mathematics

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p-Laplacian Equations in $\mathbb{R}^N_+$ with Critical Boundary Nonlinearity

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Received: 8 July 2020; Accepted: 31 August 2020; Published: 7 September 2020

Abstract: In this paper, we consider the following p-Laplacian equation in $\mathbb{R}^N_+$ with critical boundary nonlinearity. The existence of infinitely many solutions of the equation is proved via the truncation method.

Keywords: p-Laplacian equation; critical boundary nonlinearity; multiple solutions; the truncation method

1. Introduction

In this paper, we consider the following p-Laplacian equation in $\mathbb{R}^N_+$ with critical boundary nonlinearity

$$\begin{cases} -\Delta_p u = 0, & \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} + |u|^{p-2}u = |u|^{\overline{p}-2}u + \mu|u|^{q-2}u, & \text{on } \mathbb{R}^{N-1} = \partial\mathbb{R}^N_+, \end{cases} \tag{1}$$

where $1 < p < N$, $\max\{p, \overline{p}-1\} < q < \overline{p} = \frac{(N-1)p}{N-p}$, $\mu > 0$ and Δ_p is the p-Laplacian operator, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We are looking for axial solutions of the Equation (1) that are solutions of the form $u(x) = u(|y|, s)$, where we denote $x \in \overline{\mathbb{R}^N_+}$ by $x = (y, s) \in \mathbb{R}^{N-1} \times [0, \infty)$ and we identify $\mathbb{R}^{N-1} = \partial\mathbb{R}^N_+$, $y = (y, 0)$ for $y \in \mathbb{R}^{N-1}$ if there is no confusion.

Introduce in $C_0^\infty(\overline{\mathbb{R}^N_+})$ a norm by

$$\|\varphi\| = \left(\int_{\mathbb{R}^N_+} |\nabla\varphi|^p\,dx + \int_{\mathbb{R}^{N-1}} |\varphi|^p\,dy\right)^{\frac{1}{p}}.$$

Let W be the completion of $C_0^\infty(\overline{\mathbb{R}^N_+})$ with respect to this norm and W_r be the subspace of W of axial functions, that is,

$$W_r = \{u \mid u \in W,\ u(x) = u(|y|, s),\ x = (y, s) \in \mathbb{R}^N_+\}.$$

The problem (1) has a variational structure given by the functional

$$I(u) = \frac{1}{p}\left(\int_{\mathbb{R}^N_+} |\nabla u|^p\,dx + \int_{\mathbb{R}^{N-1}} |u|^p\,dy\right) - \frac{1}{\overline{p}}\int_{\mathbb{R}^{N-1}} |u|^{\overline{p}}dy - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} |u|^q\,dy,\ u \in W_r.$$

Notice that $\overline{p} = \frac{(N-1)p}{N-p}$ is the critical exponent for the Sobolev imbedding from $W^{1,p}(\mathbb{R}^N_+)$ to $L^q(\mathbb{R}^{N-1})$, $p \le q \le \overline{p}$. Moreover, the imbedding from W_r to $L^q(\mathbb{R}^{N-1})$ is continuous for $p \le q \le \overline{p}$ and compact only for $p < q < \overline{p}$ due to the dilations. Therefore, the Palais–Smale condition is not satisfied by the functional I and the problem (1) lacks the necessary compactness property. Since the pioneering work of Brezis and Nirenberg [1], significant progress has been made in recent decades for these kinds of problems lacking compactness. In particular, the authors of [2] dealt with the Laplacian equation with critical growth in the bounded domain

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u, & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a regular bounded domain, and $2^* = \frac{2N}{N-2}$. While the authors of [3] considered the Laplacian equation with subcritical nonlinear term in the whole space $\mathbb{R}^N$

$$\begin{cases} -\Delta u + a(x)u = |u|^{q-2}u, & \text{in } \mathbb{R}^N, \\ u(x) \to 0, \ \text{as } |x| \to \infty, \end{cases} \tag{3}$$

where $2 < q < 2^*$ and $a(x)$ is the potential function. As to the p-Laplacian equation, there is a lot of significant work, whether in the field of ordinary differential equations [4–6] or partial differential equations [7–9], the authors of [7] considered

$$\begin{cases} -\Delta_p u + a(x)u = |u|^{p^*-2}u + \lambda u, & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $p^* = \frac{Np}{N-p}$. All of these authors found the solutions as limits of approximated equations with subcritical growth in bounded domains. The lack of compactness due to dilations (in the case (2) and (4)) and shifts (in the case (3)) does not allow for deducing that a sequence of approximate solutions must have a convergent subsequence, but the fact that they solve the approximated problems gives, with use of a local Pohožaev identity, some extra estimates which lead to a proof of desired compactness.

In the Existing literature, some researchers considered the existence of finite multiple solutions [10,11]. While the subcritical problems in bounded domains have infinitely many solutions. In order to show the existence of multiple solutions of the original problems, we need to check that multiple solutions of approximated problems do not converge to the same solution of the limit problems. This is hard work. In both [2,3], some estimates on the Morse index are employed, which has been used as one of the possible devices to distinguish the limit of the multiple approximate solutions by their original variational characterization. For general p-Laplacian equations, we have no information on the Morse index; therefore, the approach in this last step in [2,3] can not be extended in a straightforward way to problems involving the p-Laplacian operator. Here, we will use the truncation method, as we did in [8,9]. First, we consider some truncated problems, the solutions of which will be used as approximate solutions. By a concentration–compactness analysis, similar to that in [2,3,7], in particular with use of a local Pohožaev identity, the theorem of convergence of approximate solutions is proved. We show that, by a careful choice of the approximate nonlinear terms, the approximated problems and the original problem share more and more solutions, as the approximation parameter tends to zero. For more references, we refer the readers to [12–18].

Let us describe the truncation method in more details. Let $\psi \in C_0^\infty(\mathbb{R}, [0,1])$ be an even function such that $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$ and ψ is decreasing in $[1,2]$. Define the auxiliary functions for $\lambda \in (0,1]$, $s \in \mathbb{R}$

$$\begin{aligned} &b_\lambda(s) = \psi(\lambda s), \quad m_\lambda(s) = \int_0^s b_\lambda(\tau)\, d\tau \\ &F_\lambda(s) = \frac{1}{p}|s|^q |m_\lambda(s)|^{\overline{p}-q}, \quad f_\lambda(s) = \frac{d}{ds}F_\lambda(s)\,. \end{aligned} \tag{5}$$

Instead of the original problem (1), we consider the truncated problem

$$\begin{cases} -\Delta_p u = 0, & \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} + |u|^{p-2}u = f_\lambda(u) + \mu|u|^{q-2}u, & \text{on } \mathbb{R}^{N-1}. \end{cases} \tag{6}$$

In addition, the problem (6) has a variational structure given by the functional

$$I_\lambda(u) = \frac{1}{p}\left(\int_{\mathbb{R}^N_+} |\nabla u|^p\, dx + \int_{\mathbb{R}^{N-1}} |u|^p\, dy\right) - \int_{\mathbb{R}^{N-1}} F_\lambda(u)\, dy - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} |u|^q\, dy, \ \ u \in W_r.$$

Notice that the functional $I_\lambda, \lambda > 0$ is subcritical at the infinity and the imbedding from W_r to $L^q(\mathbb{R}^{N-1}), p < q < \overline{p}$ is compact. Therefore, the functional $I_\lambda, \lambda > 0$ satisfies the Palais–Smale condition.

Here are our main results.

Theorem 1. *Assume* $\max\{p, \overline{p}-1\} < q < \overline{p}$. *Given* $L > 0$, *there exists* $\nu = \nu(L)$, *independent of* λ, *such that if* $u \in W_r$, $DI_\lambda(u) = 0$ *and* $I_\lambda(u) \leq L$, *then it holds that*

$$|u(y)| \leq \frac{1}{\nu}, \quad \textit{for } y \in \mathbb{R}^{N-1} = \partial\mathbb{R}^N_+.$$

Consequently, if $\lambda < \nu$, *then* u *is a solution of the problem* (1).

Theorem 2. *Assume* $\max\{p, \overline{p}-1\} < q < \overline{p}$. *Then, the problem* (1) *has infinitely many axial solutions.*

Throughout the paper, we use the following notations: we use $\|\cdot\|$ and $|\cdot|_q$ to denote the norms of W and $L^q(\mathbb{R}^{N-1})$, respectively, $\rightharpoonup$ and $\to$ to denote the weak and the strong convergence, respectively. In addition, we use the notations $B^+_\delta(x_0) = \{x \,|\, x \in \mathbb{R}^N_+, |x - x_0| < \delta\}$, $D_\delta(y_0) = \{y \,|\, y \in \mathbb{R}^{N-1}, |y - y_0| < \delta\}$, $B^+_\delta = B^+_\delta(0)$, $D_\delta = D_\delta(0)$.

The paper is organized as follows. In Section 2, we do the concentration–compactness analysis of the approximate solution sequence and prove the convergence Theorem 1. In Section 3, we construct a sequence of critical values of the truncated functionals by the symmetric mountain pass lemma. Finally, we prove the existence Theorem 2 by showing that approximated solutions are also solutions of the original problem for a sufficiently small parameter.

2. Concentration–Compactness Analysis

2.1. The Profile Decomposition

In this section, we analyze the concentration behavior for the solutions of the problem (6) as $\lambda \to 0$ and prove Theorem 1 . First, we list the properties of the auxiliary functions, defined in (5) in the following lemma.

Lemma 1. *It holds that for* $s \in \mathbb{R}$

(*a*) $0 \leq b_\lambda(s) \leq 1$.

(*b*) $sm_\lambda(s) \geq 0,\ 0 \leq \frac{sb_\lambda(s)}{m_\lambda(s)} \leq 1$.

(*c*) $m_\lambda(s) = s$ *for* $|s| \leq \frac{1}{\lambda}$.

(*d*) $\min\{|s|, \frac{1}{\lambda}\} \leq |m_\lambda(s)| \leq \min\{|s|, \frac{2}{\lambda}\}$.

(*e*) $|f_\lambda(s)| \leq |s|^{q-1}|m_\lambda(s)|^{\overline{p}-q} \leq |s|^{\overline{p}-1}$.

