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Stability of Ulam–Hyers and Existence of Solutions for Impulsive Time-Delay Semi-Linear Systems with Non-Permutable Matrices

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Abstract: In this paper, the stability of Ulam–Hyers and existence of solutions for semi-linear time-delay systems with linear impulsive conditions are studied. The linear parts of the impulsive systems are defined by non-permutable matrices. To obtain solution for linear impulsive delay systems with non-permutable matrices in explicit form, a new concept of impulsive delayed matrix exponential is introduced. Using the representation formula and norm estimation of the impulsive delayed matrix exponential, sufficient conditions for stability of Ulam–Hyers and existence of solutions are obtained.

Keywords: impulsive delay equation; delayed matrix exponential; stability

1. Introduction

The theory of functional differential equations has been attracted by many researchers. Delay phenomena have applications in control engineering, biology, medicine, economy and other sciences. Many processes are characterized by quick state changes. The duration of state changes are relatively short compared with the total duration of the entire process. For the theory of impulsive differential equations, we refer the reader to the monograph of Samoilenko et al. [1] and references therein.

The phenomena with time delays appear in system theory, automatic engines, and engineering systems. Recently, in [2], a concept of delayed matrix exponential is introduced providing an explicit formula of solutions for linear time-delay continuous systems with commutative matrices. Congruently [3,4], it is also used to find an explicit formula for solutions of linear discrete delay systems.

In general, it is difficult to get an explicit representation of the solution without knowing impulsive delayed fundamental matrix for impulsive linear time-delay differential equations. Therefore, in [5] authors adopted the idea of [2–4] obtaining the representation of solutions of linear time-delay continuous systems with impulses. To do so, they introduced a concept of impulsive delayed matrix function for commutative matrices.

These basic results are widely used in dealing with control theory, iterative learning control, and stability analysis for time-delay continuous\discrete and impulsive equations; for example, refer to [6–19]. For more details on the recent advances on the stability (Ulam–Hyers) of differential equations, one can observe the monographs [20–22].

However, no study exists in the literature seeking an explicit solution for linear impulsive time-delay differential equations with non-commutative matrices. Due to the double impact of impulses and time-delay, it is a challenging task to attain a representation for a solution of a time-delay impulsive differential equation of non-commutative matrices and study the stability concepts of these equations.

Motivated by the above articles, we have considered the representation of solutions of a linear time-delay impulsive differential equation of the form:

$$\begin{cases} y'(t) = Ay(t) + By(t-h) + f(t), \ \ t \in [0,T], \ h > 0, \ t \neq t_k, \\ \Delta y(t_k) = y\left(t_k^+\right) - y\left(t_k^-\right) = C_k y(t_k), \ \ k = 1,2,...p, \\ y(t) = \varphi(t), \ \ -h \le t \le 0, \end{cases} \tag{1}$$

where $A, B, C_k \in \mathbb{R}^{n\times n}$ are constant matrices, $\varphi \in C^1([-h,0],\mathbb{R}^n)$, $f \in C([0,T],\mathbb{R}^n)$, $\{t_k\}$.satisfies $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $y\left(t_k^+\right) = \lim_{\alpha\to 0^+} y(t_k+\alpha)$, $y\left(t_k^-\right) = y(t_k)$.

Moreover, we investigated existence, uniqueness, and the stability of Ulam–Hyers for the following semi-linear time-delay impulsive differential equation:

$$\begin{cases} y'(t) = Ay(t) + By(t-h) + f(t,y(t)), \ \ t \in [0,T], \ h > 0, \ t \neq t_k, \\ \Delta y(t_k) = y\left(t_k^+\right) - y\left(t_k^-\right) = C_k y(t_k), \ \ k = 1,2,...p, \\ y(t) = \varphi(t), \ \ -h \le t \le 0, \end{cases} \tag{2}$$

The main contributions were as follows:

- We introduced a novel impulsive delayed matrix exponential function (impulsive delayed exponential) and adhered its norm estimate. Using this impulsive delayed exponential and the variation of constants method, we gave an explicit representation for solutions of impulsive time-delay initial value problems with linear parts defined by non-permutable matrices.
- Based on the presentation of solutions and a norm estimation of the impulsive delayed exponential, we obtained sufficient conditions for existence, uniqueness, and the stability of Ulam–Hyers.

In the next section, we introduced the impulsive delayed matrix exponential and showed that it is the fundamental (Cauchy) matrix for linear time-delay impulsive differential equations. In Section 3, we gave explicit formulae for solutions to linear homogeneous/nonhomogeneous time-delay impulsive differential equations via an impulsive delayed matrix exponential. Section 4 is aimed at existence, uniqueness, and stability of Ulam–Hyers for system (2). In Section 5, we studied the existence of the solution for the system (2). Finally, some examples are presented in Section 6.

2. Impulsive Delayed Matrix Exponential

Let $J = [0,T]$, $J_0 = [0,t_1]$, ..., $J_{p-1} = \left(t_{p-1}, t_p\right]$, ..., $J_p = \left(t_p, T\right]$,..., $t_{p+1} = T$ Furthermore, define

$$\mathfrak{P} = PC(J,\mathbb{R}^n) := \{y : J \to \mathbb{R}^n : y \in C(J_m,\mathbb{R}^n), \ m = 0,1,...,p$$

and there exist the left limit $y\left(t_m^-\right) = y(t_m)$ and right limit $y\left(t_m^+\right)\}$. It is clear that $\mathfrak{P}$ is a Banach space endowed with norm defined by $\|y\|_{PC} = \sup_{t\in J}\|y(t)\|$.

We introduce the spaces:

- $C^1(J,\mathbb{R}^n) = \{y \in C(J,\mathbb{R}^n) : y' \in C(J,\mathbb{R}^n)\}$.
- $PC^1(J,\mathbb{R}^n) := \{y : J \to \mathbb{R}^n : y' \in PC(J,\mathbb{R}^n)\}$.

Definition 1. *A function $y \in C^1([-h,0],\mathbb{R}^n) \cup PC^1(J,\mathbb{R}^n)$ is said to be a solution of (1) if y satisfies $y(t) = \varphi(t)$, $-h \le t \le 0$ and Equation (1) on J.*

Definition 2. *[2] A function $e_h^B(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is called delayed matrix exponential if*

$$e_h^B(t) := \begin{cases} \Theta, & -\infty < t < -h, \ \ h > 0, \\ I, & -h \le t < 0, \\ I + Bt + B^2\frac{(t-h)^2}{2} + ... + B^k\frac{(t-(k-1)h)^k}{k!}, & (k-1)h \le t < kh, \end{cases} \tag{3}$$

where $k \in \mathbb{N}$, $B \in \mathbb{R}^{n\times n}$, Θ, and I are the zero and identity matrices, respectively.

