vec{n}|+1}x(\beta_k+n_k)}{c^{\alpha,\beta_k+1}_{q,\vec{n}+n_k\vec{e}_k}}\mathcal{N}^{\vec{\beta}}_{q,\vec{n}+\vec{e}_k}g_{\mathbf{n}+1}(s;\alpha) \\
&+\prod_{i=1}^{r}\frac{c^{\alpha,\beta_i}_{q,\vec{n}}}{c^{\alpha,\beta_i}_{q,\vec{n}+n_i\vec{e}_i}}\left(q^{|\vec{n}|}\sum_{i=1}^{r}\frac{x(n_i)}{q^{n_i}c^{\alpha,\beta_i}_{q,\vec{n}}}+(q-1)\prod_{i=1}^{r}\frac{x(n_i)}{c^{\alpha,\beta_i}_{q,\vec{n}}}-q^{|\vec{n}|}\frac{\alpha q^{|\vec{n}|+1}x(\beta_k+n_k)}{c^{\alpha,\beta_k+1}_{q,\vec{n}+n_k\vec{e}_k}}\right)\mathcal{N}^{\vec{\beta}}_{q,\vec{n}}g_{\mathbf{n}}(s;\alpha) \\
&-\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{q^{1/2}x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}\frac{-x(n_i)}{c^{\alpha,\beta_i}_{q,\vec{n}+n_i\vec{e}_i}}\prod_{l=1}^{r}\mathcal{N}^{\beta_l}_{q,n_l-\delta_{l,i}}g_{\mathbf{n}}(s;\alpha).
\end{aligned}
$$

Observe that

$$
\mathcal{N}^{\beta_i}_{q,n_i-1}g_{\mathbf{n}}(s;\alpha)=q^{-1/2}\frac{(q-1)\alpha q^{|\vec{n}|}x(n_i+\beta_i-1)}{c^{\alpha,\beta_i-1}_{q,\vec{n}+n_i\vec{e}_i}}\mathcal{N}^{\beta_i}_{q,n_i}g_{\mathbf{n}}(s;\alpha)+\frac{\alpha q^{|\vec{n}|}-1}{c^{\alpha,\beta_i-1}_{q,\vec{n}+n_i\vec{e}_i}}\mathcal{N}^{\beta_i}_{q,n_i-1}g_{\mathbf{n}-1}(s;\alpha),
$$

which is used in the previous expression when the indices l and i coincide. Therefore, the following expression holds

$$
\begin{aligned}
x(s)\mathcal{N}^{\vec{\beta}}_{q,\vec{n}}g_{\mathbf{n}}(s;\alpha) &= q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{c^{\alpha,\beta_i}_{q,\vec{n}}}{c^{\alpha,\beta_i}_{q,\vec{n}+n_i\vec{e}_i}}\frac{(\alpha q^{|\vec{n}|+1})x(\beta_k+n_k)}{c^{\alpha,\beta_k+1}_{q,\vec{n}+n_k\vec{e}_k}}\mathcal{N}^{\vec{\beta}}_{q,\vec{n}+\vec{e}_k}g_{\mathbf{n}+1}(s;\alpha) \\
+b_{\vec{n},k}\mathcal{N}^{\vec{\beta}}_{q,\vec{n}}g_{\mathbf{n}}(s;\alpha) &- q^{1/2}(1-\alpha q^{|\vec{n}|})\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}\frac{x(n_i)}{c^{\alpha,\beta_i}_{q,\vec{n}+n_i\vec{e}_i}c^{\alpha,\beta_i-1}_{q,\vec{n}+n_i\vec{e}_i}}\prod_{l=1}^{r}\mathcal{N}^{\beta_l}_{q,n_l-\delta_{l,i}}g_{\mathbf{n}-1}(s;\alpha).
\end{aligned}
$$

Finally, multiplying from the left both sides of the previous expression by $\mathcal{G}^{\vec{n},\vec{\beta},\alpha}_q\Gamma_q(\beta_i)/\Gamma_q(\beta_i+s)$ and using Rodrigues-type Formula (55), we obtain (71). This completes the proof of the theorem. □

4. Limit Relations as q Approaches 1

The lattice $x(s) = (q^s-1)/(q-1)$ allows to transit from the non-uniform distribution of points $(q^s-1)/(q-1)$, $s = 0,1,\ldots$, to the uniform distribution s, as q approaches 1. Under this limiting process one expects that the q-algebraic relations studied in this paper transform into the corresponding relations for discrete multiple orthogonal polynomials [28]. Indeed, the q-analogue of Rodrigues-type Formulas (31) and (55) will be transformed into their discrete counterparts (15) and (16), respectively. As a consequence, the recurrence relations (19) and (20) can be derived from (47) and (71), respectively.