(*f*) $\frac{1}{q}sf_\lambda(s) - F_\lambda(s) = \left(\frac{1}{q} - \frac{1}{\overline{p}}\right)|s|^{q+1}|m_\lambda(s)|^{\overline{p}-q-1}b_\lambda(s) \geq 0.$

(*g*) $F_\lambda(s) - \frac{1}{\overline{p}}sf_\lambda(s) = \left(\frac{1}{\overline{p}} - \frac{q}{\overline{p}^2}\right)|s|^q|m_\lambda(s)|^{\overline{p}-q-1}\left(1 - \frac{sb_\lambda(s)}{m_\lambda(s)}\right) \geq 0.$

Proof. The proof is straightforward. We verify (e)–(g). By the definition of F_λ and f_λ, we have

$$f_\lambda(s) = \frac{d}{ds}F_\lambda(s) = \frac{q}{\overline{p}}|s|^{q-2}s|m_\lambda(s)|^{\overline{p}-q} + \frac{\overline{p}-q}{\overline{p}}|s|^q|m_\lambda(s)|^{\overline{p}-q-2}m_\lambda(s)b_\lambda(s)\,. \tag{7}$$

(f) and (g) follow from (7), and (e) follows from (7) and (a), (d) of this lemma. □

Lemma 2. *Let $\lambda_n \geq 0$, $u_n \in W_r$ such that $DI_{\lambda_n}(u_n) = 0$, $I_{\lambda_n}(u_n) \leq L$. Then, $\{u_n\}$ is bounded in W_r.*

Proof. By Lemma 1 (f), we have

$$\begin{aligned} L \geq I_{\lambda_n}(u_n) &= I_{\lambda_n}(u_n) - \frac{1}{q}\langle DI_{\lambda_n}(u_n),\, u_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\left(\int_{\mathbb{R}^N_+}|\nabla u_n|^p\,dx + \int_{\mathbb{R}^{N-1}}|u_n|^p\,dy\right) + \int_{\mathbb{R}^{N-1}}\left(\frac{1}{q}u_n f_{\lambda_n}(u_n) - F_{\lambda_n}(u_n)\right)dx \\ &\geq \left(\frac{1}{p}-\frac{1}{q}\right)\left(\int_{\mathbb{R}^N_+}|\nabla u_n|^p\,dx + \int_{\mathbb{R}^{N-1}}|u_n|^p\,dy\right) = \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p\,. \end{aligned}$$

Hence, $\{u_n\}$ is bounded in W_r. □

Let $\mathcal{D} = \mathcal{D}^p(\overline{\mathbb{R}^N_+})$ be the completion of $C_0^\infty(\overline{\mathbb{R}^N_+})$ with respect to the norm

$$\|\varphi\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^N_+}|\nabla\varphi|^p\,dx\right)^{\frac{1}{p}}$$

and $\mathcal{D}_r$ be the subspace of $\mathcal{D}$ of axial functions,

$$\mathcal{D}_r = \{u \,|\, u \in \mathcal{D}, u(x) = u(|y|,s),\; x=(y,s) \in \overline{\mathbb{R}^N_+} = \mathbb{R}^{N-1}\times[0,\infty)\}\,.$$

Let D be the dilation group

$$D = \{g_\sigma \,|\, g_\sigma u(x) = \sigma^{\frac{N-p}{p}}u(\sigma x),\; x \in \overline{\mathbb{R}^N_+},\; \sigma > 0\}\,. \tag{8}$$

Notice that the operator g_σ of D is an isometry in both $\mathcal{D}$ and $L^{\overline{p}}(\mathbb{R}^{N-1})$. The imbedding from $\mathcal{D}_r$ to $L^{\overline{p}}(\mathbb{R}^{N-1})$ is compact with respect to the group D that is a sequence $\{u_n\}$ of $\mathcal{D}_r$, satisfying $g_{\sigma_n}u_n \rightharpoonup 0$ in $\mathcal{D}_r$ for any sequence $\{g_{\sigma_n}\}$ of D, denoted by $u_n \overset{D}{\rightharpoonup} 0$ in $\mathcal{D}_r$, must converge to zero in $L^{\overline{p}}(\mathbb{R}^{N-1})$.

Now, let u_n be a bounded sequence of W_r. By [19,20], we have the following profile decomposition:

$$u_n = u + \sum_{k\in\Lambda} g_{\sigma_{n,k}}U_k + r_n, \tag{9}$$

where $u \in W_r$, $U_k \in \mathcal{D}_r$, $r_n \in \mathcal{D}_r$, $\sigma_{n,k} \in (0,\infty)$ and Λ is an index set, satisfying

(a) $u_n \rightharpoonup u$ in W_r, $g^{-1}_{\sigma_{n,k}}u_n \rightharpoonup U_k$ in $\mathcal{D}_r$ as $n\to\infty$, $k\in\Lambda$.

(b) $\sigma_{n,k} \to +\infty$, $\frac{\sigma_{n,k}}{\sigma_{n,l}} + \frac{\sigma_{n,l}}{\sigma_{n,k}} \to +\infty$, as $n\to\infty$, $k,l\in\Lambda$, $k\neq l$.

(c) $\|u\|^p_{\mathcal{D}} + \sum_{k\in\Lambda}\|U_k\|^p_{\mathcal{D}} \leq \liminf_{n\to\infty}\|u_n\|^p_{\mathcal{D}}$.

(d) $r_n \rightharpoonup 0$ in $\mathcal{D}_r$ as $n\to\infty$, consequently $r_n \to 0$ in $L^{\overline{p}}(\mathbb{R}^{N-1})$ as $n\to\infty$.

We refer to [19,20] for general concepts of compactness and the profile decomposition and relevant results. For reader's convenience, we consider the compactness of the imbedding from $\mathcal{D}_r$ to $L^{\overline{p}}(\mathbb{R}^{N-1})$ with respect to the dilation group D.

Lemma 3. *Assume $\lambda_n > 0$, $\lambda_n \to 0$ as $n\to\infty$, $u_n \in W_r$, $DI_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \leq L$. Assume that the profile decomposition (9) holds. Then,*

(1) $v = |u|$ *satisfies the inequality*

$$\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla \varphi\, dx + \int_{\mathbb{R}^{N-1}} v^{p-1}\varphi\, dy \le \int_{\mathbb{R}^{N-1}} v^{\overline{p}-1}\varphi\, dy + \mu \int_{\mathbb{R}^{N-1}} v^{q-1}\varphi\, dy, \qquad (10)$$

for $\varphi \ge 0$, $\varphi \in W_r$. *Consequently, for some* $c > 1$,

$$\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla \varphi\, dx \le c \int_{\mathbb{R}^{N-1}} v^{\overline{p}-1}\varphi\, dy, \;\; \text{for } \varphi \ge 0,\; \varphi \in W_r. \qquad (11)$$

(2) $V_k = |U_k|$ *satisfies the inequality*

$$\int_{\mathbb{R}^N_+} |\nabla V_k|^{p-2}\nabla V_k \nabla \varphi\, dx \le \int_{\mathbb{R}^{N-1}} V_k^{\overline{p}-1}\varphi\, dy \text{ for } \varphi \ge 0,\, \varphi \in \mathcal{D}_r\,. \qquad (12)$$

Proof. We prove the conclusion for the function V_k. u_n satisfies the equation in the weak form

$$\begin{aligned} &\int_{\mathbb{R}^N_+} |\nabla u_n|^{p-2}\nabla u_n \nabla \varphi\, \mathrm{d}x + \int_{\mathbb{R}^{N-1}} |u_n|^{p-2}u_n\varphi\, \mathrm{d}y \\ &= \int_{\mathbb{R}^{N-1}} f_{\lambda_n}(u_n)\varphi\, \mathrm{d}y + \mu \int_{\mathbb{R}^{N-1}} |u_n|^{q-2}u_n\varphi\, \mathrm{d}y, \quad \varphi \in W_r. \end{aligned} \qquad (13)$$

Denote $\widetilde{u}_n = g_{\sigma_{n,k}}^{-1}u_n$. For $\varphi \in W_r$, take $g_{\sigma_{n,k}}\varphi$ as a test function in (13). By a variable change, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N_+} |\nabla \widetilde{u}_n|^{p-2}\nabla \widetilde{u}_n \nabla \varphi\, dx + \sigma_{n,k}^{-(p-1)} \int_{\mathbb{R}^{N-1}} |\widetilde{u}_n|^{p-2}\widetilde{u}_n\varphi\, dy \\ &= \int_{\mathbb{R}^{N-1}} f_{\widetilde{\lambda}_n}(\widetilde{u}_n)\varphi\, dy + \mu\sigma_{n,k}^{\frac{N-p}{p}q-(N-1)} \int_{\mathbb{R}^{N-1}} |\widetilde{u}_n|^{q-2}\widetilde{u}_n\varphi\, dy,\;\; \varphi \in W_r \end{aligned} \qquad (14)$$

where $\widetilde{\lambda}_n = \lambda_n\sigma_{n,k}^{\frac{N-p}{p}}$. In the above, we have used the fact that

$$\nu^{-(\overline{p}-1)}f_\lambda(\nu s) = f_{\lambda\nu}(s),\;\; \lambda,\, \nu > 0,\, s \in \mathbb{R}$$

which can be proved by the very definition of the function f_λ.

Since $\int_{\mathbb{R}^{N-1}} |\widetilde{u}_n|^{\overline{p}}\, dy = \int_{\mathbb{R}^{N-1}} |u_n|^{\overline{p}}\, dy$ is bounded and $\widetilde{u}_n$ is axial, for any $y \in \mathbb{R}^{N-1} \setminus \{0\}$

$$\lim_{s\to 0} \limsup_{n\to\infty} \int_{D_y} |\widetilde{u}_n|^{\overline{p}}\, dy = 0\,.$$

Choose $\delta = \delta(y)$, independent of n, such that

$$\int_{D_{4\delta}(y)} |\widetilde{u}_n|^{\overline{p}}\, dy \le \frac{1}{2}\overline{S}_p^{\frac{N-1}{p-1}}$$

where $\overline{S}_p$ is the Sobolev constant of the imbedding $\mathcal{D} \hookrightarrow L^{\overline{p}}(\mathbb{R}^{N-1})$. By Lemma A4, $\widetilde{u}_n$ is uniformly bounded in $D_{2\delta}(y)$. Consequently, by Equation (14) and the following elementary inequality (15), $\widetilde{u}_n$ converges in $W^{1,p}(B_\delta(y))$ and in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N_+)$. The following inequality (15) is useful for problems involving the p-Laplacian operator [21]. There exists a constant c_p such that, for ξ, $\eta \in \mathbb{R}^N$,

$$\begin{aligned} &(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \ge c_p|\xi - \eta|^p, \text{ if } p \ge 2, \\ &(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \ge c_p|\xi - \eta| \cdot (|\xi|^p + |\eta|^p)^{\frac{p-2}{p}}, \text{ if } 1 < p < 2\,. \end{aligned} \qquad (15)$$

Let $\widetilde{v}_n = |\widetilde{u}_n|$, $\widetilde{v}_n$ converge to $V_k = |U_k|$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^N_+)$ and satisfy the inequality

$$\begin{aligned}&\int_{\mathbb{R}^N_+} |\nabla \widetilde{v}_n|^{p-2}\nabla \widetilde{v}_n \nabla \varphi\, dx + \sigma_{n,k}^{-(p-1)} \int_{\mathbb{R}^{N-1}} \widetilde{v}_n^{p-1} \varphi\, dy \\ \le &\int_{\mathbb{R}^{N-1}} \widetilde{v}_n^{\overline{p}-1}\varphi\, dy + \mu \sigma_{n,k}^{\frac{N-p}{p}q-(N-1)} \int_{\mathbb{R}^{N-1}} \widetilde{v}_n^{q-1}\varphi\, dy\end{aligned} \tag{16}$$

for $\varphi \in W_r,\ \varphi \ge 0$. Assume $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}_r$. Taking the limit $n \to \infty$ in (16), we obtain

$$\int_{\mathbb{R}^N_+} |\nabla V_k|^{p-2}\nabla V_k \nabla \varphi\, dx \le \int_{\mathbb{R}^{N-1}} V_k^{\overline{p}-1}\varphi\, dy\,. \tag{17}$$

By a density argument, (17) holds for $\varphi \in \mathcal{D}$, $\varphi \ge 0$. □

Lemma 4. *The index set Λ in the profile decomposition (9) is finite.*

Proof. By Lemma 3, V_k satisfies the inequality (12). Choose $\varphi = V_k$ in (12). By the Sobolev imbedding theorem

$$\int_{\mathbb{R}^N_+} |\nabla V_k|^p\, dx \le \int_{\mathbb{R}^{N-1}} V_k^{\overline{p}}\, dy \le \left(\bar{S}_p^{-1}\int_{\mathbb{R}^N_+} |\nabla V_k|^p\, dx\right)^{\frac{\overline{p}}{p}}$$

hence

$$\int_{\mathbb{R}^N_+} |\nabla U_k|^p\, dx = \int_{\mathbb{R}^N_+} |\nabla V_k|^p\, dx \ge \bar{S}_p^{\frac{N-1}{p-1}}\,.$$

By the property (3) of the decomposition (9), Λ is a finite set. □

2.2. Safe Regions

Assume the profile decomposition (9) with a finite index set Λ. Denote

$$\sigma_n = \min\{\sigma_{n,k}\,|\, k \in \Lambda\}$$

and define the so-called safe regions [2]

$$\begin{aligned}A_n^i &= \{x\,|\,x \in \mathbb{R}^N_+,\, i\sigma_n^{-\frac{1}{p}} < |x| < (7-i)\sigma_n^{-\frac{1}{p}}\},\\ T_n^i &= \{y\,|\,y \in \mathbb{R}^{N-1},\, i\sigma_n^{-\frac{1}{p}} < |y| < (7-i)\sigma_n^{-\frac{1}{p}}\},\ i = 1,2,3\,.\end{aligned} \tag{18}$$

For these regions, we have a good estimate.