For $k \geq 0$, we define

$$X_0(t,s) = e^{A(t-s)}, \ \ t \geq s,$$
$$X_1(t,s+h) = \begin{cases} \int_{s+h}^{t} e^{A(t-r)} B X_0(r-h,s)\,dr, & s+h \leq t, \\ \Theta, & s+h > t. \end{cases}$$
$$X_k(t,s+kh) = \begin{cases} \int_{s+kh}^{t} e^{A(t-r)} B X_{k-1}(r-h,s+(k-1)h)\,dr, & s+kh \leq t, \\ \Theta, & s+kh > t. \end{cases}$$

Definition 3. *Let $A, B \in \mathbb{R}^{n\times n}$. Delayed perturbation of matrix exponential function $X_h^{A,B} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ generated by A, B is defined by*

$$X_h^{A,B}(t,s) = \begin{cases} \Theta, & -\infty < t-s < 0, \\ I, & t = s, \\ e^{A(t-s)} + X_1(t,s+h) + \ldots + X_k(t,s+kh), & kh \leq t-s < (k+1)h,\ k = 0,1,2,\ldots \end{cases} \tag{4}$$

Lemma 1. *Let $X_h^{A,B}(t,s)$ be defined as in Equation (4). Then, the following holds true:*

(i) *if $A = \Theta$, then $X_h^{A,B}(t,0) = e_h^{B}(t-h)$, $kh \leq t < (k+1)h$,*
(ii) *if $B = \Theta$, then $X_h^{A,B}(t,s) = e^{A(t-s)}$,*
(iii) *if $AB = BA$, then $X_h^{A,B}(t,s) = e^{A(t-s)} e_h^{B_1(t-h-s)}$, $B_1 = \exp(-Ah)B$, $kh \leq t-s < (k+1)h$.*

Proof. (i) If $A = \Theta$, then

$$X_0(t,s) = I, \ \ X_1(t,s+h) = \int_{s+h}^{t} B dr = B(t-h-s),$$
$$X_2(t,s+2h) = \int_{s+2h}^{t} B^2 (r-2h-s)\,dr = B^2 \frac{(t-2h-s)^2}{2!},$$
$$X_k(t,s+kh) = B^k \frac{(t-kh-s)^k}{k!}, \ \ s+kh \leq t < s+(k+1)h.$$

Thus,

$$X_h^{A,B}(t,s) = \sum_{j=0}^{k} B^j \frac{(t-jh-s)^j}{j!}, \ \ s+kh \leq t < s+(k+1)h.$$

(ii) If $B = \Theta$, then

$$X_0(t,s) = e^{A(t-s)}, \ \ X_k(t,s+kh) = \Theta, \ k = 1,2,\ldots,$$

and

$$X_h^{A,B}(t,s) = e^{A(t-s)}.$$

(iii) We assumed A and B as commutative; consequently, $e^{A(t-s)}B = Be^{A(t-s)}$. Using this property, we obtained

$$X_0(t,s) = e^{A(t-s)}, \ \ X_1(t,s+h) = \int_{s+h}^{t} e^{A(t-r)} B e^{A(r-h-s)} dr = e^{A(t-s)} B e^{-Ah}(t-h-s),$$
$$X_2(t,s+2h) = \int_{s+2h}^{t} e^{A(t-r)} B e^{A(r-2h-s)} B(t-2h-s)\,dr = e^{A(t-s)} B^2 e^{-A2h} \frac{(t-2h-s)^2}{2!}$$
$$X_k(t,s+kh) = e^{A(t-s)} B^k e^{-Akh} \frac{(t-kh-s)^k}{k!}, \ \ s+kh \leq t < s+(k+1)h.$$

It follows that

$$X_h^{A,B}(t,s) = \sum_{j=0}^{k} X_j(t, s+jh) = \sum_{j=0}^{k} e^{A(t-s)} B^j e^{-Ajh} \frac{(t-jh-s)^j}{j!}$$
$$= e^{A(t-s)} e_h^{B_1(t-h-s)}, \quad s+kh \le t < s+(k+1)h.$$

The lemma is proved. □

Lemma 2. *For all $t, s \in \mathbb{R}$, we have*

$$\frac{\partial}{\partial t} X_h^{A,B}(t,s) = A X_h^{A,B}(t,s) + B X_h^{A,B}(t-h,s).$$

Proof. The proof is based on the following formula:

$$\frac{\partial}{\partial t} \int_{s+jh}^{t} e^{A(t-r)} B X_{j-1}(r-h, s+(j-1)h)\, dr$$
$$= A \int_{s+jh}^{t} e^{A(t-r)} B X_{j-1}(r-h, s+(j-1)h)\, dr + B X_{j-1}(t-h, s+(j-1)h).$$

Indeed, for $kh \le t-s < (k+1)h$, we have

$$\begin{aligned}
\frac{\partial}{\partial t} X_h^{A,B}(t,s) &= \frac{\partial}{\partial t} \sum_{j=0}^{k} X_j(t, s+jh) \\
&= \sum_{j=0}^{k} \frac{\partial}{\partial t} \int_{s+jh}^{t} e^{A(t-r)} B X_{j-1}(r-h, s+(j-1)h)\, dr \\
&= \sum_{j=0}^{k} \left[A \int_{s+jh}^{t} e^{A(t-r)} B X_{j-1}(r-h, s+(j-1)h)\, dr + B X_{j-1}(t-h, s+(j-1)h) \right] \\
&= A \sum_{j=0}^{k} X_j(t, s+jh) + B \sum_{j=0}^{\infty} X_{j-1}(t-h, s+(j-1)h) \\
&= A \sum_{j=0}^{k} X_j(t, s+jh) + B \sum_{j=0}^{\infty} X_j(t-h, s+jh).
\end{aligned}$$

□

We, then, introduced an impulsive analogue $Y_h^{A,B,C}(t,s)$ of the delayed matrix exponential $X_h^{A,B}(t,s)$. Since in Equation (1), the impulse has the linear form $\Delta y(t_k) = C_k y(t_k)$, the impulsive Cauchy matrix has to contain the matrices C_k, being the reason why we introduced the following impulsive delayed matrix:

Definition 4. *Let $A, B, C_k \in \mathbb{R}^{n\times n}$ be constant matrices. Impulsive delayed matrix exponential function $Y_h^{A,B}(t,s)$ is defined by*

$$Y_h^{A,B,C}(t,s) := \begin{cases} \Theta, \quad t < s, \\ I, \ t = s, \\ X_h^{A,B}(t,s) + \sum\limits_{s<t_k<t} X_h^{A,B}(t,t_k)\, C_k Y_h^{A,B,C}(t_k,s). \end{cases} \tag{5}$$

It should be emphasized that if, in a commutative case, A, B, C_k were commutative matrices, the impulsive delayed matrix exponential function was then introduced in [13].