We begin by analyzing the Rodrigues-type formulas, which then can be used for addressing the limit relations involving other algebraic properties.

Proposition 3. *The following limiting relations for q-Meixner multiple orthogonal polynomials of the first kind* (31) *and second kind* (55) *hold:*

$$
\lim_{q\to1}M^{\vec{\alpha},\beta}_{q,\vec{n}}(s)=(\beta)_{|\vec{n}|}\prod_{i=1}^{r}\left(\frac{\alpha_i}{\alpha_i-1}\right)^{n_i}\frac{\Gamma(\beta)\Gamma(s+1)}{\Gamma(\beta+s)}\prod_{i=1}^{r}\alpha_i^{-s}\,\nabla^{n_i}\,\alpha_i^{s}\left(\frac{\Gamma(\beta+|\vec{n}|+s)}{\Gamma(\beta+|\vec{n}|)\Gamma(s+1)}\right), \tag{75}
$$

$$
\lim_{q\to1}M^{\alpha,\vec{\beta}}_{q,\vec{n}}(s)=\left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left(\prod_{i=1}^{r}(\beta_i)_{n_i}\right)\frac{\Gamma(s+1)}{\alpha^s}\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+s)}\,\nabla^{n_i}\,\frac{\Gamma(\beta_i+n_i+s)}{\Gamma(\beta_i+n_i)}\left(\frac{\alpha^s}{\Gamma(s+1)}\right). \tag{76}
$$

The right-hand side limiting results are the corresponding discrete multiple orthogonal polynomials $M^{\vec{\alpha},\beta}_{\vec{n}}(s)$ and $M^{\alpha,\vec{\beta}}_{\vec{n}}(s)$ given in (15) *and* (16)*, respectively.*

Proof. We begin by proving (75). Let us rewrite the m-th action of the difference operator ∇ on a function $f(s)$ defined on the q-lattice $x(s)$ as follows (see formula (3.2.29) from [38])

$$\nabla^m f(s) = q^{\binom{m+1}{2}/2-ms} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} (-1)^k q^{\binom{m-k}{2}} f(s-k), \tag{77}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}}, \qquad m = 1,2,\ldots,$$
$$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j) \quad \text{for} \quad k>0, \quad \text{and} \quad (a;q)_0 = 1.$$

Here the expression $(a;q)_k$ denotes the q-analogue of the Pochhammer symbol [37,38,43,44]. Moreover, expression (77) is a q-analogue of (11).

In (31) we have the following expression

$$M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \mathcal{G}_q^{\vec{n},\vec{\alpha},\beta} \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \prod_{i=1}^{r} (\alpha_i)^{-s} \nabla^{n_i} (\alpha_i q^{n_i})^s \left(\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \right),$$

where the normalizing coefficient $\mathcal{G}_q^{\vec{n},\vec{\alpha},\beta}$ is given in (33) and it tends to the following expression, as q approaches to 1

$$(\beta)_{|\vec{n}|} \left(\prod_{i=1}^{r} \left(\frac{\alpha_i}{\alpha_i - 1} \right)^{n_i} \right).$$

Without loss of generality, let us consider a multi-index $\vec{n} = (n_1, n_2)$ and rewrite the above expression in accordance with Formula (77); that is, we first need to express $\nabla^{n_1}(\alpha_1 q^{n_1})^s \Gamma_q(\beta+|\vec{n}|+s)/(\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1))$ in terms of a finite sum and then compute the action of ∇^{n_2} on the product formed by this resulting expression and $(\alpha_2 q^{n_2})^s$. Namely,

$$\begin{aligned} M_{q,n_1,n_2}^{\alpha_1,\alpha_2,\beta}(s) &= \mathcal{G}_q^{n_1,n_2,\alpha_1,\alpha_2,\beta} \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \left(\alpha_2^{-s} \nabla^{n_2} (\alpha_2 q^{n_2})^s\right) \left(\alpha_1^{-s} \nabla^{n_1} (\alpha_1 q^{n_1})^s\right) \frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \\ &= \mathcal{G}_q^{n_1,n_2,\alpha_1,\alpha_2,\beta} q^{(\binom{n_1+1}{2}+\binom{n_2+1}{2})/2} \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)\Gamma_q(\beta+n_1+n_2)} \\ &\times \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} (-1)^{l+k} \begin{bmatrix} n_2 \\ l \end{bmatrix} \begin{bmatrix} n_1 \\ k \end{bmatrix} \frac{q^{\binom{n_2-l}{2} - ln_2 + \binom{n_1-k}{2} - kn_1}}{\alpha_2^l \alpha_1^k} \frac{\Gamma_q(\beta+n_1+n_2-k-l+s)}{\Gamma_q(s-k-l+1)}. \end{aligned} \tag{78}$$