Proposition 1. *There exists a constant c, independent of n, such that*

$$|u_n(x)| \le c \quad \text{for } x \in A_n^2 \cup T_n^2\,.$$

Corollary 1. *There exists a constant c, independent of n, such that*

$$\int_{A_n^3} |\nabla u_n|^p\, dx \le c\,.$$

In order to prove these estimates, we start with the following definition.

Definition 1. *Suppose $1 \le p_2 < \overline{p} < p_1$, $\sigma > 1$ and $\alpha > 0$. Consider the following system of inequality*

$$\begin{cases} |u_1|_{p_1} \le \alpha, \\ |u_2|_{p_2} \le \alpha\sigma^{\frac{N-1}{\overline{p}} - \frac{N-1}{p_2}} A \end{cases} . \tag{19}$$

Define the norm $|\cdot|_{p_1,p_2,\sigma}$ by

$$|u|_{p_1,p_2,\sigma} = \inf\{\alpha \,|\, \text{there exist } u_1,\, u_2 \text{ such that } |u| \le u_1 + u_2 \text{ and (19) holds}\}.$$

Proposition 2. *Assume $\lambda_n > 0$, $\lambda_n \to 0$ as $n \to \infty$, $u_n \in W_r$, $DI_{\lambda_n}(u_n) = 0$ and $I_{\lambda_n}(u_n) \le L$. Assume the profile decomposition (9) holds. Denote $\sigma_n = \min\{\sigma_{n,k} \,|\, k \in \Lambda\}$. Then, for any p_1, p_2 satisfying*

$$(1 - \frac{1}{p})\overline{p} < p_2 < \overline{p} < p_1,$$

there exists a constant $c = c(p_1,\, p_2)$ such that

$$|u_n|_{p_1,p_2,\sigma_n} \le c\,.$$

Proof. By Lemma 3, $v = |u|$ satisfies the inequality

$$\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla\varphi\, dx \le c\int_{\mathbb{R}^{N-1}} v^{\overline{p}-1}\varphi\, dy,\;\; \varphi \ge 0,\; \varphi \in W_r\,.$$

By Lemma A4, $u \in L^\infty(\mathbb{R}^{N-1})$, hence for $p_1 > \overline{p}$,

$$|u|_{p_1} \le |u|_\infty^{1-\frac{\overline{p}}{p_1}} |u|_{\overline{p}}^{\frac{\overline{p}}{p_1}} \le c\,. \tag{20}$$

By Lemma 3, $V_k = |U_k|$ satisfies the inequality

$$\int_{\mathbb{R}^N_+} |\nabla v|^{p-2}\nabla v \nabla\varphi\, dx \le \int_{\mathbb{R}^{N-1}} v^{\overline{p}-1}\varphi\, dy,\;\; \varphi \ge 0,\; \varphi \in W_r\,.$$

By Theorem 2.2 of [22], there exists a constant c such that

$$|U_k(y)| = V_k(y) \le c\Big(1 + |y|^{\frac{p}{p-1}}\Big)^{-\frac{N-p}{p}},\; y \in \mathbb{R}^{N-1}\,.$$

Hence, for $(1 - \frac{1}{p})\overline{p} < p_2 < \overline{p}$, we have

$$\begin{aligned} |g_{\sigma_{n,k}} U_k|_{p_2} &\le \left(\int_{\mathbb{R}^{N-1}} (\sigma_{n,k}^{\frac{N-p}{p}} (1 + |\sigma_{n,k} y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}})^{p_2}\, dy\right)^{\frac{1}{p_2}} \\ &= c\sigma_{n,k}^{\frac{N-p}{p} - \frac{N-1}{p_2}} \left(\int_{\mathbb{R}^{N-1}} (1 + |y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}p_2} dy\right)^{\frac{1}{p_2}} \\ &\le c\sigma_{n,k}^{\frac{N-1}{\overline{p}} - \frac{N-1}{p_2}} \le c\sigma_n^{\frac{N-1}{\overline{p}} - \frac{N-1}{p_2}}\,. \end{aligned} \tag{21}$$

By (20) and (21), we have

$$|u|_{p_1,p_2,\sigma_n} + \sum_{k\in\Lambda} |g_{\sigma_{n,k}} U_k|_{p_1,p_2,\sigma_n} \le c\,. \tag{22}$$

Define $w, W_k, R \in \mathcal{D}_r$ by

$$\begin{cases} -\Delta_p w = 0, & \text{in } \mathbb{R}^N_+, \\ |\nabla w|^{p-2} = w^{\overline{p}-1}, & \text{on } \mathbb{R}^{N-1}, \end{cases} \tag{23}$$

$$\begin{cases} -\Delta_p W_k = 0, & \text{in } \mathbb{R}^N_+, \\ |\nabla W_k|^{p-2}\dfrac{\partial W_k}{\partial n} = W_k^{\overline{p}-1}, & \text{on } \mathbb{R}^{N-1}, \end{cases} \tag{24}$$

$$\begin{cases} -\Delta_p R = 0, & \text{in } \mathbb{R}^N_+, \\ |\nabla R|^{p-2}\dfrac{\partial R}{\partial n} = |r_n|^{\overline{p}-1}, & \text{on } \mathbb{R}^{N-1}, \end{cases} \tag{25}$$

By the Wolff potential estimate([2], Corollary 4.13), we have

$$|u_n| = v_n \le c\big(w + \sum_{k\in\Lambda} W_k + R\big)\,. \tag{26}$$

By Lemma A3, for $\left(1-\frac{1}{p}\right)\overline{p} < p_2 < \overline{p} < p_1$, we have

$$\begin{aligned} |w|_{p_1,p_2,\sigma_n} &\le c|v^{\overline{p}-p}|^{\frac{1}{p-1}}_{\frac{N-1}{p-1}} \cdot |v|_{p_1,p_2,\sigma_n} \\ &\le c|u|^{\frac{p}{N-p}}_{\overline{p}} \cdot |u|_{p_1,p_2,\sigma_n} \le c|u|_{p_1,p_2,\sigma_n} \le c \end{aligned} \tag{27}$$

$$\begin{aligned} |W_k|_{p_1,p_2,\sigma_n} &\le c|g_{\sigma_{n,k}}U_k|^{\frac{p}{N-p}}_{\overline{p}}|g_{\sigma_{n,k}}U_k|_{p_1,p_2,\sigma_n} \\ &\le c|g_{\sigma_{n,k}}U_k|_{p_1,p_2,\sigma_n} \le c \end{aligned} \tag{28}$$

and

$$\begin{aligned} |R|_{p_1,p_2,\sigma_n} &\le c|r_n|^{\frac{p}{N-p}}_{\overline{p}}|r_n|_{p_1,p_2,\sigma_n} \\ &= o(1)|r_n|_{p_1,p_2,\sigma_n} \\ &\le o(1)\big(|u_n|_{p_1,p_2,\sigma_n} + |u|_{p_1,p_2,\sigma_n} + \sum_{k\in\Lambda}|g_{\sigma_{n,k}}U_k|_{p_1,p_2,\sigma_n}\big) \\ &= o(1) + o(1)|u_n|_{p_1,p_2,\sigma_n}\,. \end{aligned} \tag{29}$$

We have

$$\begin{aligned} |u_n|_{p_1,p_2,\sigma_n} &\le c\big(|w|_{p_1,p_2,\sigma_n} + \sum_{k\in\Lambda}|W_k|_{p_1,p_2,\sigma_n} + |r_n|_{p_1,p_2,\sigma_n}\big) \\ &\le c + o(1)|u_n|_{p_1,p_2,\sigma_n} \end{aligned}$$

and

$$|u_n|_{p_1,p_2,\sigma_n} \le c\,.$$

□

Lemma 5. *Assume $\lambda_n > 0$, $u_n \in W_r$, $DI_{\lambda_n}(u_n) \le L$. Assume the profile decomposition (9) holds. Then, for $\gamma \in \left(p-1, \frac{(p-1)p\overline{p}}{(p-1)p+\overline{p}}\right)$ there exists $c = c(\gamma)$, independent of n, such that*

$$\left(\gamma^{-N}\int_{B_r^+}|u_n|^\gamma\,dx + \gamma^{-N+1}\int_{D_r}|u_n|^\gamma\,dy\right)^{\frac{1}{\gamma}} \le c \quad \textit{for } \gamma \ge \sigma_n^{-\frac{1}{p}}\,.$$

Proof. By Lemma A6 for $\gamma < 1$, we have

$$
\begin{aligned}
&\left(\gamma^{-N}\int_{B_r^+}|u_n|^\gamma\,dx+\gamma^{-N+1}\int_{D_r}|u_n|^\gamma\,dy\right)^{\frac{1}{\gamma}}\\
\le& c\left(\int_{B_1^+}|u_n|^\gamma\,dx+\int_{D_1}|u_n|^\gamma\,dy\right)^{\frac{1}{\gamma}}+c\int_r^1\left(\int_{D_t}|u_n|^{\overline{p}-1}\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\\
\le& c+c\int_r^1\left(\int_{D_t}|u_n|^{\overline{p}-1}\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}.
\end{aligned}
\tag{30}
$$

By Proposition 2, we have $|u_n|_{p_1,p_2,\sigma_n}\le c$ for any p_1,p_2 such that $(1-\frac{1}{p})\overline{p}<p_2<\overline{p}<p_1$. Let $p_2=\overline{p}-1$, $p_1=N\overline{p}$. Choose v_1,v_2 such that $|u_n|\le v_1+v_2$, $|v_1|_{p_1}\le c$, $|v_2|_{p_2}\le c\sigma_n^{\frac{N-1}{\overline{p}}-\frac{N-1}{p_2}}$. Then,

$$
\begin{aligned}
&\int_r^1\left(\int_{D_t}v_1^{\overline{p}-1}\,dt\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\\
\le&\int_r^1\left(\int_{D_t}v_1^{N\overline{p}}\,dy\right)^{\frac{\overline{p}-1}{N\overline{p}(p-1)}}\left(\int_{D_t}dy\right)^{\frac{N\overline{p}-\overline{p}+1}{N\overline{p}(p-1)}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\\
\le& c\int_r^1\frac{dt}{t^{\overline{p}}}\le c
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
&\int_r^1\left(\int_{D_t}v_1^{\overline{p}-1}\,dt\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\\
\le&\int_r^1\sigma_n^{(\overline{p}-1)\left(\frac{N-1}{\overline{p}}-\frac{N-1}{\overline{p}-1}\right)-\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\\
=& c\sigma_n^{-\frac{N-p}{p(p-1)}}\int_r^1\frac{dt}{t^{1+\frac{N-p}{p-1}}}\le c\left(\sigma_n^{\frac{1}{p}}t\right)^{-\frac{N-p}{p-1}}\le c
\end{aligned}
\tag{32}
$$

provided $r\ge\sigma_n^{-\frac{1}{p}}$. Hence,

$$
\begin{aligned}
&\left(\gamma^{-N}\int_{B_r^+}|u_n|^\gamma\,dx+\gamma^{-N+1}\int_{D_r}|u_n|^\gamma\,dy\right)^{\frac{1}{\gamma}}\\
\le& c+c\int_r^1\left(\int_{D_t}|u_n|^{\overline{p}-1}\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\\
\le& c+c\int_r^1\left(\int_{D_t}|v_1|^{\overline{p}-1}dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}+c\int_r^1\left(\int_{D_t}|v_2|^{\overline{p}-1}dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\le c\quad\text{for } r\ge\sigma_n^{-\frac{1}{p}}.
\end{aligned}
$$