Definition 5. *[13] If A, B, C_k are commutative matrices, then impulsive delayed matrix exponential function is defined as follows:*

$$\begin{aligned} V(t,s) &= e^{A(t-s)} X(t, s+h), \\ X(t,s+h) &= e_h^{B_1(t-h-s)} + \sum_{s<t_k<t} C_k e_h^{B_1(t-h-t_k)} X(t_k, s+h), \ B_1 = \exp(-Ah)B. \end{aligned} \tag{6}$$

Lemma 3. *Let $Y_h^{A,B,C}(t,s)$ be defined by (5). If A, B, C_k are commutative, then $Y_h^{A,B,C}(t,s) = V(t,s)$.*

Proof. Since $AB = BA$, then, by Lemma 1, we have $X_h^{A,B}(t,s) = \exp(A(t-s)) e_h^{B_1(t-h-s)}$. Thus,

$$\begin{aligned} Y_h^{A,B,C}(t,s) &= X_h^{A,B}(t,s) + \sum_{s<t_k<t} X_h^{A,B}(t,t_k) C_k Y_h^{A,B,C}(t_k,s) \\ &= e^{A(t-s)} e_h^{B_1(t-h-s)} + \sum_{s<t_k<t} e^{A(t-t_k)} e_h^{B_1(t-h-t_k)} C_k Y_h^{A,B,C}(t_k,s) \\ &= e^{A(t-s)} \left(e_h^{B_1(t-h-s)} + \sum_{s<t_k<t} C_k e_h^{B_1(t-h-t_k)} e^{A(s-t_k)} Y_h^{A,B,C}(t_k,s) \right) \\ &= e^{A(t-s)} \left(e_h^{B_1(t-h-s)} + \sum_{s<t_k<t} C_k e_h^{B_1(t-h-t_k)} X(t_k, s+h) \right) \\ &= e^{A(t-s)} X(t,s+h) = V(t,s). \end{aligned}$$

□

Lemma 4. *Impulsive delayed matrix exponential function $Y_h^{A,B,C}(t,s)$ satisfies*

$$\frac{\partial}{\partial t} Y_h^{A,B,C}(t,s) = A Y_h^{A,B,C}(t,s) + B Y_h^{A,B,C}(t-h,s), \quad t \neq t_k, \tag{7}$$

$$Y_h^{A,B,C}(t_k^+,s) = Y_h^{A,B,C}(t_k,s) + C_k Y_h^{A,B,C}(t_k,s), \tag{8}$$

$$\frac{\partial}{\partial t} Y_h^{A,B,C}(t_k^+,s) = \frac{\partial}{\partial t} Y_h^{A,B,C}(t_k,s) + A C_k Y_h^{A,B,C}(t_k,s). \tag{9}$$

Proof. Step 1: We verify that $Y_h^{A,B,C}(t,s)$ satisfies the differential Equation (7).

$$\begin{aligned} \frac{\partial}{\partial t} Y_h^{A,B,C}(t,s) &= \frac{\partial}{\partial t} X_h^{A,B}(t,s) + \sum_{s<t_k<t} \frac{\partial}{\partial t} X_h^{A,B}(t,t_k) C_k Y_h^{A,B,C}(t_k,s) + \sum_{s<t_k<t} \frac{\partial}{\partial t} X_h^{A,B}(t,t_k) \phi_k \\ &= A X_h^{A,B}(t,s) + \sum_{s<t_k<t} A X_h^{A,B}(t,t_k) C_k Y_h^{A,B,C}(t_k,s) + \sum_{s<t_k<t} A X_h^{A,B}(t,t_k) \phi_k \\ &+ B X_h^{A,B}(t-h,s) + \sum_{s<t_k<t-h} B X_h^{A,B}(t-h,t_k) C_k Y_h^{A,B,C}(t_k,s) + \sum_{s<t_k<t-h} B X_h^{A,B}(t-h,t_k) \phi_k \\ &= A Y_h^{A,B,C}(t,s) + B Y_h^{A,B,C}(t-h,s). \end{aligned}$$

Step 2: We verify the equality (8). Note that $X_h^{A,B}(t^+,s) = X_h^{A,B}(t,s)$. Then,

$$\begin{aligned} Y_h^{A,B,C}(t_m^+,s) &= X_h^{A,B}(t_m^+,s) + \sum_{s<t_k<t_m^+} X_h^{A,B}(t_m^+,t_k) C_k Y_h^{A,B,C}(t_k,s) \\ &= X_h^{A,B}(t_m^-,s) + \sum_{s<t_k<t_m^-} X_h^{A,B}(t_m^+,t_k) C_k Y_h^{A,B,C}(t_k,s) \\ &+ X_h^{A,B}(t_m^+,t_m) C_m Y_h^{A,B,C}(t_m,s) \\ &= Y_h^{A,B,C}(t_m^-,s) + C_m Y_h^{A,B,C}(t_m,s). \end{aligned}$$

Step 3: The proof of (9) is similar to that of (8). □

3. Representation of Solutions

In this section, we looked for an explicit formula for the solutions of the linear impulsive inhomogeneous delay system, adopting the classical ideas in finding solutions for linear ordinary differential equations.

Firstly, we drive two explicit formulae of solutions to a linear impulsive homogeneous delay system:

Theorem 1. *Let $\varphi \in C^1([-h,0],\mathbb{R}^n)$. Then, the solution of the initial value problem (1) with $f = 0$ has the form*

$$y(t) = Y_h^{A,B,C}(t,-h)\,\varphi(-h) + \int_{-h}^{0} Y_h^{A,B,C}(t,s)\left[\varphi'(s) - A\varphi(s)\right] ds,\ \ t \geq -h, \tag{10}$$

$$y(t) = Y_h^{A,B,C}(t,0)\,\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)\,B\varphi(s)\,ds,\quad t \geq 0. \tag{11}$$

Proof. To prove the formula (10), we looked for the solution in the form of

$$y(t) = Y_h^{A,B,C}(t,-h)\,g(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s)\,g(s)\,ds,\ t \geq 0, \tag{12}$$

where $g(t) : [-h,0] \to \mathbb{R}^n$ is an unknown continuously differentiable function and that it satisfies the initial condition $y(t) = \varphi(t)$, $-h \leq t \leq 0$:

$$y(t) = Y_h^{A,B,C}(t,-h)\,g(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s)\,g(s)\,ds = \varphi(t),\ \ -h \leq t \leq 0.$$