Applying limit in the above expression as q approaches to 1 yields

$$\begin{aligned} \lim_{q\to 1} M_{q,n_1,n_2}^{\alpha_1,\alpha_2,\beta}(s) &= (\beta)_{n_1+n_2} \left(\frac{\alpha_1}{\alpha_1-1} \right)^{n_1} \left(\frac{\alpha_2}{\alpha_2-1} \right)^{n_2} \frac{\Gamma(\beta)\Gamma(s+1)}{\Gamma(\beta+s)} \\ &= \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} (-1)^{l+k} \binom{n_2}{l} \binom{n_1}{k} \frac{1}{\alpha_2^l \alpha_1^k} \frac{\Gamma(\beta+n_1+n_2-k-l+s)}{\Gamma(s-k-l+1)}. \end{aligned} \tag{79}$$

Using (11), one rewrites Equation (79) such that it involves the product of raising operators as in (13) to obtain

$$\begin{aligned}\lim_{q\to 1} M_{q,n_1,n_2}^{\alpha_1,\alpha_2,\beta}(s) &= (\beta)_{n_1+n_2}\left(\frac{\alpha_1}{\alpha_1-1}\right)^{n_1}\left(\frac{\alpha_2}{\alpha_2-1}\right)^{n_2}\frac{\Gamma(\beta)\Gamma(s+1)}{\Gamma(\beta+s)} \\ &\times\left(\alpha_2^{-s}\bigtriangledown^{n_2}\alpha_2^{s}\right)\left(\alpha_1^{-s}\bigtriangledown^{n_1}\alpha_1^{s}\right)\frac{\Gamma(\beta+n_1+n_2+s)}{\Gamma(\beta+n_1+n_2)\Gamma(s+1)} \\ &= M_{n_1,n_2}^{\alpha_1,\alpha_2,\beta}(s),\end{aligned}$$

which coincides with (15) for $\vec{n}=(n_1,n_2)$. Observe that repeating the aforementioned procedure for a multi-index $\vec{n}$ of dimension r, we obtain for the polynomial

$$\begin{aligned}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) &= \mathcal{G}_q^{\vec{n},\vec{\alpha},\beta} q^{\sum_{i=1}^{r}\binom{n_i+1}{2}/2}\frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)\Gamma_q(\beta+|\vec{n}|)} \\ &\times\sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r}(-1)^{|\vec{k}|}\begin{bmatrix}n_r\\k_r\end{bmatrix}\cdots\begin{bmatrix}n_1\\k_1\end{bmatrix}\frac{q^{\binom{n_r-k_r}{2}-k_rn_r+\cdots+\binom{n_1-k_1}{2}-k_1n_1}}{\alpha_r^{k_r}\cdots\alpha_1^{k_1}}\frac{\Gamma_q(\beta+|\vec{n}-\vec{k}|+s)}{\Gamma_q(s-|\vec{k}|+1)},\end{aligned}$$

where $\vec{k}=(k_1,\ldots,k_r)$, the following relation

$$\begin{aligned}\lim_{q\to 1} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) &= (\beta)_{|\vec{n}|}\left(\prod_{i=1}^{r}\left(\frac{\alpha_i}{\alpha_i-1}\right)^{n_i}\right)\frac{\Gamma(\beta)\Gamma(s+1)}{\Gamma(\beta+s)}\prod_{i=1}^{r}\alpha_i^{-s}\bigtriangledown^{n_i}\alpha_i^{s}\left(\frac{\Gamma(\beta+|\vec{n}|+s)}{\Gamma(\beta+|\vec{n}|)\Gamma(s+1)}\right), \\ &= M_{\vec{n}}^{\vec{\alpha},\beta}(s).\end{aligned}$$

This proves the expression (75).