□

Proof of Proposition 1 and Corollary 1. Let $w_n(x)=\sigma_n^{-\frac{N-p}{p^2}}|u_n|(\sigma_n^{-\frac{1}{p}}x)$, w_n satisfy

$$
\begin{cases}
-\Delta_p w_n\le 0, & \text{in } \mathbb{R}^N_+,\\
|\nabla w_n|^{p-2}\frac{\partial w_n}{\partial n}\le cw_n^{\overline{p}-1}, & \text{on } \mathbb{R}^{N-1}.
\end{cases}
$$

By the profile decomposition (9), we have

$$
\begin{aligned}
&\int_{1\le|y|\le 6}|w_n|^{\overline{p}}dy=\int_{T_n^1}|u_n|^{\overline{p}}dy\\
\le&\int_{T_n^1}|u|^{\overline{p}}dy+c\sum_{k\in\Lambda}\int_{T_n^1}|g_{\sigma_{n,k}}U_k|^{\overline{p}}dy+c\int_{T_n^1}|r_n|^{\overline{p}}dy\\
\le& c\sigma_n^{-\frac{N-1}{p}}+c\sum_{k\in\Lambda}\int_{|y|\ge\sigma_n^{-\frac{1}{p}}}\left|\sigma_{n,k}^{\frac{N-p}{p}}U_k(\sigma_{n,k}y)\right|^{\overline{p}}dy+o(1)\\
\le& o(1)+c\sum_{k\in\Lambda}\int_{|y|\ge\sigma_n^{1-\frac{1}{p}}}\left(1+|y|^{\frac{p}{p-1}}\right)^{-\frac{N-p}{p}\cdot\overline{p}}dy=o(1)\,.
\end{aligned}
\tag{33}
$$

By Lemma A4 and Lemma 5, for $2\le x\le 5$, $x\in\mathbb{R}_+^N\cup\mathbb{R}^{N-1}$, we have

$$
\begin{aligned}
w_n(x)&\le c\left(\int_{1\le|x|\le 6}w_n^\gamma(x)dx+\int_{1\le|x|\le 6}w_n^\gamma(y)dy\right)^{\frac{1}{\gamma}}\\
&=c\sigma_n^{-\frac{N-p}{p^2}}\left(\sigma_n^{-\frac{N}{p}}\int_{A_n^1}|u_n|^\gamma dx+\sigma_n^{-\frac{N-1}{p}}\int_{T_n^1}|u_n|^\gamma dy\right)^{\frac{1}{\gamma}}\\
&\le c\sigma_n^{-\frac{N-p}{p^2}}\,.
\end{aligned}
$$

Hence,

$$
|u_n(x)|=\sigma_n^{\frac{N-p}{p^2}}w_n(\sigma_n^{\frac{1}{p}}x)\le c\quad\text{for } x\in A_n^2\cup T_n^2\,.
$$

We complete the proof of Proposition 1. To prove Corollary 1, we choose a function $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x)=1$ for $x\in A_n^3\cup T_n^3$ and $\varphi(x)=0$ for $x\notin A_n^2\cup T_n^3$ and $|\nabla\varphi|\le 2\sigma_n^{\frac{1}{p}}$. Testing the Equation (13) by $\varphi^p u_n$, we obtain

$$
\begin{aligned}
&\int_{A_n^2}|\nabla u_n|^p\varphi^p\,dx=\int_{A_n^2}|\nabla u_n|^{p-2}\nabla(u_n\varphi)\,dx-p\int_{A_n^2}|\nabla u_n|^{p-2}\nabla u_n\,u_n\varphi^{p-1}\nabla\varphi\,dx\\
\le&\int_{T_n^2}(-|u_n|^p+|u_n|^{\overline{p}}+\mu|u_n|^q)\varphi^p\,dy+\frac{1}{2}\int_{A_n^2}|\nabla u_n|^p\varphi^p\,dx+c\int_{A_n^2}|u_n|^p|\nabla\varphi|^p\,dx\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{A_n^3}|\nabla u_n|^p\,dx\le\int_{A_n^2}|\nabla u_n|^p\varphi^p\,dx\\
\le& c\int_{T_n^2}|u_n|^{\overline{p}}dy+c\int_{A_n^2}|u_n|^p|\nabla\varphi|^p\,dx\\
\le& c\sigma_n^{-\frac{1}{p}(N-1)}+c\sigma_n^{-\frac{N}{p}+\frac{1}{p}\cdot p}\le c\sigma_n^{1-\frac{N}{p}}\,.
\end{aligned}
$$

□

2.3. Pohožaev Identity

In the remainder of this section, following the idea of [2,3], we apply the local Pohožaev identity to prove the convergence Theorem 1.

Lemma 6. *(Local Pohožaev identity) Assume that $u\in W$ satisfies the equation*

$$
\begin{cases}
-\Delta_p u=0, & \text{in } \mathbb{R}_+^N,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial n}+|u|^{p-2}u=f_\lambda(u)+\mu|u|^{q-2}u, & \text{on } \mathbb{R}^{N-1}\,.
\end{cases}
\tag{34}
$$

Let $\varphi\in C_0^\infty(\mathbb{R}^N)$, then

$$
\begin{aligned}
&-\frac{p-1}{p}\int_{\mathbb{R}^{N-1}}|u|^p\varphi\,dy+\int_{\mathbb{R}^{N-1}}\left((N-1)F_\lambda(u)-\frac{N-p}{p}f_\lambda(u)u\right)dy\\
&+\left(\frac{N-1}{q}-\frac{N-p}{p}\right)\mu\int_{\mathbb{R}^{N-1}}|u|^q\varphi\,dy\\
=&\frac{1}{p}\int_{\mathbb{R}^N_+}|\nabla u|^p(x,\nabla\varphi)\,dx-\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}(\nabla u,x)(\nabla u,\nabla\varphi)\,dx-\frac{N-p}{p}\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}u(\nabla u,\,\nabla\varphi)\,dx\\
&+\int_{\mathbb{R}^{N-1}}\left(\frac{1}{p}|u|^p-F_\lambda(u)-\frac{\mu}{q}|u|^q\right)(y,\,\nabla_y\varphi)\,dy\,.
\end{aligned}
\tag{35}
$$

Proof. Multiplying (34) by $(x,\nabla u)\varphi$ and integration by parts, we obtain

$$
\begin{aligned}
&(N-1)\int_{\mathbb{R}^{N-1}}\left(-\frac{1}{p}|u|^p+F_\lambda(u)+\frac{\mu}{q}|u|^q\right)\varphi\,dy\\
=&\frac{N-p}{p}\int_{\mathbb{R}^N_+}|\nabla u|^p\varphi\,dx-\int_{\mathbb{R}^N_+}|\nabla u|^{p-2}(\nabla u,x)(\nabla u,\,\nabla\varphi)\,dx+\frac{1}{p}\int_{\mathbb{R}^N_+}|\nabla u|^p(x,\nabla\varphi)\,dx\\
&+\int_{\mathbb{R}^{N-1}}\left(\frac{1}{p}|u|^p-F_\lambda(u)-\frac{\mu}{q}|u|^q\right)(y,\,\nabla_y\varphi)\,dy\,.
\end{aligned}
\tag{36}
$$

Multiplying (34) by $u\varphi$ and integration by parts, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{N-1}}(|u|^p-f_\lambda(u)u-\mu|u|^q)\varphi\,dy\\
=&-\int_{\mathbb{R}^N_+}|\nabla u|^p\varphi\,dx-\int_{\mathbb{R}^{N-1}}|\nabla u|^{p-2}u(\nabla u,\,\nabla\varphi)\,dx\,.
\end{aligned}
\tag{37}
$$

Eliminating the term $\int_{\mathbb{R}^N_+}|\nabla u|^p\varphi\,dx$, we obtain the local Pohožaev identity. □

Proof of the convergence Theorem 1. We apply the local Pohožaev identity to the function u_n. Let

$$
B_n^+=\{x|\,x\in\mathbb{R}^N_+,\,|x|<4\sigma_n^{-\frac{1}{p}}\},
$$

$$
D_n=\{y|\,y\in\mathbb{R}^{N-1},\,|y|<4\sigma_n^{-\frac{1}{p}}\}\,.
$$

Choose $\varphi\in C_0^\infty(\mathbb{R}^N,\,[0,1])$ such that $\varphi(x)=1$ for $|x|\le 3\sigma_n^{-\frac{1}{p}}$, $\varphi(x)=0$ for $|x|\ge 4\sigma_n^{-\frac{1}{p}}$ and $|\nabla\varphi|\le 2\sigma_n^{\frac{1}{p}}$. By Lemma 6, the local Pohožaev identity, we have

$$
\begin{aligned}
&-\frac{p-1}{p}\int_{D_n}|u_n|^p\varphi\,dy+\int_{D_n}\left((N-1)F_{\lambda_n}(u_n)-\frac{N-p}{p}f_{\lambda_n}(u_n)u_n\right)\varphi\,dy\\
&+\left(\frac{N-p}{q}-\frac{N-p}{p}\right)\mu\int_{D_n}|u_n|^q\varphi\,dx\\
=&\frac{1}{p}\int_{B_n^+}|\nabla u_n|^p(x,\nabla\varphi)\,dx-\int_{B_n^+}|\nabla u_n|^{p-2}(\nabla u_n,\,x)(\nabla u_n,\,\nabla\varphi)\,dx\\
&-\frac{N-p}{p}\int_{B_n^+}|\nabla u_n|^{p-2}u_n(\nabla u_n,\,\nabla\varphi)\,dx\\
&+\int_{D_n}\left(\frac{1}{p}|u_n|^p-F_\lambda(u_n)-\frac{\mu}{q}|u_n|^q\right)(y,\nabla_y\varphi)\,dy\,.
\end{aligned}
\tag{38}
$$

We estimate (38). Notice that the integrals of the right-hand side of (38) are taken over the domains A_n^3 and T_n^3. By Proposition 1 and Corollary 1, we know

$$\begin{aligned}\text{RHS of (38)} &\le c\int_{A_n^3} |\nabla u_n|^p\,|x|\,|\nabla\varphi|\,dx + c\int_{A_n^3} |\nabla u_n|^{p-1}\,|\nabla\varphi|\,dx + c\int_{T_n^3} |y|\,|\nabla_y\varphi|\,dy \\ &\le c\sigma_n^{1-\frac{N}{p}} + c\sigma_n^{-\frac{N-1}{p}} \le c\sigma_n^{1-\frac{N}{p}}.\end{aligned} \tag{39}$$

On the other hand, by Lemma 1 (7), we have

$$\begin{aligned}\text{LHS of (38)} &\ge \left(\frac{N-1}{q} - \frac{N-p}{p}\right)\mu\int_{D_n} |u_n|^q\varphi\,dy - \frac{p-1}{p}\int_{D_n} |u_n|^p\varphi\,dy \\ &\ge \frac{1}{2}\left(\frac{N-1}{q} - \frac{N-p}{p}\right)\int_{D_n} |u_n|^q\varphi\,dy - c\int_{D_n} dy \\ &\ge c\int_{D_n} |u_n|^q\varphi\,dy - c\sigma_n^{-\frac{N-1}{p}}.\end{aligned} \tag{40}$$