If $t = -h$, we have

$$Y_h^{A,B,C}(-h,-h)\,g(0) + \int_{-h}^{0} Y_h^{A,B,C}(-h,s)\,g(s)\,ds = g(0) = \varphi(-h).$$

Thus, $g(0) = \varphi(-h)$. On the interval $-h \leq t \leq 0$, one can easily derive that

$$\begin{aligned} \varphi(t) &= Y_h^{A,B,C}(t,-h)\,\varphi(-h) + \left(\int_{-h}^{t} + \int_{t}^{0}\right) Y_h^{A,B,C}(t,s)\,g(s)\,ds \\ &= e^{A(t+h)}\varphi(-h) + \int_{-h}^{t} e^{A(t-s)} g(s)\,ds. \end{aligned}$$

Differentiating the above equality, we have

$$\begin{aligned} \varphi'(t) &= Ae^{A(t+h)}\varphi(-h) + A\int_{-h}^{t} e^{A(t-s)} g(s)\,ds + g(t) \\ &= A\varphi(t) + g(t). \end{aligned}$$

Therefore,

$$g(t) = \varphi'(t) - A\varphi(t).$$

Next, we prove the equivalence of (10) and (11). To do this, we use the integration by parts formula

$$\begin{aligned}\int_{-h}^{0} Y_h^{A,B,C}(t,s)\,d\varphi(s) &= Y_h^{A,B,C}(t,s)\,\varphi(s)\Big|_{s=-h}^{s=0} - \int_{-h}^{0}\frac{\partial}{\partial s}Y_h^{A,B,C}(t,s)\,\varphi(s)\,ds\\ &= Y_h^{A,B,C}(t,0)\,\varphi(0) - Y_h^{A,B,C}(t,-h)\,\varphi(-h)\\ &+ \int_{-h}^{0} Y_h^{A,B,C}(t,s)\,A\varphi(s)\,ds + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)\,B\varphi(s)\,ds.\end{aligned}$$

Thus, we obtained

$$\begin{aligned}y(t) &= Y_h^{A,B,C}(t,-h)\,\varphi(-h) + \int_{-h}^{0} Y_h^{A,B,C}(t,s)\left[\varphi'(s) - A\varphi(s)\right]ds\\ &= Y_h^{A,B,C}(t,0)\,\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)\,B\varphi(s)\,ds.\end{aligned}$$

□

Next, we attained an explicit formula of solutions to linear impulsive a non-homogeneous delay system with a zero initial condition.

Theorem 2. *The solution $y_p(t)$ of (8) satisfying a zero initial condition has a form*

$$y_p(t) = \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)\,f(s)\,ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\,f(s)\,ds,\quad t \geq 0. \tag{13}$$

Proof. We looked for the solution $y_p(t)$ in the form

$$y_p(t) = \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)\,g_j(s)\,ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\,g_k(s)\,ds,$$

whereas $g_j(s)$, $j = 0,1,...,k$ are unknown vector functions. We split the proof into several steps:

Step 1: $0 < t \leq t_1$. In this case, we have

$$y_p(t) = \int_0^t Y_h^{A,B,C}(t,s)\,g_0(s)\,ds.$$

We differentiated y_p and used the property $Y_h^{A,B}(t-h,s) = \Theta$, $t-h < s$, to obtain

$$\begin{aligned}y_p'(t) &= A\int_0^t Y_h^{A,B,C}(t,s)\,g_0(s)\,ds + B\int_0^t Y_h^{A,B,C}(t-h,s)\,g_0(s)\,ds + g_0(t)\\ &= A\int_0^t Y_h^{A,B,C}(t,s)\,g_0(s)\,ds + B\left(\int_0^{t-h} + \int_{t-h}^{t}\right) Y_h^{A,B,C}(t-h,s)\,g_0(s)\,ds + g_0(t)\\ &= A\int_0^t Y_h^{A,B,C}(t,s)\,g_0(s)\,ds + B\int_0^{t-h} Y_h^{A,B,C}(t-h,s)\,g_0(s)\,ds + g_0(t)\\ &= Ay_p(t) + By_p(t-h) + f(t).\end{aligned}$$

It follows that $g_0(t) = f(t)$.

Step 2: $t_1 < t \leq t_2$. In this case,

$$y_p(t) = \int_0^{t_1} Y_h^{A,B,C}(t,s)\,f(s)\,ds + \int_{t_1}^{t} Y_h^{A,B,C}(t,s)\,g_1(s)\,ds.$$

We differentiated $y_p(t)$ again to obtain

$$\begin{aligned} y_p'(t) &= \int_0^{t_1} \left[A Y_h^{A,B,C}(t,s) + B Y_h^{A,B,C}(t-h,s) \right] f(s)\, ds \\ &\quad + \int_{t_1}^{t} \left[A Y_h^{A,B,C}(t,s) + B Y_h^{A,B,C}(t-h,s) \right] g_1(s)\, ds + g_1(t) \\ &= A y_p(t) + B y_p(t-h) + f(t), \end{aligned}$$

which implies that $g_1(t) = f(t)$.

Step 3: Suppose that $g_{k-1}(t) = f(t)$ holds on the subintervals $(t_{k-1}, t_k]$, $k = 2, 3, \ldots$ Then, for any $t_k < t \le t_{k+1}$, we have

$$y_p(t) = \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s) f(s)\, ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s) g_k(s)\, ds.$$

We differentiated $y_p(t)$ again to obtain

$$\begin{aligned} y_p'(t) &= A y_p(t) + B \left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t-h,s) f(s)\, ds \right] \\ &\quad + \int_{t_k}^{t-h} Y_h^{A,B,C}(t-h,s) g_k(s)\, ds + g_k(t) \\ &= A y_p(t) + B y_p(t-h) + f(t). \end{aligned}$$

It follows that $g_k(t) = f(t)$.

According to the mathematical induction, we obtained $g_k(t) = f(t)$, $k = 0, 1, 2, \ldots$ Thus, the formula (13) is derived. □

Combining Theorems 1 and 2, we obtained the following representation formula:

Theorem 3. *Let $\varphi \in C^1([-h,0], \mathbb{R}^n)$, $f \in C([0,T], \mathbb{R}^n)$. Then, the solution of the initial value problem (1) has the form*

$$y(t) = \begin{cases} \varphi(t), & -h \le t \le 0 \\ Y_h^{A,B,C}(t,0)\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h) B\varphi(s)\, ds & \\ + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s) f(s)\, ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s) f(s)\, ds, & t \ge 0, \end{cases}$$

where k is the number of points t_j in the interval $(0,t)$.