Next, we will prove the second limiting relation (76). Notice that the normalizing coefficient $\mathcal{G}_q^{\vec{n},\vec{\beta},\alpha}$ given in (57) has the following limit expression, as q approaches 1,

$$\begin{aligned}\lim_{q\to 1}\mathcal{G}_q^{\vec{n},\vec{\beta},\alpha} &= \lim_{q\to 1}(-1)^{|\vec{n}|}\left(\alpha q^{|\vec{n}|}\right)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{\prod_{j=1}^{n_i} q^{\beta_i+j-1}}{\prod_{j=1}^{n_i}\left(\alpha q^{|\vec{n}|+\beta_i+j-1}-1\right)}\right)\left(\prod_{i=1}^{r}[-\beta_i]_q^{(n_i)}\right) \\ &= \left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left(\prod_{i=1}^{r}(\beta_i)_{n_i}\right).\end{aligned}$$

From (55) and (77) we have

$$\begin{aligned}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) &= \mathcal{G}_q^{\vec{n},\vec{\beta},\alpha} q^{\sum_{i=1}^{r}\binom{n_i+1}{2}/2}\prod_{i=1}^{r}\frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i+n_i)}\frac{\Gamma_q(s+1)}{\alpha^s} \\ &\times\sum_{k_r=0}^{n_r}\cdots\sum_{k_1=0}^{n_1}(-1)^{|\vec{k}|}\begin{bmatrix}n_r\\k_r\end{bmatrix}\cdots\begin{bmatrix}n_1\\k_1\end{bmatrix}\frac{q^{\binom{n_r-k_r}{2}-k_rn_r+\cdots+\binom{n_1-k_1}{2}-k_1n_1}}{\Gamma_q(\beta_r+s)\Gamma_q(\beta_{r-1}+s-k_r)\cdots\Gamma_q(\beta_1+s-k_r-\cdots-k_2)} \\ &\quad\times\frac{\left(\alpha q^{|\vec{n}|}\right)^{s-|\vec{k}|}\Gamma_q(\beta_r+n_r+s-k_r)\cdots\Gamma_q(\beta_2+n_2+s-k_r-\cdots-k_2)\Gamma_q(\beta_1+n_1+s-|\vec{k}|)}{\Gamma_q(s-|\vec{k}|+1)}.\end{aligned}$$

Therefore, we evaluate the following limit:

$$\lim_{q\to 1} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left(\prod_{i=1}^{r}(\beta_i)_{n_i}\right)\frac{\Gamma(s+1)}{\alpha^s}$$
$$\times \sum_{k_r=0}^{n_r}\cdots\sum_{k_1=0}^{n_1}(-1)^{|\vec{k}|}\binom{n_r}{k_r}\cdots\binom{n_1}{k_1}\frac{1}{\Gamma(\beta_r+s)\Gamma(\beta_{r-1}+s-k_r)\cdots\Gamma(\beta_1+s-k_r-\cdots-k_2)}$$
$$\times \frac{\alpha^{s-|\vec{k}|}\Gamma(\beta_r+n_r+s-k_r)\cdots\Gamma(\beta_2+n_2+s-k_r-\cdots-k_2)\Gamma(\beta_1+n_1+s-|\vec{k}|)}{\Gamma(s-|\vec{k}|+1)}.$$

Finally, using (11) one rewrites the right-hand side as follows

$$\lim_{q\to 1} M_{q,\vec{n}}^{\alpha,\vec{\beta}}(s) = \left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left(\prod_{i=1}^{r}(\beta_i)_{n_i}\right)\frac{\Gamma(s+1)}{\alpha^s}\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+s)}\nabla^{n_i}\frac{\Gamma(\beta_i+n_i+s)}{\Gamma(\beta_i+n_i)}\left(\frac{\alpha^s}{\Gamma(s+1)}\right)$$
$$= M_{\vec{n}}^{\alpha,\vec{\beta}}(s).$$

This completes the proof of expression (76). □

5. Appendix: AT-Property for the Studied Discrete Measures

Lemma 9. *The system of functions*

$$\alpha_1^s, x(s)\alpha_1^s, \ldots, x(s)^{n_1-1}\alpha_1^s, \ldots, \alpha_r^s, x(s)\alpha_r^s, \ldots, x(s)^{n_r-1}\alpha_r^s, \tag{80}$$

with $\alpha_i > 0$, $i = 1, 2, \ldots, r$, *with all the* α_i *different, and* $(\alpha_i/\alpha_j) \neq q^k$, $k \in \mathbb{Z}$, $i, j = 1, \ldots, r$, $i \neq j$, *forms a Chebyshev system on* $\mathbb{R}^+$ *for every* $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$.