Without loss of generality, assume $\sigma_{n,1} = \sigma_n = \min\{\sigma_{n,k}\,|\,k \in \Lambda\}$. Choose L large enough such that

$$\int_{D_L} |U_1|^q\,dy = \beta > 0$$

where $D_L = \{y\,|\,y \in \mathbb{R}^{N-1},\ |y| < L\}$. Since $\widetilde{u}_n = \sigma_n^{-\frac{N-p}{p}} u_n(\sigma_n^{-1}\cdot)$ weakly converges to U_1 in $\mathcal{D}_r$, we have

$$\begin{aligned}\int_{D_n} |u_n|^q\varphi\,dy &\ge \int_{|y|\ge L\sigma_n^{-1}} |u_n|^q\,dy \\ &= \sigma_n^{\frac{N-p}{p}q-(N-1)}\int_{D_L} |\widetilde{u}_n|^q\,dy \\ &\sim \sigma_n^{\frac{N-p}{p}q-(N-1)}\cdot\beta\end{aligned} \tag{41}$$

we arrive at a contradiction

$$\sigma_n^{\frac{N-p}{p}q-(N-1)} \le x\sigma_n^{-\frac{N-p}{p}}$$

for σ_n large enough, since $q+1 > \overline{p} = \frac{(N-1)p}{N-p}$. The index set Λ in the profile decomposition (9) must be empty, and (9) reduces to

$$u_n = u + r_n, \quad \text{and } r_n \to 0 \text{ in } L^{\overline{p}}(\mathbb{R}^{N-1}) \text{ as } n \to \infty. \tag{42}$$

That is, $u_n \to u$ in $L^{\overline{p}}(\mathbb{R}^{N-1})$. By Lemma A4, u_n is uniformly bounded, and there exists $\nu = \nu(L)$ such that

$$|u_n(y)| \le \frac{1}{\nu} \quad \text{for } y \in \mathbb{R}^{N-1}.$$

□

3. Existence of Multiple Solutions

We define a sequence of critical values of the truncated functional I_λ, $\lambda > 0$ by the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz, and prove that the corresponding critical points are solutions of the original problem (1) for sufficiently small parameter λ.

Definition 2. *Define the critical values of I_λ,*

$$c_k(\lambda) = \inf_{A\in\Gamma_k}\sup_{u\in A} I_\lambda(u),\ k = 1, 2, \cdots$$

where

$$\Gamma_k = \{A|\ A \subset W_r,\ A \text{ compact},\ -A = A,\ \gamma(A \cap \sigma^{-1}(S_\rho)) \ge k\}$$

$$G = \{\sigma|\ \sigma \in C(W_r, W_r),\ \sigma \text{ odd},\ \sigma(u) = u \text{ if } I_1(u) \le 0\}$$

and

$$S_\rho = \{u |\, u \in W_r,\ \|u\| = \rho\},$$

ρ is chosen as a suitable positive constant such that

$$I(u) \geq \beta > 0\,.$$

In fact, for $u \in W_r,\ \|u\| = \rho$, we have

$$\begin{aligned} I_\lambda(u) \geq I(u) &= \frac{1}{p}\left(\int_{\mathbb{R}^N_+} |\nabla u|^p\,dx + \int_{\mathbb{R}^{N-1}} |u|^p\,dy\right) - \frac{1}{\overline{p}}\int_{\mathbb{R}^{N-1}} |u|^{\overline{p}}\,dy - \frac{\mu}{q}\int_{\mathbb{R}^{N-1}} |u|^q\,dy \\ &\geq \frac{1}{p}\left(\int_{\mathbb{R}^N_+} |\nabla u|^p\,dx + \int_{\mathbb{R}^{N-1}} |u|^p\,dy\right) - \int_{\mathbb{R}^{N-1}} (c|u|^{\overline{p}} + \varepsilon|u|^q)\,dy \\ &\geq c_0\rho^p - c_1\rho^{\overline{p}} \geq \frac{1}{2}c_0\rho^p := \beta > 0 \end{aligned}$$

provided $\frac{1}{2}c_0\rho^p \geq c_1\rho^{\overline{p}}$.

Lemma 7. *The functional I_λ, $\lambda > 0$ satisfies the Palais–Smale condition.*

Proof. Let u_n be a Palais–Smale sequence of I_λ, and we have

$$\begin{aligned} &I_\lambda(u_n) - \frac{1}{q}\langle DI_\lambda(u_n),\, u_n\rangle \\ =&(\frac{1}{p} - \frac{1}{q})\left(\int_{\mathbb{R}^N_+} |\nabla u_n|^p\,dx + \int_{\mathbb{R}^{N-1}} |u_n|^p\,dy\right) + \int_{\mathbb{R}^{N-1}} \left(\frac{1}{q}f_\lambda(u_n)u_n - F_\lambda(u_n)\right)\,dy \\ \geq&(\frac{1}{p} - \frac{1}{q})\left(\int_{\mathbb{R}^N_+} |\nabla u_n|^p\,dx + \int_{\mathbb{R}^{N-1}} |u_n|^p\,dy\right) = (\frac{1}{p} - \frac{1}{q})\|u_n\|^p \end{aligned}$$

hence u_n is bounded in W_r. Since the imbedding from W_r to $L^q(\mathbb{R}^{N-1})$ is compact, we assume $u_n \rightharpoonup u$ in W_r, $u_n \to u$ in $L^q(\mathbb{R}^{N-1})$. By Lemma 1, we have

$$\begin{aligned} &\int_{\mathbb{R}^N_+} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m,\, \nabla u_n - \nabla u_m)\,dx \\ &+ \int_{\mathbb{R}^{N-1}} (|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m)\,dy \\ =&\int_{\mathbb{R}^{N-1}} (f_\lambda(u_n) - f_\lambda(u_m))\,(u_n - u_m)\,dy + \mu\int_{\mathbb{R}^{N-1}} (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m)(u_n - u_m)\,dy \\ &+ \langle DI_\lambda(u_n) - DI_\lambda(u_m), u_n - u_m\rangle \\ \leq& c\left((\frac{2}{\lambda})^{\overline{p}-q} + 1\right)\int_{\mathbb{R}^{N-1}} (|u_n|^{q-1} + |u_m|^{q-1})|u_n - u_m|\,dy + o(1) \\ \leq& c|u_n - u_m|_q + o(1) \to 0, \quad \text{as } n, m \to \infty\,. \end{aligned}$$

By the elementary inequalities (15), u_n is a Cauchy sequence. □

The following proposition is well known ([23,24]).

Proposition 3. *(Ambrosetti–Rabinowitz) Assume $0 < \lambda \leq 1$. Then,*

(1) *$c_k(\lambda) \geq \beta > 0,\ k = 1, 2, \cdots$ are critical values of I_λ.*
(2) *If $c_k(\lambda) = c_{k+1}(\lambda) = \cdots = c_{k+m-1}(\lambda) = c$, then $\gamma(K_c(\lambda)) \geq m$, where*

$$K_c(\lambda) = \{u |\, u \in W_r,\ I_\lambda(u) = c, DI_\lambda(u) = 0\}\,.$$

Proof of Theorem 2. Given an integer k, let $u_j(\lambda) \in W_r$, $j = 1, \cdots, k$ such that $I_\lambda(u_j(\lambda)) = c_j(\lambda)$, $DI_\lambda(u_j(\lambda)) = 0$. By Proposition 3 (2), we assume that $u_j(\lambda)$, $j = 1, \cdots, k$ are different from each other. Since $I_\lambda \leq I_1$ for $0 < \lambda \leq 1$, we have $c_1(\lambda) \leq \cdots \leq c_k(\lambda) \leq c_k(1)$. By Theorem 1, there exists ν_k such that

$$|u_j(\lambda)(y)| \leq \frac{1}{\nu_k}, \quad j = 1, \cdots, k, \; y \in \mathbb{R}^{N-1}.$$

Now, for $\lambda < \nu_k$, we have

$$|u_j(\lambda)(y)| \leq \frac{1}{\nu_k} < \frac{1}{\lambda}, \; j = 1, \cdots, k, \; y \in \mathbb{R}^{N-1}$$

hence $f_\lambda(u_j(\lambda)) = f(u_j(\lambda))$, $u_j(\lambda)$, $j = 1, \cdots, k$ are solutions of the original problem (1). Since the integer k is arbitrary, the problem (1) has infinitely many solutions. □

For more details and background material, we refer the readers to the Appendices A–C of this paper.

4. Results

The main results of this paper are Theorem 1 and Theorem 2.

Author Contributions: Conceptualization, J.Z. and X.L.; methodology, J.Z.; formal analysis, X.M.; investigation, X.M.; writing—original draft preparation, X.M.; writing—review and editing, X.M.; funding acquisition, J.Z. and X.L. All authors have read and agreed to the published version of the manuscript.

Funding: X.M. and J.Z. were supported by NSFC 11601493 and the Fundamental Research Funds for the Central Universities 2652018058, and X.L. was supported by NSFC 11361077 and the Yunnan Province, Young Academic and Technical Leaders Program (2015HB028), and J.L. was supported by NSFC 11671364.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Estimates on Solutions of p-Laplacian Equations in $\mathbb{R}^N_+$

Lemma A1. *Let $u \in \mathcal{D}$ satisfy the equation*

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^{N-1}} f \varphi \, dy, \quad \varphi \in \mathcal{D} \tag{A1}$$

where $f \geq 0$, $f \in L^q(\mathbb{R}^{N-1}) \cap L^{\frac{\overline{p}}{\overline{p}-1}}(\mathbb{R}^{N-1})$, $1 < q < \frac{N-1}{p-1}$. Then, there exists a constant $c = c(p,q)$ such that

$$|u|_s \leq c|f|_q^{\frac{1}{p-1}} \tag{A2}$$

with $\frac{1}{s} = \frac{1}{(p-1)q} - \frac{1}{N-1}$.

Proof. Denote

$$\gamma = \frac{q(p-1)}{qp - \overline{p}(q-1)}$$

then

$$\gamma > 1 - \frac{1}{p}, \; (1 + p(\gamma - 1)) \frac{q}{q-1} = \gamma \overline{p} = \frac{(N-1)(p-1)q}{N - 1 - \overline{p}(q-1)} = s. \tag{A3}$$

First, we assume $\gamma \geq 1$. Take the test function $\varphi = u \cdot u_T^{p(\gamma-1)} \in \mathcal{D}$ in (A1), where $u_T = \min\{u, T\}$ for $T > 0$. We have

$$\begin{aligned} &\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{u>T} |\nabla u|^p u_T^{p(\gamma-1)} \, dx + (1 + p(\gamma-1)) \int_{u \leq T} |\nabla u|^p u^{p(\gamma-1)} \, dx \\ \geq & c \int_{\mathbb{R}^N_+} |\nabla (u u_T^{\gamma-1})|^p \, dx \geq c \left(\int_{\mathbb{R}^{N-1}} (u u_T^{\gamma-1})^{\overline{p}} \, dy \right)^{\frac{p}{\overline{p}}} \end{aligned} \tag{A4}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} f\varphi\,dx &= \int_{\mathbb{R}^{N-1}} fuu_T^{p(\gamma-1)}\,dy \\ &\le |f|_q \left(\int_{\mathbb{R}^{N-1}} \left(uu_T^{p(\gamma-1)}\right)^{\frac{q}{q-1}}\,dy\right)^{\frac{q-1}{q}} \\ &\le |f|_q \left(\int_{\mathbb{R}^{N-1}} (uu_T^{\gamma-1})^{\overline{p}}\,dy\right)^{\frac{q-1}{q}}\end{aligned} \tag{A5}$$

In the above, we have used (A3) and $\frac{q}{q-1} \le \overline{p}$ if $\gamma \ge 1$. By (A4) and (A5), we have

$$\left(\int_{\mathbb{R}^{N-1}} (uu_T^{\gamma-1})^{\overline{p}}\,dy\right)^{\frac{p}{\overline{p}}} \le c|f|_q \left(\int_{\mathbb{R}^{N-1}} (uu_T^{\gamma-1})^{\overline{p}}\,dy\right)^{\frac{q-1}{q}}. \tag{A6}$$

Notice that $\frac{p}{\overline{p}} - \frac{q-1}{q} = (p-1)s$, $(1+p(\gamma-1))\frac{q}{q-1} = s$. Letting $T \to \infty$ in (A6), we obtain (A2).