4. Ulam–Hyers Stability

In this section, we discussed the stability of Ulam–Hyers for (2). In the stability of Ulam–Hyers, we compared the solution for the given differential equation with the solution of other differential inequality. The solution for the differential equation was the stability of Ulam–Hyers if it stayed close to a solution of other differential inequality in relation with the original equation. The stability of Ulam–Hyers did not imply the asymptotic stability in general.

For problem (2), for some $\varepsilon > 0$, we focus on the following inequalities:

$$\begin{aligned} \left\| y'(t) - Ay(t) - By(t-h) - f(t, y(t)) \right\| \le \varepsilon, \ 0 \le t \le T, \\ \left\| \Delta y(t_k) - C_k y(t_k) \right\| \le \varepsilon, \ k = 1, \ldots, p. \end{aligned} \tag{14}$$

Definition 6. *Equation (2) is Ulam–Hyers stable on* $[-h,T]$ *if for every* $y \in PC[-h,T] \cap PC^1[0,T]$ *satisfying (14), there exists a solution* $x \in PC([-h,T],\mathbb{R}^n) \cap PC^1([0,T],\mathbb{R}^n)$ *of (2) with* $\|y-x\|_{PC} \le L\varepsilon$*, for all* $t \in [-h,T]$.

Proposition 1. *A function* $y \in PC^1([0,T],\mathbb{R}^n)$ *satisfies (14) if and only if there is a function* $\phi \in PC([-h,T],\mathbb{R}^n)$ *and a sequence* g_k *depending on* y *such that*

(i) $\|\phi\|_{PC} \le \varepsilon$ *for all* $t \in [-h,T]$, $\|g_k\| \le \varepsilon$ *for all* $k = 1,...,p$;
(ii) $y'(t) = Ay(t) + By(t-h) + f(t,y(t)) + \phi(t),\ \ 0 \le t \le T$;
(iii) $\Delta y(t_k) = C_k y(t_k) + g_k,\ \ k = 1,...,p.$

Lemma 5. *For* $s < t$*, we have*

$$\left\| X_h^{A,B}(t,s) \right\| \le e^{(\|A\|+\|B\|)(t-s)}.$$

Proof. For $k = 1$, we have

$$\begin{aligned}\|X_1(t,s+h)\| &\le \|B\| \int_{s+h}^{t} \left\| e^{A(t-r)} \right\| \left\| e^{A(r-h-s)} \right\| dr \le \|B\| \int_{s+h}^{t} e^{\|A\|(t-r)} e^{\|A\|(r-h-s)} dr \\ &\le \|B\| \int_{s+h}^{t} e^{\|A\|(t-h-s)} dr = \|B\| e^{\|A\|(t-h-s)} (t-h-s).\end{aligned}$$

For $k = 2$, we get

$$\begin{aligned}\|X_2(t,s+2h)\| &\le \int_{s+2h}^{t} \left\| e^{A(t-r)} \right\| \|B\| \|X_1(r-h,s+h)\| dr \\ &\le \int_{s+2h}^{t} e^{\|A\|(t-r)} \|B\| \|B\| e^{\|A\|(r-2h-s)} (r-2h-s) dr \\ &\le \|B\|^2 e^{\|A\|(t-2h-s)} \int_{s+2h}^{t} (r-2h-s) dr \\ &= \|B\|^2 e^{\|A\|(t-2h-s)} \frac{(t-2h-s)^2}{2}.\end{aligned}$$

By the mathematical induction assuming

$$\|X_{k-1}(t,s+(k-1)h)\| \le \|B\|^{k-1} e^{\|A\|(t-(k-1)h-s)} \frac{(t-(k-1)h-s)^k}{k!},$$

one can get

$$\begin{aligned}\|X_k(t,s+kh)\| &\le \int_{s+kh}^{t} \left\| e^{A(t-r)} \right\| \|B\| \|X_{k-1}(r-h,s+(k-1)h)\| dr \\ &\le \int_{s+kh}^{t} e^{\|A\|(t-r)} \|B\| \|B\|^{k-1} e^{\|A\|(r-(k-1)h-s)} \frac{(r-kh-s)^{k-1}}{(k-1)!} dr \\ &\le \|B\|^k e^{\|A\|(t-kh-s)} \frac{(t-kh-s)^k}{k!}.\end{aligned}$$

Thus, for $s + kh \leq t < s + (k+1)\,h$, we get

$$\begin{aligned}\left\| X_h^{A,B}(t,s) \right\| &\leq \sum_{j=0}^{k} \left\| X_j(t, s+jh) \right\| \\ &\leq \sum_{j=0}^{k} \|B\|^j e^{\|A\|(t-jh-s)} \frac{(t-jh-s)^j}{j!} \\ &= e^{\|A\|(t-kh-s)} \sum_{j=0}^{k} \|B\|^j \frac{(t-jh-s)^j}{j!} \\ &\leq e^{(\|A\|+\|B\|)(t-s)}.\end{aligned}$$

□

The impulsive delayed matrix exponential $Y_h^{A,B,C}(t,s)$ for the problem in Proposition 1 was defined as follows:

$$Y_h^{A,B,C}(t,s) := \begin{cases} \Theta, \quad t < s, \\ I, \; t = s, \\ X_h^{A,B}(t,s) + \sum\limits_{s<t_k<t} X_h^{A,B}(t,t_k)\left(C_k Y_h^{A,B,C}(t_k,s) + g_k\right). \end{cases}$$

Lemma 6. *For $s < t$, we have the following estimation:*

$$\left\| Y_h^{A,B,C}(t,s) \right\| \leq \prod_{s<t_k<t} \left(1 + \|g_k\| + \|C_k\|\right) e^{(\|A\|+\|B\|)(t-s)}. \tag{15}$$

Proof. Our proof is based on the mathematical induction. We may assume that $t_m < s \leq t_{m+1}$ and $t_{m+n} < t \leq t_{m+n+1}$ for some natural number n.