Proof. For a Chebyshev system every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x(s))\alpha_i^s$ has at most $|\vec{n}|-1$ zeros on $\mathbb{R}^+$ for every $Q_{n_i-1}(x(s)) \in \mathbb{P}_{n_i-1}\setminus\{0\}$. Since $x(s) = c_1 q^s + c_3$, where c_1, c_3 are constants, we consider $\sum_{i=1}^{r} Q_{n_i-1}(q^s)\alpha_i^s$, instead. Thus, the system (80) transforms into

$$a_{1,0}^s, a_{1,1}^s, \ldots, a_{1,n_1-1}^s, \ldots, a_{r,0}^s, a_{r,1}^s, \ldots, a_{r,n_r-1}^s,$$

where $a_{i,k} = (q^k\alpha_i)$, with $k = 0, \ldots, n_i-1$, $i = 1, \ldots, r$. Observe that $a_{j,m} \neq a_{l,p}$ for $j \neq l$, $m \neq p$. Hence, identity $a_{i,k} = e^{\log a_{i,k}}$ yields the well-known Chebyshev system (see [34], p. 138)

$$e^{s\log a_{1,0}}, e^{s\log a_{1,1}}, \ldots, e^{s\log a_{1,n_1-1}}, \ldots, e^{s\log a_{r,0}}, e^{s\log a_{r,1}}, \ldots, e^{s\log a_{r,n_r-1}}.$$

Then, we conclude that the functions (80) form a Chebyshev system on $\mathbb{R}^+$. □

Lemma 10. *Let* $\beta_i > 0$ *and* $\beta_i - \beta_j \notin \mathbb{Z}$ *whenever* $i \neq j$. *Assume* $v(s)$ *is a continuous function with no zeros on* $\mathbb{R}^+$, *then the functions*

$$\begin{array}{c} v(s)\Gamma_q(s+\beta_1), v(s)x(s)\Gamma_q(s+\beta_1), \ldots, v(s)x(s)^{n_1-1}\Gamma_q(s+\beta_1), \\ \vdots \\ v(s)\Gamma_q(s+\beta_r), v(s)x(s)\Gamma_q(s+\beta_r), \ldots, v(s)x(s)^{n_r-1}\Gamma_q(s+\beta_r), \end{array} \tag{81}$$

form a Chebyshev system on Ω *for every* $\vec{n} \in \mathbb{N}^r$.

Proof. For the system of functions (81) we have a Chebyshev system on Ω for every $\vec{n} \in \mathbb{N}^r$ if and only if every linear combination of these functions (except the one with each coefficient equals 0) has at most $|\vec{n}| - 1$ zeros. This linear combination can be rewritten as a function of the system

$$\begin{aligned}
&v(s)\Gamma_q(s+\beta_1), v(s)[s+\beta_1]_q^{(1)}\Gamma_q(s+\beta_1), \ldots, \\
&\qquad v(s)[s+\beta_1+n_1-2]_q^{(n_1-1)}\Gamma_q(s+\beta_1), \\
&\qquad v(s)\Gamma_q(s+\beta_r), v(s)[s+\beta_r]_q^{(1)}\Gamma_q(s+\beta_r), \ldots, \\
&\qquad\qquad v(s)[s+\beta_1+n_r-2]_q^{(n_r-1)}\Gamma_q(s+\beta_r),
\end{aligned}$$

where $[s+\beta_i]_q^{(n_i)}, \quad i = 1, \ldots, r$, is given in (6).

Observe that

$$[s+k-1]_q^{(k)}\Gamma_q(s) = \Gamma_q(s+k),$$

holds. Therefore, the above system transforms into

$$\begin{aligned}
&v(s)\Gamma_q(s+\beta_1), v(s)\Gamma_q(s+\beta_1+1), \ldots, v(s)\Gamma_q(s+\beta_1+n_1-1), \\
&\qquad\vdots \\
&v(s)\Gamma_q(s+\beta_r), v(s)\Gamma_q(s+\beta_r+1), \ldots, v(s)\Gamma_q(s+\beta_r+n_r-1).
\end{aligned} \tag{82}$$

Thus, it is sufficient to prove that these systems (82) form a Chebyshev system on Ω for every $\vec{n} \in \mathbb{N}^r$. If we define the matrix $\mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right)$ by

$$\begin{pmatrix}
\Gamma_q(s_1+\beta_1) & \Gamma_q(s_2+\beta_1) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_1\right) \\
\vdots & \vdots & & \vdots \\
\Gamma_q(s_1+\beta_1+n_1-1) & \Gamma_q(s_2+\beta_1+n_1-1) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_1+n_1-1\right) \\
\vdots & \vdots & & \vdots \\
\Gamma_q(s_1+\beta_r) & \Gamma_q(s_2+\beta_r) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_r\right) \\
\vdots & \vdots & & \vdots \\
\Gamma_q(s_1+\beta_r+n_1-1) & \Gamma_q(s_2+\beta_r+n_1-1) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_r+n_1-1\right)
\end{pmatrix},$$