Next, we assume $1 - \frac{1}{p} < \gamma < 1$. Let $\varphi = (u+\theta)^{1+p(\gamma-1)} - \theta^{1+p(\gamma-1)}$, $\theta > 0$, then $\varphi \in \mathcal{D}$. In fact,

$$|\nabla\varphi| = (1+p(\gamma-1))(u+\theta)^{p(\gamma-1)}|\nabla u| \le \theta^{p(\gamma-1)}|\nabla u|\,.$$

Taking φ as a test function in (A1), we have

$$\begin{aligned}&\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx = c\int_{\mathbb{R}^N} (u+\theta)^{p(\gamma-1)}|\nabla u|^p\,dx \\ =&c\int_{\mathbb{R}^N} |\nabla((u+\theta)^\gamma - \theta^\gamma)|^p\,dx \ge c\left(\int_{\mathbb{R}^{N-1}} ((u+\theta)^\gamma - \theta^\gamma)^{\overline{p}}\,dy\right)^{\frac{p}{\overline{p}}}\end{aligned} \tag{A7}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} f\varphi\,dy &= \int_{\mathbb{R}^{N-1}} f\left((u+\theta)^{1+p(\gamma-1)} - \theta^{1+p(\gamma-1)}\right)dy \\ &\le |f|_q \left(\int_{\mathbb{R}^{N-1}} \left((u+\theta)^{1+p(\gamma-1)} - \theta^{1+p(\gamma-1)}\right)^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \\ &\le c|f|_q \left(\int_{\mathbb{R}^{N-1}} ((u+\theta)^\gamma - \theta^\gamma)^{\overline{p}}\,dy\right)^{\frac{q-1}{q}}.\end{aligned} \tag{A8}$$

In the above, we used the following elementary inequality:

$$\left((1+s)^{1+p(\gamma-1)} - 1\right)^{\frac{q}{q-1}} \le c\left((1+s)^\gamma - 1\right)^{\overline{p}}, \quad \text{for } s \ge 0\,.$$

By (A7) and (A8), we obtain

$$\left(\int_{\mathbb{R}^{N-1}} ((u+\theta)^\gamma - \theta^\gamma)^{\overline{p}}\,dy\right)^{\frac{p}{\overline{p}}} \le c|f|_q \left(\int_{\mathbb{R}^{N-1}} ((u+\theta)^\gamma - \theta^\gamma)^{\overline{p}}\,dy\right)^{\frac{q-1}{q}}. \tag{A9}$$

Letting $T \to \infty$ in (A9), we obtain (A2). □

Lemma A2. *Given $f \in L^4(\mathbb{R}^{N-1}) \cap L^{\frac{\overline{p}}{p-1}}(\mathbb{R}^{N-1})$, $1 < q < \frac{N-1}{p-1}$. Then, there exists a unique function $u \in \mathcal{D}$ satisfying the equation*

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u\nabla\varphi\,dx = \int_{\mathbb{R}^{N-1}} f\varphi\,dx, \quad \varphi \in \mathcal{D}\,.$$

Moreover, $u \in L^s(\mathbb{R}^{N-1})$, $|u|_s \le c|f|_q^{\frac{1}{p-1}}$, where $\frac{1}{s} = \frac{1}{(p-1)q} - \frac{1}{N-1}$.

Proof. Consider the functional J defined on $\mathcal{D}$

$$J(u) = \frac{1}{p}\left(\int_{\mathbb{R}^N_+} |\nabla u|^p \, dx + \int_{\mathbb{R}^{N-1}} |u|^p \, dy\right) - \int_{\mathbb{R}^{N-1}} f u \, dy, \quad u \in \mathcal{D}. \tag{A10}$$

J is lower semi-continuous and bounded from below. Therefore, J assumes its infimum at a function $u \in \mathcal{D}$, which solves the equation. By the elementary inequalities (15), the solution is unique. □

Lemma A3. *Let $u \in \mathcal{D}$ and satisfy the equation*

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^{N-1}} a|v|^{p-2} v \varphi \, dy, \quad u \in \mathcal{D} \tag{A11}$$

where $a \in L^{\frac{N-1}{p-1}}(\mathbb{R}^{N-1})$, $v \in L^q(\mathbb{R}^{N-1}) \cap L^{\overline{p}}(\mathbb{R}^{N-1})$, $q > \overline{p}(1 - \frac{1}{p})$. Then, there exists a constant $c = c(p,q) > 0$ such that

$$|u|_q \le c|a|^{\frac{1}{p-1}}_{\frac{N-1}{p-1}} |v|_q \,. \tag{A12}$$

Consequently, for $p(1 - \frac{1}{p}) < p_2 < p_1$, $\sigma > 0$, we have

$$|u|_{p_1, p_2, \sigma} \le c|a|^{\frac{1}{p-1}}_{\frac{N-1}{p-1}} |v|_{p_1, p_2, \sigma} \,.$$

Proof. Let $\frac{1}{\sigma} = \frac{p-1}{N-1} + \frac{p-1}{q}$. Then, $1 < \sigma < \frac{N-1}{p-1}$ and $\frac{1}{q} = \frac{1}{(p-1)\sigma} - \frac{1}{N-1}$. By Lemma A1 and the Hölder inequality, we have

$$|u|_q \le c\big|a|v|^{p-2}v\big|^{\frac{1}{\gamma-1}}_{\sigma} \le c|a|^{\frac{1}{p-1}}_{\frac{p-1}{N-1}} |v|_q \,.$$

□

Lemma A4. *Let $u \ge 0$, $u \in \mathcal{D}$ and satisfy the inequality*

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \le \int_{\mathbb{R}^{N-1}} |u|^{\overline{p}-1} \varphi \, dy, \quad \varphi \ge 0, \, \varphi \in \mathcal{D}. \tag{A13}$$

Assume

$$\int_{D_R} u^{\overline{p}} \, dy \le (\overline{S}_p - \delta)^{\frac{N-p}{p-1}}. \tag{A14}$$

Then, for any $\gamma > 0$, there exists a constant $c = c(p, \gamma, \delta)$ such that

$$|u|_{L^\infty(D_{\frac{1}{2}R})} + |u|_{L^\infty(B^+_{\frac{1}{2}R})} \le c\Big(R^{-\frac{N-1}{\gamma}} |u|_{L^\gamma(D_R)} + R^{-\frac{N}{\gamma}} |u|_{L^\gamma(B^+_R)}\Big) \,. \tag{A15}$$

Proof. We only need to consider the case $R = 1$. The general case can be obtained by a rescaling $u(x) \mapsto R^{\frac{N-p}{p}} u(Rx)$. Then, the proof is a standard Moser's iteration and divided into three steps:

Step 1. There exists $\varepsilon > 0$, $\widetilde{p} = (1+\varepsilon)p$ such that

$$\left(\int_{D_R} u^{\widetilde{p}} dy\right)^{\frac{1}{\widetilde{p}}} \le \frac{c}{(1-R)^{\frac{p}{\widetilde{p}}}} \left(\int_{B^+_1} u^{p^*} \, dx\right)^{\frac{1}{p^*}}, \; 0 < R < 1. \tag{A16}$$

Let $\varepsilon > 0$, $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\eta(x) = 1$, $|x| \le R$; $\eta(x) = 0$, $|x| \ge 1$ and $|\nabla \eta| \le \frac{2}{1-R}$. Take $\varphi = u\, u_T^{p\varepsilon} \eta^p$ as test function in (a_{15}), where $T > 0$, $u_T = \min\{u, T\}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx \\
&= (1+p\varepsilon)\int_{u\le T} |\nabla u|^p u^{p\varepsilon}\eta^p \, dx + \int_{u>T} |\nabla u|^p T^{p\varepsilon}\eta^p \, dx + p\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u u_T^{(p-1)\varepsilon}\eta^{p-1} u u_T \nabla \eta \, dx \\
&\ge \frac{1}{(1+\varepsilon)^p}\int_{\mathbb{R}^N_+} |\nabla (u u_T^\varepsilon \eta)|^p \, dx - c_\varepsilon \int_{\mathbb{R}^N_+} (u u_T^\varepsilon)^p |\nabla \eta|^p \, dx \\
&\ge \frac{\overline{S}_p}{(1+\varepsilon)^p}\left(\int_{\mathbb{R}^{N-1}} (u u_T^\varepsilon \eta)^{\overline{p}} \, dy\right)^{\frac{p}{\overline{p}}} - \frac{c_\varepsilon}{(1-R)^p}\int_{B_1^+} (u u_T^\varepsilon)^p \, dx
\end{aligned} \tag{A17}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^{N-1}} u^{\overline{p}-1}\varphi \, dy &= \int_{\mathbb{R}^{N-1}} u^{\overline{p}-p}(u u_T^\varepsilon \eta)^p \, dx \\
&\le \left(\int_{D_1} u^{\overline{p}} \, dy\right)^{\frac{\overline{p}-p}{p}} \left(\int_{\mathbb{R}^{N-1}} (u u_T^\varepsilon \eta)^{\overline{p}} \, dy\right)^{\frac{p}{\overline{p}}} \\
&\le (\overline{S}_p - \delta)\left(\int_{\mathbb{R}^{N-1}} (u u_T^\varepsilon \eta)^{\overline{p}} \, dy\right)^{\frac{p}{\overline{p}}}.
\end{aligned} \tag{A18}$$

Choose $\varepsilon > 0$ such that

$$\frac{\overline{S}_p}{(1+\varepsilon)^p} \ge \overline{S}_p - \frac{1}{2}\delta, \ (1+\varepsilon)p < p^*. \tag{A19}$$

By (A17) and (A18), we have

$$\left(\int_{D_R} (u u_T^\varepsilon)^{\overline{p}} \, dy\right)^{\frac{p}{\overline{p}}} \le \left(\int_{\mathbb{R}^{N-1}} (u u_T^\varepsilon \eta)^{\overline{p}} \, dy\right)^{\frac{p}{\overline{p}}} \le \frac{c_{\varepsilon,\delta}}{(1-R)^p}\int_{B_1^+} (u u_T^\varepsilon)^p \, dx \, . \tag{A20}$$

Letting $T \to \infty$, we obtain

$$\left(\int_{D_R} u^{(1+\varepsilon)\overline{p}} \, dy\right)^{\frac{p}{\overline{p}}} \le \frac{c_{\varepsilon,\delta}}{(1-R)^p}\int_{B_1^+} u^{(1+\varepsilon)p} \, dx \le \frac{c_{\varepsilon,\delta}}{(1-R)^p}\left(\int_{B_1^+} u^{p^*} \, dx\right)^{\frac{p^*}{(1+\varepsilon)p}}$$

and (A16) follows.