(i) $t_m < s < t \leq t_{m+1}$. By Lemma 5

$$\begin{aligned} Y_h^{A,B,C}(t,s) &= X_h^{A,B}(t,s), \\ \left\| Y_h^{A,B,C}(t,s) \right\| &\leq e^{(\|A\|+\|B\|)(t-s)}. \end{aligned}$$

(ii) $t_{m+1} < t \leq t_{m+2}:$ Then,

$$\begin{aligned} Y_h^{A,B,C}(t,s) &= X_h^{A,B}(t,s) + X_h^{A,B}(t,t_{m+1})\left(C_{m+1} Y_h^{A,B,C}(t_{m+1},s) + g_{m+1}\right), \\ \left\| Y_h^{A,B,C}(t,s) \right\| &\leq e^{(\|A\|+\|B\|)(t-s)} \\ &\quad + e^{(\|A\|+\|B\|)(t-t_{m+1})}\left(\|C_{m+1}\| e^{(\|A\|+\|B\|)(t_{m+1}-s)} + \|g_{m+1}\|\right) \\ &\leq \left(1 + \|C_{m+1}\|\right) e^{(\|A\|+\|B\|)(t-s)} + \|g_{m+1}\| e^{(\|A\|+\|B\|)(t-t_{m+1})} \\ &\leq \left(1 + \|g_{m+1}\| + \|C_{m+1}\|\right) e^{(\|A\|+\|B\|)(t-s)}. \end{aligned}$$

(iii) For $t_{m+2} < t \leq t_{m+3}$, we have

$$\begin{aligned} Y_h^{A,B,C}(t,s) = {} & X_h^{A,B}(t,s) + X_h^{A,B}(t,t_{m+1})\left(C_{m+1} Y_h^{A,B,C}(t_{m+1},s) + g_{m+1}\right) \\ & + X_h^{A,B}(t,t_{m+2})\left(C_{m+2} Y_h^{A,B,C}(t_{m+2},s) + g_{m+2}\right). \end{aligned}$$

Consequently,

$$\begin{aligned}\left\|Y_h^{A,B,C}(t,s)\right\| &\le e^{(\|A\|+\|B\|)(t-s)} + e^{(\|A\|+\|B\|)(t-t_{m+1})}\left(\|C_{m+1}\| e^{(\|A\|+\|B\|)(t_{m+1}-s)} + \|g_{m+1}\|\right)\\ &+e^{(\|A\|+\|B\|)(t-t_{m+2})}\left(\|C_{m+2}\|\left(1+\|g_{m+1}\|+\|C_{m+1}\|\right)e^{(\|A\|+\|B\|)(t_{m+2}-s)} + \|g_{m+2}\|\right)\\ &\le e^{(\|A\|+\|B\|)(t-s)}\left(\left(1+\|C_{m+1}\|+\|g_{m+1}\|\right)\right)\\ &+e^{(\|A\|+\|B\|)(t-s)}\left(\|C_{m+2}\|\left(1+\|g_{m+1}\|+\|C_{m+1}\|\right)+\|g_{m+2}\|\right)\\ &\le e^{(\|A\|+\|B\|)(t-s)}\left(1+\|g_{m+1}\|+\|C_{m+1}\|\right)\left(1+\|g_{m+2}\|+\|C_{m+2}\|\right).\end{aligned}$$

We may use the mathematical induction on n to get (15). □

Lemma 7. *Every $y \in PC([-h,T],\mathbb{R}^n)$ that satisfies (14) also satisfies the following inequality:*

$$\begin{aligned}&\Bigg\| y(t) - Y_h^{A,B,C}(t,0)\varphi(0) - \int_{-h}^{0} Y_h^{A,B,C}(t,s)B\varphi(s)\,ds\\ &-\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s) f(s)\,ds - \int_{t_k}^{t} Y_h^{A,B,C}(t,s) f(s,y(s))\,ds\Bigg\| \le C\varepsilon\end{aligned}$$

for all $t \in [0,T]$, where k is the number of points t_j in the interval $(0,t)$ and

$$C := \left(\frac{1}{\|A\|+\|B\|}\prod_{0<t_k<T}\left(1+\|g_k\|+\|C_k\|\right)\left(e^{(\|A\|+\|B\|)T}-1\right)+\sum_{j=0}^{k-1} e^{(\|A\|+\|B\|)(t-t_j)}\right). \tag{16}$$

Proof. If $y \in PC([-h,T],\mathbb{R}^n)$ satisfies (14), then, by Proposition 1, we have

$$\begin{aligned}&\|\phi\|_{PC} \le \varepsilon \text{ for all } t\in[0,T],\ \|g_k\| \le \varepsilon \text{ for all } k = 1,...,p;\\ &y'(t) = Ay(t) + By(t-h) + f(t,y(t)) + \phi(t),\ \ 0\le t\le T;\\ &\Delta y(t_k) = C_k y(t_k) + g_k,\quad k = 1,\ldots,p.\end{aligned}$$

Then, by Theorem 3, we have the following representation formula for the above problem:

$$\begin{aligned}y(t) &= Y_h^{A,B,C}(t,0)\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)B\varphi(s)\,ds + \sum_{j=0}^{k} X_h^{A,B}(t,t_j)\,g_j\\ &+\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)\left[f(s,y(s))+\phi(s)\right]ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\left[f(s,y(s))+\phi(s)\right]ds,\ t\in(t_k,t_{k+1}].\end{aligned}$$

It follows that

$$\begin{aligned}
&\left\| y(t) - Y_h^{A,B,C}(t,0)\,\varphi(0) - \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)\,B\varphi(s)\,ds \right.\\
&\left. - \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)\,f(s,y(s))\,ds - \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\,f(s,y(s))\,ds \right\| \\
&\leq \int_0^t \left\| Y_h^{A,B,C}(t,s) \right\| \|\phi(s)\|\,ds + \sum_{j=0}^{k-1} \left\| X_h^{A,B}(t,t_j) \right\| \|g_j\| \\
&\leq \int_0^t \prod_{s<t_k<t}(1+\|g_k\|+\|C_k\|)\,e^{(\|A\|+\|B\|)(t-s)}ds\,\|\phi\|_{PC} + \sum_{j=0}^{k-1} e^{(\|A\|+\|B\|)(t-t_j)}\,\|g_j\| \\
&\leq \left(\frac{1}{\|A\|+\|B\|}\prod_{0<t_k<T}(1+\|g_k\|+\|C_k\|)\left(e^{(\|A\|+\|B\|)T}-1\right) + \sum_{j=0}^{k-1} e^{(\|A\|+\|B\|)(t-t_j)} \right)\varepsilon.
\end{aligned}$$

□

Now, we are able to present our second main result on the stability of Ulam–Hyers.

Theorem 4. *If $f:[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition: there exists $L_f > 0$ such that, for all $(t,y_1),(t,y_2) \in [0,T]\times\mathbb{R}^n$*

$$\|f(t,y_1) - f(t,y_2)\| \leq L_f\,\|y_1 - y_2\|\,.$$

Then,

(i) Equation (2) has a unique solution in $PC([-h,T],\mathbb{R}^n) \cap PC^1([-h,T],\mathbb{R}^n)$;
(ii) Equation (2) is stable in the sense of Ulam–Hyers.