the proof is reduced to showing that $\det \mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right) \neq 0$, for every $|\vec{n}|$, and different points $s_1, \ldots, s_{|\vec{n}|}$ in Ω, because $|v| > 0$ on Ω. Now we replace the q-gamma function in $\mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right)$ by the integral representation

$$\Gamma_q(s) = \int_0^{\frac{1}{1-q}} t^{s-1} E_q^{-qt} d_q t = \int_0^{x(\infty)} t^{s-1} E_q^{-qt} d_q t, \quad s > 0, \tag{83}$$

where

$$E_q^z = {}_0\varphi_0\left(-;-;q,-(1-q)z\right)$$

denotes the q-analogue of the exponential function. From multilinearity of the determinant we take $|\vec{n}|$ integrations out of $|\vec{n}|$ rows to obtain

$$\det \mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right) = \underbrace{\int_0^{x(\infty)} \cdots \int_0^{x(\infty)}}_{|\vec{n}| \text{ times}} \prod_{1 \le i \le |\vec{n}|} E_q^{-qt_i} t_i^{s_i - 1} \times \det \mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) d_q t_1 \ldots d_q t_{|\vec{n}|}, \tag{84}$$

where

$$\mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) = \begin{pmatrix} t_1^{\beta_1} & t_2^{\beta_1} & \cdots & t_{|\vec{n}|}^{\beta_1} \\ \vdots & \vdots & & \vdots \\ t_1^{\beta_1+n_1-1} & t_2^{\beta_1+n_1-1} & \cdots & t_{|\vec{n}|}^{\beta_1+n_1-1} \\ \vdots & \vdots & & \vdots \\ t_1^{\beta_r} & t_2^{\beta_r} & \cdots & t_{|\vec{n}|}^{\beta_r} \\ \vdots & \vdots & & \vdots \\ t_1^{\beta_r+n_r-1} & t_2^{\beta_r+n_r-1} & \cdots & t_{|\vec{n}|}^{\beta_r+n_r-1} \end{pmatrix}.$$

Notice that, from ([34], p. 138, example 4) we know that the functions

$$t^{\beta_1}, \ldots, t^{\beta_1+n_1-1}, \ldots, t^{\beta_r}, \ldots, t^{\beta_r+n_r-1},$$

form a Chebyshev system on $\mathbb{R}^+$ if all the exponents are different, which is in accordance with our choice $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$. Moreover, if all $n_i < N+1$, then the exponents involved in the above matrix are different for $\beta_i - \beta_j \notin \{0, 1, \ldots, N\}$ whenever $i \neq j$. Hence, $\det \mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right)$ does not vanish for distinct $t_1, \ldots, t_{|\vec{n}|}$. Now, for a permutation σ of $\{1, \ldots, |\vec{n}|\}$ we make a change of variables $t_i \mapsto t_{\sigma(i)}$ in the integral (84). Thus, we have

$$\det \mathcal{A}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) = \underbrace{\int_0^{x(\infty)} \cdots \int_0^{x(\infty)}}_{|\vec{n}| \text{ times}} \prod_{1 \le i \le |\vec{n}|} E_q^{-qt_i} \det \mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) \times \operatorname{sgn}(\sigma) \prod_{1 \le j \le |\vec{n}|} t_{\sigma(j)}^{s_j - 1} d_q t_1 \ldots d_q t_{|\vec{n}|}. \tag{85}$$

We average (85) over all permutation σ, i.e.,

$$\det \mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right) = \frac{1}{n!} \sum_{\sigma \in S_{|\vec{n}|}} \underbrace{\int_0^{x(\infty)} \cdots \int_0^{x(\infty)}}_{|\vec{n}| \text{ times}} \prod_{1 \le i \le |\vec{n}|} E_q^{-qt_i} \times \det \mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) \operatorname{sgn}(\sigma) \prod_{1 \le j \le |\vec{n}|} t_{\sigma(j)}^{s_j - 1} d_q t_1 \ldots d_q t_{|\vec{n}|},$$

being $S_{|\vec{n}|}$ the permutation group. Now, relabeling the choice of points, i.e., $t_1, \ldots, t_{|\vec{n}|}$, where $0 < t_1 < \cdots < t_{|\vec{n}|}$, we have

$$\det \mathcal{A}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) = \frac{1}{n!} \underbrace{\int_0^{x(\infty)} \cdots \int_0^{x(\infty)}}_{0<t_1<\cdots<t_{|\vec{n}|}} \prod_{1\leq i\leq|\vec{n}|} E_q^{-qt_i} \det \mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right)$$
$$\times \sum_{\sigma\in S_{|\vec{n}|}} \operatorname{sgn}(\sigma) \prod_{1\leq j\leq|\vec{n}|} t_{\sigma(j)}^{s_j-1} d_q t_1 \ldots d_q t_{|\vec{n}|}. \tag{86}$$