Step 2. Assume $0 < r < R \le R_0 < 1$. Then, there exists $c_{R_0} > 0$ such that

$$|u|_{L^\infty(D_r)} + |u|_{L^\infty(B_r^+)} \le \frac{c_{R_0}}{(R-r)}\left(|u|_{L^p(D_R)} + |u|_{L^{p^*}(B_R^+)}\right). \tag{A21}$$

Let $\varphi = u u_T^{p(s-1)}\eta^p$, $s > 1$, $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $\eta(x) = 1$, $|x| \le r$; $\eta(x) = 0$, $|x| \ge R$ and $|\nabla \eta| \le \frac{2}{R-r}$. Taking φ as a test function in (a_{15}), we have

$$\begin{aligned}
\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx &= \frac{1+p(s-1)}{s^p}\int_{v\le T} |\nabla u u_T^{s-1}|^p \eta^p \, dx + \int_{v>T} |\nabla u u_T^{s-1}|^p \eta^p \, dx \\
&\quad + p\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u u_T^{(p-1)(s-1)}\eta^{p-1} u u_T^{s-1}\nabla \eta \, dx \\
&\ge \frac{1}{2s^p}\int_{\mathbb{R}^N_+} |\nabla (u u_T^{s-1}\eta)|^p \, dx - c\int_{\mathbb{R}^N_+} (u u_T^{s-1})^p |\nabla \eta|^p \, dx
\end{aligned} \tag{A22}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} u^{\overline{p}-1}\varphi\,dy &= \int_{\mathbb{R}^{N-1}} u^{\overline{p}-p}(uu_T^{s-1}\eta)^p\,dy \\ &\le \left(\int_{D_{R_0}} u^{\widetilde{p}}\,dy\right)^{\frac{\overline{p}-p}{\widetilde{p}}}\cdot\left(\int_{\mathbb{R}^{N-1}} (uu_T^{s-1}\eta)^{\frac{p\widetilde{p}}{\widetilde{p}-\overline{p}+p}}\,dy\right)^{\frac{\widetilde{p}-\overline{p}+p}{\widetilde{p}}} \\ &\le c_{R_0}\left(\int_{\mathbb{R}^{N-1}} (uu_T^{s-1}\eta)^{\overline{p}d}\,dy\right)^{\frac{p}{\overline{p}d}}\end{aligned} \tag{A23}$$

where $d = \frac{p\widetilde{p}}{\overline{p}(\widetilde{p}-\overline{p}+p)} < 1$. It follows from (A22), (A23), and the Sobolev imbedding theorem

$$\begin{aligned}&\left(\int_{D_r} (uu_T^{s-1}\eta)^{\overline{p}}\,dy\right)^{\frac{1}{s\overline{p}}} + \left(\int_{B_1^+} (uu_T^{s-1})^{p^*}\,dx\right)^{\frac{1}{sp^*}} \\ &\le \left(\int_{\mathbb{R}^{N-1}} (uu_T^{s-1}\eta)^{\overline{p}}\,dy\right)^{\frac{1}{s\overline{p}}} + \left(\int_{\mathbb{R}^N_+} (uu_T^{s-1})^{p^*}\,dx\right)^{\frac{1}{sp^*}} \\ &\le \left(c\int_{\mathbb{R}^N_+} |\nabla(uu_T^{s-1}\eta)|^p\,dx\right)^{\frac{1}{sp}} \\ &\le (cs)^{\frac{1}{s}}\left(\int_{\mathbb{R}^{N-1}} (uu_T^{s-1}\eta)^{\overline{p}d}\,dy\right)^{\frac{1}{s\overline{p}d}} + \left(\int_{\mathbb{R}^N_+} (uu_T^{s-1})^p|\nabla\eta|^p\,dx\right)^{\frac{1}{sp}} \\ &\le (cs)^{\frac{1}{s}}\left(\left(\int_{D_R} (uu_T^{s-1})^{\overline{p}d}\,dy\right)^{\frac{1}{s\overline{p}d}} + \frac{1}{(R-r)^{\frac{1}{s}}}\left(\int_{B_R^+} (uu_T^{s-1})^p\,dx\right)^{\frac{1}{sp}}\right) \\ &\le \left(\frac{cs}{R-r}\right)^{\frac{1}{s}}\left(\left(\int_{D_R} (uu_T^{s-1})^{\overline{p}d}\,dy\right)^{\frac{1}{s\overline{p}d}} + \left(\int_{B_r^+} (uu_T^{s-1})^{p^*d}\,dx\right)^{\frac{1}{p^*d}}\right).\end{aligned} \tag{A24}$$

In the above, we have used $p < p^*d$. Assume

$$\int_{D_R} u^{s\overline{p}d}\,dy < +\infty, \quad \int_{B_R^+} u^{sp^*d}\,dx < +\infty.$$

Letting $T \to \infty$ in (A24), we obtain

$$\left(\int_{D_r} u^{s\overline{p}}\,dy\right)^{\frac{1}{s\overline{p}}} + \left(\int_{B_R^+} u^{sp^*}\,dx\right)^{\frac{1}{sp^*}} \le \left(\frac{cs}{R-r}\right)^{\frac{1}{s}}\left(\left(\int_{D_R} u^{s\overline{p}d}dy\right)^{\frac{1}{s\overline{p}d}} + \left(\int_{B_R^+} u^{sp^*d}\,dx\right)^{\frac{1}{sp^*d}}\right). \tag{A25}$$

Let $\chi = \frac{1}{d}$, $x_j = \chi^j$, $r_j = r + \frac{1}{2^{j-1}}(R-r)$, $j = 1, 2, \cdots$. By Moser's iteration, for some $t > 0$, we obtain

$$|u|_{L^\infty(D_r)} + |u|_{L^\infty(B_r^+)} \le \frac{c}{(R-r)^t}\Big(|u|_{L^{\overline{p}}(D_R)} + |u|_{L^{p^*}(B_R^+)}\Big). \tag{A26}$$

Step 3. By (A26), there exists t', c such that

$$|u|_{L^\infty(D_r)} + |u|_{L^\infty(B_r^+)} \le \frac{1}{2}\Big(|u|_{L^\infty(D_R)} + |u|_{L^\infty(B_R^+)}\Big) + \frac{c}{(R-r)^{t'}}\Big(|u|_{L^\gamma(D_R)} + |u|_{L^\gamma(B_R^+)}\Big).$$

By iteration, we obtain

$$|u|_{L^\infty(D_r)} + |u|_{L^\infty(B_r^+)} \le \frac{c'}{(R-r)^{t'}}\Big(|u|_{L^\gamma(D_R)} + |u|_{L^\gamma(B_R^+)}\Big).$$

In particular

$$|u|_{L^\infty(D_{\frac{1}{2}})} + |u|_{L^\infty(B^+_{\frac{1}{2}})} \le c\Big(|u|_{L^\gamma(D_{\frac{3}{4}})} + |u|_{L^\gamma(B^+_{\frac{3}{4}})}\Big)$$
$$\le c\Big(|u|_{L^\gamma(D_1)} + |u|_{L^\gamma(B^+_1)}\Big).$$

□

We also have inner estimate

Lemma A5. *Let $u \ge 0$, $u \in \mathcal{D}$ and satisfy*

$$\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx \le 0, \;\; \varphi \ge 0, \;\; \varphi \in \mathcal{D}.$$

Then, for any $\gamma > 0$, there exists $c = c(p, \gamma)$ such that

$$|u|_{L^\infty(B_{\frac{1}{2}R})} \le cR^{-\frac{N}{\gamma}} |u|_{L^\gamma(B_R)}.$$

Appendix B. Estimate via the Wolff Potential

Lemma A6. *Let $f \ge 0$, $u \in \mathcal{D}$ satisfy the equation*

$$\begin{cases} -\Delta_p u = 0, & \text{in } \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} = f, & \text{on } \mathbb{R}^{N-1}. \end{cases} \tag{A27}$$

Then, for $\gamma \in \Big(p-1, \frac{(p-1)p\overline{p}}{(p-1)p+\overline{p}}\Big)$, there exists a constant $c = c(p, \gamma)$ such that

$$\begin{aligned} &\left(\gamma^{-N}\int_{B^+_1} |u|^\gamma \, dx + \gamma^{-N+1}\int_{D_r} |u|^\gamma \, dx\right)^{\frac{1}{\gamma}} \\ &\le c\left(\int_{B^+_1} |u|^\gamma \, dx + \int_{D_1} |u|^\gamma \, dy\right)^{\frac{1}{\gamma}} + c\int_r^1 \left(\int_{D_t} f \, dy\right)^{\frac{1}{p-1}} \frac{dt}{t^{1+\frac{N-p}{p-1}}}, \;\; 0 < r < 1. \end{aligned} \tag{A28}$$

Proof. Let $0 < R \le 1$, $r_j = 2^{1-j}R$, $j = 1, 2, \cdots$ and a_0. Define

$$a_{j+1} = a_j + \frac{1}{\delta}\left(r_{j+1}^{-N}\int_{B^+_{j+1}\cap\{u>a_j\}} + r_{j+1}^{-N+1}\int_{D_{j+1}\cap\{u>a_j\}} (u-a_j)^\gamma \, dy\right), \tag{A29}$$

where $B^+_j = \{x | \, x \in \mathbb{R}^N_+, \; |x| < r_j\}$, $D_j = \{y | \, y \in \mathbb{R}^{N-1}, \; |y| < r_j\}$, δ is a small positive constant. By Lemma A2 of [22](and refer [25]) for δ small enough, there exists a constant $c = c(p, \gamma)$ such that

$$a_k \le 2a_1 + c\sum_{j=1}^k \left(\frac{1}{r_j^{N-p}}\int_{D_{r_j}} f \, dy\right)^{\frac{1}{p-1}}. \tag{A30}$$

We have

$$a_1 = \frac{1}{\delta}\left(\int_{B^+_R} |u|^\gamma \, dx + R^{-N+1}\int_{D_R} |u|^\gamma \, dx\right) \tag{A31}$$

and

$$\sum_{j=1}^{k}\left(\frac{1}{r_j^{N-p}}\int_{D_{r_j}} f\,dy\right)^{\frac{1}{p-1}} \le c\sum_{j=1}^{k}\int_{r_j}^{r_{j-1}}\left(\int_{D_t} f\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}$$
$$= c\int_{r_k}^{R}\left(\int_{D_t} f\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}} \tag{A32}$$

and by the proof of Proposition 3 [22], we have

$$\left(r_k^{-N}\int_{B_{r_k}^+}|u|^\gamma\,dx + r_k^{-N+1}\int_{D_{r_k}}|u|^\gamma\,dy\right)^{\frac{1}{\gamma}} \le c a_k\,. \tag{A33}$$

Now, it follows that

$$\left(r_k^{-N}\int_{B_{r_k}^+}|u|^\gamma\,dx + r_k^{-N+1}\int_{D_{r_k}}|u|^\gamma\,dy\right)^{\frac{1}{\gamma}}$$
$$\le c\left(R^{-N}\int_{B_R^+}|u|^\gamma\,dx + R^{-N+1}\int_{D_R}|u|^\gamma\,dy\right)^{\frac{1}{\gamma}} + c\int_{r_k}^{R}\left(\int_{D_t} f\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\,. \tag{A34}$$

Assume $2^{-k} < r \le 2^{-k+1}$. Let $R = 2^{k-1}r$, $\frac{1}{2} < R \le 1$. By (A30)–(A34), we obtain

$$\left(r^{-N}\int_{B_r^+}|u|^\gamma\,dx + r^{-N+1}\int_{D_r}|u|^\gamma\,dy\right)^{\frac{1}{\gamma}}$$
$$\le c\left(R^{-N}\int_{B_R^+}|u|^\gamma\,dx + R^{-N+1}\int_{D_R}|u|^\gamma\,dy\right)^{\frac{1}{\gamma}} + c\int_{r}^{R}\left(\int_{D_t} f\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}$$
$$\le c\left(\int_{B_1^+}|u|^\gamma\,dx + \int_{D_1}|u|^\gamma\,dy\right)^{\frac{1}{\gamma}} + c\int_{r}^{1}\left(\int_{D_t} f\,dy\right)^{\frac{1}{p-1}}\frac{dt}{t^{1+\frac{N-p}{p-1}}}\,.$$

□

Appendix C. The Sobolev Imbedding Theorem

Lemma A7. $W^{1,p}(\mathbb{R}_+^N) \subset W \subset W^{1,p}(\mathbb{R}^{N-1}\times(0,1))$.