Proof. We define

$$\begin{aligned}
\Pi y(t) &= Y_h^{A,B,C}(t,0)\,\varphi(0) + \int_{-h}^{0} Y_h^{A,B,C}(t,s+h)\,B\varphi(s)\,ds \\
&+ \sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} Y_h^{A,B,C}(t,s)\,f(s,y(s))\,ds + \int_{t_k}^{t} Y_h^{A,B,C}(t,s)\,f(s,y(s))\,ds
\end{aligned}$$

on the space $PC([-h,T],\mathbb{R}^n)$. We applied the contraction mapping theorem to show that Π has a unique fixed point. At first glance, it seems natural to use the supremum norm. However, the choice of supremum norm only leads us to a local solution defined in the subinterval of $[-h,T]$. The idea was to use the weighted supremum norm

$$\|y\|_\delta := \sup\left\{ e^{-\delta t}\,\|y(t)\| : -h \leq t \leq T \right\}$$

on $PC([-h,T],\mathbb{R}^n)$. Observe that $PC([-h,T],\mathbb{R}^n))$ is a Banach space with this norm since it is equivalent to the supremum norm.

(i) We showed that Π is a contraction on $PC([-h,T],\mathbb{R}^n)$. Indeed, for any $y, x \in PC([-h,T],\mathbb{R}^n)$, we have

$$\begin{aligned}
&e^{-\delta t}\left\|\Pi x(t)-\Pi y(t)\right\| \\
&\leq e^{-\delta t}\int_0^t \left\|Y_h^{A,B,C}(t,s)\right\| e^{\delta s}e^{-\delta s}\left\|f(s,x(s))-f(s,y(s))\right\| ds \\
&\leq e^{-\delta t}\int_0^t e^{\delta s}\left\|Y_h^{A,B,C}(t,s)\right\| dsL_f\left\|x-y\right\|_\delta \\
&\leq \prod_{s<t_k<T}(1+\|C_k\|)\int_0^t e^{(\|A\|+\|B\|-\delta)(t-s)}dsL_f\left\|x-y\right\|_\delta \\
&= \frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}(1+\|C_k\|)\left(1-e^{(\|A\|+\|B\|-\delta)T}\right)\left\|x-y\right\|_\delta .
\end{aligned} \tag{17}$$

Taking supremum over $[0,T]$, we get

$$\left\|\Pi x-\Pi y\right\|_\delta \leq \frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}(1+\|C_k\|)\left(1-e^{(\|A\|+\|B\|-\delta)T}\right)\left\|x-y\right\|_\delta .$$

We can choose $\delta > \|A\|+\|B\|$ so that the coefficient of $\|x-y\|_\delta$ become strictly less that one. Hence, Π is a contractive operator and, by the Banach contraction principle, Π is a unique fixed point in $PC([-h,T],\mathbb{R}^n)$, and Equation (2) has a unique solution.

(ii) Let $y \in PC([-h,T],\mathbb{R}^n)$ be a solution (14), and let x be a unique solution of (2). We see that $\|y(t)-x(t)\| = 0$ for $-h \leq t \leq 0$. For $t \in [0,T]$, we have

$$\begin{aligned}
&\|y(t)-x(t)\| = \|y(t)-\Pi x(t)\| \\
&\leq \|y(t)-\Pi y(t)\| + \|\Pi y(t)-\Pi x(t)\| .
\end{aligned}$$

Now, we use Lemma 7 and inequality (17) to get

$$e^{-\delta t}\|y(t)-x(t)\| \leq C\varepsilon + \frac{1}{\delta-\|A\|-\|B\|}\prod_{s<t_k<T}(1+\|C_k\|)\left(1-e^{(\|A\|+\|B\|-\delta)T}\right)\left\|x-y\right\|_\delta ,$$

where C is defined by (16). Consequently,

$$\|x-y\|_\delta \leq \frac{C}{1-\frac{1}{\delta-\|A\|-\|B\|}\prod\limits_{s<t_k<T}(1+\|C_k\|)\left(1-e^{(\|A\|+\|B\|-\delta)T}\right)}\varepsilon.$$

Hence, Equation (2) is Ulam–Hyers stable. □

5. Existence Results

Our next result is based on the Schaefer's fixed point theorem. For obtaining the desired results, we assume the following:

(H_1) The function $f: J\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous.

(H_2) There exists a constant $M_f > 0$ such that

$$\|f(t,y)\| \leq M_f(1+\|y\|), \quad \text{for } t\in J \text{ and } y \in \mathbb{R}^n.$$

Theorem 5. *If the assumptions (H_1) and (H_2) are satisfied, then problem (2) has at least one solution.*

Proof. Consider the operator Π defined in Theorem 4. We used Schaefer's fixed point theorem to show that Π has a fixed point. The proof is split into four steps.

Step 1. Π is continuous.

Take a sequence $\{y_n\} \subset \mathfrak{P}$, such that y_n converges to $y \in \mathfrak{P}$ as $n \to \infty$. Then, for $t \in J_m$, we have

$$\begin{aligned}
&\|(\Pi y_n)(t) - (\Pi y)(t)\| \\
&\le \int_{t_m}^{t} \left\| Y_h^{A,B,C}(t,s) \right\| \|f(s, y_n(s)) - f(s, y(s))\| \, ds \\
&\le \int_{0}^{T} \left\| Y_h^{A,B,C}(T,s) \right\| \|f(s, y_n(s)) - f(s, y(s))\| \, ds.
\end{aligned}$$

As a consequence of the Lebesgue dominated convergent theorem, the right-hand side of the above inequality tends to zero as $n \to \infty$, hence

$$\|(\Pi y_n)(t) - (\Pi y)(t)\| \to 0 \text{ as } n \to \infty,$$

which implies that

$$\|\Pi y_n - \Pi y\|_{PC} \to 0 \text{ as } n \to \infty.$$

Thus, T is continuous on J.

Step 2. Π maps bounded sets into bounded sets in $\mathfrak{P}$.