As a result, from the definition of determinant we have

$$\sum_{\sigma\in S_{|\vec{n}|}} \operatorname{sgn}(\sigma) \prod_{1\leq j\leq|\vec{n}|} t_{\sigma(j)}^{s_j-1} = \begin{vmatrix} t_1^{s_1-1} & t_1^{s_2-1} & \cdots & t_1^{s_{|\vec{n}|}-1} \\ t_2^{s_1-1} & t_2^{s_2-1} & \cdots & t_2^{s_{|\vec{n}|}-1} \\ \vdots & \vdots & & \vdots \\ t_{|\vec{n}|}^{s_1-1} & t_{|\vec{n}|}^{s_2-1} & \cdots & t_{|\vec{n}|}^{s_{|\vec{n}|}-1} \end{vmatrix}. \tag{87}$$

Taking into account that $t_1, \ldots, t_{|\vec{n}|}$ are strictly positive and different, then using the result in ([34], p. 138, example 3) with multi-index $(1, \ldots, 1)$, will imply that (87) is different from zero if all the $s_1, \ldots, s_{|\vec{n}|}$ are different. Accordingly, for distinct $s_1, \ldots, s_{|\vec{n}|}$, the integrand of Equation (86) has a constant sign in the region of integration and hence $\det \mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right)$ does not vanish. □

As a consequence of Lemma 10 the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (51) forms an AT system on Ω.

6. Concluding Remarks

We have studied two families of multiple orthogonal polynomials on a non-uniform lattice, i.e., q-Meixner multiple orthogonal polynomials of the first and second kind, respectively. They are derived from two systems of q-discrete measures. Each system forms an AT-system. For these families of multiple q-orthogonal polynomials we have obtained the Rodrigues-type Formulas (31) and (55) as well as the recurrence relations (47) and (71), and the q-difference equations (41) and (69). The use of some q-difference operators has played an important role in deriving the aforementioned algebraic properties. Finally, in the limit situation $q \to 1$, we have obtained the multiple Meixner polynomials given in [28].

In closing, we address some research directions and open problems:

Problem 1. *A description of the main term of the logarithm asymptotics of the q-analogues of multiple Meixner polynomials deserves special attention. For such a purpose, we will use an algebraic function formulation for the solution of the equilibrium problem with constraints [45–47] to describe the zero distribution of multiple orthogonal polynomials [48]. This approach has been recently developed for multiple Meixner polynomials in [21] (see [49] as well as [17,50] for other approaches). Moreover, by analyzing the limiting behavior of the coefficients of the recurrence relations for such polynomials we expect to obtain the main term of their asymptotics.*

Problem 2. *In [51] the authors use the annihilation and creation operators a_i, $a_i^\dagger$ $(i = 1, \ldots, r)$ satisfying the commutation relations*

$$[a_i, a_j^\dagger] = \delta_{i,j}, \qquad [a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0, \quad i, j = 1, \ldots, r.$$

The generated Lie algebra is formed by r copies of the Heisenberg–Weyl algebra $W_i = span\{a_i, a_i^\dagger, 1\}$. For a more detailed and technical information about orthogonal polynomials in the Lie algebras see [52] as well as [53] for quantum mechanics and polynomials of a discrete variable.

The normalized simultaneous eigenvectors of the r number operators $N_i = a_i^\dagger a_i$ *are denoted by*

$$|n_1, n_2, \dots, n_r\rangle = |n_1\rangle|n_2\rangle \cdots |n_r\rangle,$$

Indeed,

$$\begin{aligned} N_i|n_1, n_2, \dots, n_r\rangle &= n_i|n_1, n_2, \dots, n_r\rangle, \\ \langle m_1, m_2, \dots, m_r|n_1, n_2, \dots, n_r\rangle &= \delta_{m_1,n_1} \cdots \delta_{m_r,n_r}. \end{aligned}$$

Moreover,

$$\begin{aligned} a_i^\dagger|n_1, n_2, \dots, n_r\rangle &= \sqrt{n_i+1}|n_1, \dots, n_i+1, \dots, n_r\rangle, \\ a_i|n_1, n_2, \dots, n_r\rangle &= \sqrt{n_i}|n_1, \dots, n_i-1, \dots, n_r\rangle, \end{aligned}$$