In fact, we just need to show that, for any positive constant r_1, there existed a constant $r_2 > 0$ such that, for each

$$y \in B_{r_1} := \left\{ y \in \mathfrak{P} : \|y\|_{PC} \le r_1 \right\},$$

we have $\|\Pi y\|_{PC} \le r_2$. For $t \in J_m$, $m = 0, 1, ..., p$, we have

$$\begin{aligned}
\|\Pi y(t)\| \le& \left\| Y_h^{A,B,C}(t,0) \right\| \|\varphi(0)\| + \int_{-h}^{0} \left\| Y_h^{A,B,C}(t, s+h) \right\| \|B\| \|\varphi(s)\| \, ds \\
&+ \int_{t_m}^{t} \left\| Y_h^{A,B,C}(t,s) \right\| \|f(s, y(s))\| \, ds \\
&\le C_0 + \left\| Y_h^{A,B,C}(T,0) \right\| M_f T \left(1 + \|y\|_{PC}\right) \\
&\le C_0 + \left\| Y_h^{A,B,C}(T,0) \right\| M_f T (1 + r_1) := r_2,
\end{aligned}$$

which implies that

$$\|\Pi y\|_{PC} \le r_2.$$

Step 3. Π maps bounded set into equicontinuous set of $\mathfrak{P}$.

Let $t_1, t_2 \in J_m$, $m = 0, 1, ..., p$, with $t_1 < t_2$ and B_{r_1} be a ball as in the second step. Then, for $y \in \mathfrak{P}$, we have

$$\begin{aligned}
\|\Pi y(t_2) - \Pi y(t_1)\| \le& \left\| Y_h^{A,B,C}(t_2,0) - Y_h^{A,B,C}(t_1,0) \right\| \|\varphi(0)\| \\
&+ \int_{-h}^{0} \left\| Y_h^{A,B,C}(t_2, s+h) - Y_h^{A,B,C}(t_1, s+h) \right\| \|B\| \|\varphi(s)\| \, ds \\
&+ \int_{t_m}^{t_1} \left\| Y_h^{A,B,C}(t_2,s) - Y_h^{A,B,C}(t_1,s) \right\| \|f(s, y(s))\| \, ds \\
&+ \int_{t_1}^{t_2} \left\| Y_h^{A,B,C}(t_2,s) \right\| \|f(s, y(s))\| \, ds.
\end{aligned}$$

We saw that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$, since $Y_h^{A,B}(t,s)$ is continuous in $t \in J_m$ and f is bounded on B_{r_1}.

We can conclude that Π is completely continuous from Step 1–Step 3 with the Arzela–Ascoli theorem.

Step 4. A priori bound.

Now, in the final step, we showed that the set defined by

$$W = \{y \in \mathfrak{P} : y = \lambda \Pi (y) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $y \in W$, the for some $0 < \lambda < 1$, $y = \lambda \Pi (y)$. Therefore, for $t \in J$, as in Step 2, we have

$$\|y (t)\| = \lambda \|\Pi y (t)\| \leq C_0 + \left\|Y_h^{A,B,C} (T, 0)\right\| M_f T + \left\|Y_h^{A,B,C} (T, 0)\right\| M_f \int_0^t \|y (s)\| ds.$$

Gronwall's inequality yields

$$\|y (t)\| \leq C_0 + \left\|Y_h^{A,B,C} (T, 0)\right\| M_f T \exp \left(\left\|Y_h^{A,B,C} (T, 0)\right\| M_f T\right) < \infty.$$

Then, the set W is bounded.

Thus, using Schaefer's fixed point theorem, we concluded that Π has a fixed point, which is the corresponding solution of the proposed problem (2). □

6. Illustrative Examples

Example 1. *Consider the linear problem (1):*

$$\begin{cases} y' (t) = Ay (t) + By (t - h) + f (t), \quad t \in [0, T], \ h > 0, \ t \neq t_k, \\ \Delta y (t_k) = y \left(t_k^+\right) - y \left(t_k^-\right) = C_k y (t_k), \quad k = 1, 2, ...p, \\ y (t) = \varphi (t), \quad -h \leq t \leq 0, \end{cases}$$

where $A, B, C_k \in \mathbb{R}^{n \times n}$ are constant matrices, $\varphi \in C^1 ([-h, 0], \mathbb{R}^n)$, $f \in C ([0, T], \mathbb{R}^n)$, $\{t_k\}$.satisfies $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$. This problem satisfies the conditions of Theorem 4 and this linear impulsive system is stable in Ulam–Hyers sense.

Example 2. *Consider (2) with $h = 0.2$*

$$A = \begin{pmatrix} -0.3 & 1 \\ 0 & -0.3 \end{pmatrix}, \ B = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{pmatrix}, \ C_j = \begin{pmatrix} 1.2 & 0.5 \\ 0.2 & 1.5 \end{pmatrix}, \ j = 1, 2, ...$$

$$\phi(t) = \begin{pmatrix} e^{-3} \\ e^{-4} \end{pmatrix}, \quad f(t, x) = \begin{pmatrix} 0.25 \sin(x_1) \\ 0.25 \sin(x_2) \end{pmatrix}$$

and $AB \neq BA$, $AC_j \neq C_j A$ and $BC_j \neq C_j B$. Obviously, f satisfies the Lipschitz condition $L_f = 0.25 > 0$, the conditions of Theorem 4 are satisfied and Equation (2) has a uniqueness solution in $PC[-h, 1] \cap PC^1[0, 1]$ which is Ulam–Hyers stable on $[-h, 1]$.

Example 3. *We consider the following fractional problem:*

$$\begin{cases} y'(t) = \begin{pmatrix} -60 & 0 \\ 0 & -5.5 \end{pmatrix} y(t) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} y(t-h) + f(t, y(t)), & t \in [0,1], \; h = 0.2, \; t \neq t_k, \\ \Delta y(t_k) = y(t_k^+) - y(t_k^-) = \begin{pmatrix} 2 + \frac{1}{k} & 0 \\ 1 & 2 \end{pmatrix} y(t_k), & k = 1, 2, \ldots 4, \\ y(t) = \begin{pmatrix} e^{-3} \\ e^{-4} \end{pmatrix}, & -h \leq t \leq 0, \end{cases}$$

Obviously, A, B, and C_j are mutually non-commutative

$$AB \neq BA, AC_j \neq C_jA, BC_j \neq C_jB, \quad j = 1, 2.$$

Assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is any continuous function satisfying (H_2) . Then, by Theorem 5, Equation (2) has al least one solution on $[-h, 1]$.

7. Conclusions

The main contribution of this paper is to introduce an impulsive delayed matrix exponential for non-permutable matrices and use it to construct explicit solutions for the impulsive delay systems with linear parts defined by non-permutable matrices. We applied fixed point methods to establish existence, uniqueness, and the stability of Ulam–Hyers for the impulsive delay system. The study on representation and stability of delay differential systems with impulses provides a potential for the future research on fractional impulsive delay systems, on fractional multiple delay impulsive problems, or in problems with delayed nonlinear terms.

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