The Bargmann realization in terms of coordinates z_i*,* $i = 1, \dots, r$*, in* $\mathbb{C}^r$ *has*

$$\begin{aligned} a_i &= \frac{\partial}{\partial z_i}, \quad a_i^\dagger = z_i, \\ \langle z_1, z_2, \dots, z_r|n_1, n_2, \dots, n_r\rangle &= \frac{z_1^{n_1} \cdots z_r^{n_r}}{\sqrt{n_1! \cdots n_r!}}. \end{aligned}$$

For the model in [51]

$$H_i^{\vec{\alpha},\beta} = a_i + \sum_{k=1}^{r} \frac{N_k}{1-\alpha_k} + \left(\frac{\alpha_i}{1-\alpha_i} + \sum_{j=1}^{r} \frac{\alpha_j}{(1-\alpha_j)^2} a_j^\dagger\right) \left(\sum_{k=1}^{r} N_k + \beta\right), \quad i = 1, \dots, r,$$

represent the set of non-Hermitian operators defined in the universal enveloping algebra formed by the r copies W_i*.*

The operators making up the H_i *generate an isomorphic Lie algebra to that of the diffeomorphisms in* $\mathbb{C}^r$ *spanned by vector fields of the form*

$$Z = \sum_{i=1}^{r} f_i(\vec{z}) \frac{\partial}{\partial z_i} + g(\vec{z}), \quad \vec{z} = (z_1, \dots, z_r).$$

The authors indicated that although in the coordinate realization where

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right),$$

the operators H_i *are third order differential operators, they can be considered as Hamiltonians and are simultaneously diagonalized by the multiple Meixner polynomials of the first kind.*

Consider the states $|x, \vec{\alpha}, \beta\rangle$ *defined by means of the combination of states* $|n_1, \dots, n_r\rangle$ *as:*

$$|x, \vec{\alpha}, \beta\rangle = N^r_{x,\vec{\alpha},\beta} \sum_{\vec{n}} \frac{M_{\vec{n}}^{\vec{\alpha},\beta}(x)}{\sqrt{n_1! \cdots n_r!}} |n_1, n_2, \dots, n_r\rangle, \quad x \in \mathbb{N}.$$

Thus,

$$H_i^{\vec{\alpha},\beta}|x,\vec{\alpha},\beta\rangle = N^r_{x,\vec{\alpha},\beta}\sum_{\vec{n}}\frac{1}{\sqrt{n_1!\cdots n_r!}}\Big[M^{\vec{\alpha},\beta}_{\vec{n}+\vec{e}_i}(x)$$
$$+\left((\beta+|\vec{n}|)\left(\frac{\alpha_i}{1-\alpha_i}\right)+\sum_{k=1}^{r}\frac{n_k}{1-\alpha_k}\right)M^{\vec{\alpha},\beta}_{\vec{n}}(x)$$
$$+\sum_{j=1}^{r}\frac{\alpha_j n_j(\beta+|\vec{n}|-1)}{(\alpha_j-1)^2}M^{\vec{\alpha},\beta}_{\vec{n}-\vec{e}_j}(x)\Big]|n_1,n_2,\ldots,n_r\rangle.$$

In [51], by using the recurrence relation (19) for multiple Meixner polynomials of the first kind, the following relation

$$H_i^{\vec{\alpha},\beta}|x,\vec{\alpha},\beta\rangle = x|x,\vec{\alpha},\beta\rangle,$$

holds.

Despite the fact the operators are non-Hermitian, they have a real spectrum given by the lattice, i.e., the non-negative integers. The states $|x,\vec{\alpha},\beta\rangle$ are uniquely defined as the joint eigenstates of the Hamiltonian operators with eigenvalues equal to x. Moreover,

$$[H_i^{\vec{\alpha},\beta},H_j^{\vec{\alpha},\beta}]|x,\vec{\alpha},\beta\rangle = 0.$$

However, these Hamiltonians do not commute pairwise. Indeed,

$$[H_i^{\vec{\alpha},\beta},H_j^{\vec{\alpha},\beta}] = a_i - a_j + \frac{\alpha_i-\alpha_j}{(1-\alpha_i)(1-\alpha_j)}\left(\beta+\sum_{k=1}^{r}N_k\right).$$

Finally, because they do not commute and yet have common eigenvectors, the authors in [51] say that they form a 'weakly' integrable system.

The physical model described above motivates the study of a q-deformed model, which is currently being considered by using the results of the present paper involving the q-analogue of multiple Meixner polynomials of the first kind. In particular, the recurrence relation (47).

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