Proof. (1) By the Sobolev imbedding theorem, the imbedding from $W^{1,p}(\mathbb{R}_+^N)$ to $L^q(\mathbb{R}^{N-1})$, $p \le q \le \overline{p}$ is continuous, hence $W^{1,p}(\mathbb{R}_+^N) \subset W$. On the other hand, there exist functions that belong to W but not to $W^{1,p}(\mathbb{R}_+^N)$. Here, we give an example. Let $\varphi \in C_0^\infty(B_1, [0,1])$. Define

$$\varphi_n(x) = n^{-\frac{N-p}{p}}\varphi(n^{-1}(x-2^{n+1}e)),\ x \in \overline{\mathbb{R}_+^N},\ n = 1,2,\cdots,$$

$$u_n = \sum_{k=1}^{n}\frac{1}{n}\varphi_n$$

where $e = (0,\cdots,0,1) \in \mathbb{R}^N$, $\varphi_n(n = 1,2,\cdots)$ are supports, and we have

$$\int_{\mathbb{R}_+^N}|\nabla\varphi_n|^p\,dx = \int_{\mathbb{R}^N}|\nabla\varphi|^p\,dx,\quad \int_{\mathbb{R}_+^N}\varphi_n^p\,dx = n^p\int_{\mathbb{R}^N}\varphi^p\,dx,\ \int_{\mathbb{R}^{N-1}}\varphi_n^p\,dy = 0\,.$$

Hence,

$$\|u_n - u_m\|_W^p = \sum_{k=n+1}^{m}\frac{1}{u^p}\int_{\mathbb{R}^N}|\nabla\varphi|^p\,dx \to 0$$

$$|u_n|_p^p = \sum_{k=1}^{n} \frac{1}{n^p} \cdot n^p \int_{\mathbb{R}^N} \varphi^p \, dx \to \infty \, .$$

Let $u = \lim_{n\to\infty} u_n$, $u \in W$ but $u \notin W^{1,p}(\mathbb{R}^N_+)$.

(2) Letting $u \in W$, we have

$$\begin{aligned} |u|^p(y,s) &= |u|^p(y,0) + \int_0^s \frac{\partial}{\partial s} |u|^p(y,\tau) \, d\tau \\ &= |u|^p(y,0) + p \int_0^s |u|^{p-2} u \cdot \frac{\partial u}{\partial s} d\tau \\ &\le |u|^p(y,0) + c \int_0^1 \left| \frac{\partial u}{\partial s} \right|^p d\tau + \varepsilon \int_0^1 |u|^p \, d\tau \, . \end{aligned}$$

Integrating over $x \in \mathbb{R}^{N-1} \times (0,1)$, we obtain

$$\int_{\mathbb{R}^{N-1}\times(0,1)} |u|^p \, dx \le \int_{\mathbb{R}^{N-1}\times(0,1)} |u|^p(y,0) \, dy + c \int_{\mathbb{R}^{N-1}\times(0,1)} \left| \frac{\partial u}{\partial s} \right|^p dx + \varepsilon \int_{\mathbb{R}^{N-1}\times(0,1)} |u|^p \, dx,$$

hence

$$\int_{\mathbb{R}^{N-1}\times(0,1)} |u|^p \, dx \le c \left(\int_{\mathbb{R}^{N-1}} |u|^p \, dy + \int_{\mathbb{R}^N_+} |\nabla u|^p \, dx \right) = c\|u\|_W^p$$

and

$$\|u\|_{W^{1,p}(\mathbb{R}^{N-1}\times(0,1))} \le c\|u\|_W \, .$$

□

Lemma A8. *The imbedding from W_r to $L^q(\mathbb{R}^{N-1})$, $p < q < \overline{p}$ is compact.*

Proof. Denote $Q = (-1,\, 1)^{N-1} \subset \mathbb{R}^{N-1}$. For $y \in \mathbb{R}^{N-1}$, $|y| \ge R$, we find orthogonal transformation $\tau_i \in O(N-1) \subset O(N)$, $i = 1, \cdots, N(R)$ such that $\tau_i = Id$ and $\tau_i(Q_+)$, $i = 1, \cdots, N(R)$ are mutually disjoint. Obviously, $N(R) \to +\infty$ as $R \to +\infty$.

Let $u \in W_r$, $z \in \mathbb{R}^{N-1}$, $|z| \ge R$. We have

$$\int_{Q+z} |u|^q \, dy = \frac{1}{N(R)} \sum_{i=1}^{N(R)} \int_{\tau_i(Q+y)} |u|^q \, dy \le \frac{1}{N(R)} |u|^q \, dy \tag{A35}$$

and

$$\begin{aligned} \int_{Q+z} |u|^q \, dy &\le c \int_{(Q+z)\times(0,1)} (|\nabla u|^p + |u|^p) \, dx \left(\int_{Q+z} |u|^q \, dy \right)^{1-\frac{p}{q}} \\ &\le c \left(\int_{(Q+z)\times(0,1)} |\nabla u|^p \, dx + \int_{Q+z} |u|^p \, dy \right) \left(\frac{1}{N(R)} \int_{\mathbb{R}^{N-1}} |u|^q \, dy \right)^{1-\frac{p}{q}} . \end{aligned} \tag{A36}$$

Taking sum over $z \in \mathbb{R}^{N-1}$, $|z| \ge R$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}\setminus D_R} |u|^q \, dy &\le c \left(\int_{\mathbb{R}^N_+} |\nabla u|^p \, dx + \int_{\mathbb{R}^{N-1}} |u|^p \, dy \right) \cdot \left(\frac{1}{N(R)} \int_{\mathbb{R}^{N-1}} |u|^q \, dy \right)^{1-\frac{p}{q}} \\ &\le c\|u\|_W^q \, N(R)^{-\left(1-\frac{p}{q}\right)} = c\|u\|_W^q \cdot o_R(1) \, . \end{aligned} \tag{A37}$$

Now, assume $u_n \in W_r$, $u_n \rightharpoonup 0$ in W_r. Then,

$$\int_{\mathbb{R}^{N-1}} |u_n|^q \, dy = \int_{D_R} |u_n|^q \, dy + \int_{\mathbb{R}^{N-1}\setminus D_R} |u_n|^q \, dy \le \int_{D_R} |u_n|^q \, dy + c o_R(1) \to 0 \text{ as } n \to \infty \, .$$

□

Proposition A1. *The imbedding from D_r to $L^{\overline{p}}(\mathbb{R}^{N-1})$ is compact with respect to the dilation group D (defined by (8)).*

Proof. (Adapted from [19]) Choose $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ such that $\chi(t) = |t|$, $1 \le |t| \le 2^{\frac{N-p}{p}}$; $chi(t) = 0$, $|t| \le 2^{-\frac{N-p}{p}}$ or $|t| \ge 2^{2\cdot\frac{N-p}{p}}$. $Q = (-1, 1)^{N-1}$ and $N(R)$ as defined in Lemma A8. Assume $u_n \overset{D}{\rightharpoonup} 0$ in $\mathcal{D}_r$.

Step 1. $\int_{\mathbb{R}^{N-1}} \chi^{\overline{p}}(u_n)\, dy \to 0$ as $n \to \infty$.
For $z \in \mathbb{R}^{N-1}$, $|z| \ge R$, we have

$$\begin{aligned}\int_{Q+z} \chi^{\overline{p}}(u_n)\, dy &\le c \int_{(Q+z)\times(0,1)} \left(|\nabla \chi(u_n)|^p + \chi^p(u_n)\right)\, dx \left(\int_{Q+z} \chi^{\overline{p}}(u_n)\, dy\right)^{1-\frac{p}{\overline{p}}} \\ &\le c\left(\int_{(Q+z)\times(0,1)} |u_n|^p\, dx + \int_{Q+z} \chi^p(u_n)\, dy\right)\left(\frac{1}{N(R)}\int_{\mathbb{R}^{N-1}} \chi^{\overline{p}}(u_n)\, dy\right)^{1-\frac{p}{\overline{p}}} \\ &\le c\left(\int_{(Q+z)\times(0,1)} |\nabla u_n|^p\, dx + \int_{Q+z} |u_n|^{\overline{p}}\, dy\right)\left(\frac{1}{N(R)}\int_{\mathbb{R}^{N-1}} |u_n|^{\overline{p}}\, dy\right)^{1-\frac{p}{\overline{p}}}.\end{aligned} \tag{A38}$$

Taking sum over $z \in \mathbb{R}^{N-1}$ and $|z| \ge R$,

$$\begin{aligned}\int_{\mathbb{R}^{N-1}\setminus D_R} \chi^{\overline{p}}(u_n)\, dy &\le c\left(\int_{\mathbb{R}^N_+} |\nabla u_n|^p\, dx + \int_{\mathbb{R}^{N-1}} |u_n|^{\overline{p}}\, dy\right)\left(\frac{1}{N(R)}\int_{\mathbb{R}^{N-1}} |u_n|^{\overline{p}}\, dy\right)^{1-\frac{p}{\overline{p}}} \\ &\le cN(R)^{-\left(1-\frac{p}{\overline{p}}\right)} = o_R(1)\end{aligned} \tag{A39}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \chi^{\overline{p}}(u_n)\, dy &= \int_{D_R} \chi^{\overline{p}}(u_n)\, dy + \int_{\mathbb{R}^{N-1}\setminus D_R} \chi^{\overline{p}}(u_n)\, dy \\ &\le c\int_{D_R} |u_n|^p\, dy + o_R(1) \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{A40}$$

Step 2. For $j \in Z$, define $h_j \in D$ by $h_j u(x) = z^{j\cdot\frac{N-p}{p}} u(2^j x)$. Then, for any sequence $j_n \in Z$, $h_{j_n} u_n \overset{D}{\rightharpoonup} 0$ in $\mathcal{D}_r$. By Step 1, we have

$$\int_{\mathbb{R}^{N-1}} \chi^{\overline{p}}(h_{j_n} u_n)\, dy \to 0, \quad \text{as } n \to \infty. \tag{A41}$$

Step 3. We estimate $\int_{\mathbb{R}^{N-1}} |u_n|^p\, dy$. Since

$$\begin{aligned}&\int_{2^{j\cdot\frac{N-p}{p}} \le |u_n| \le 2^{(j+1)\cdot\frac{N-p}{p}}} |u_n|^{\overline{p}}\, dy \\ &\le \int_{\mathbb{R}^{N-1}} \left(2^{j\cdot\frac{N-p}{p}} \chi(2^{-j\cdot\frac{N-p}{p}} u_n(x))\right)^{\overline{p}} dy \\ &= \int_{\mathbb{R}^{N-1}} \left(\chi(2^{-j\cdot\frac{N-p}{p}} u_n(2^{-j}x))\right)^{\overline{p}} dy \\ &\le c\int_{\mathbb{R}^N_+} \left|\nabla\left(2^{j\cdot\frac{N-p}{p}} \chi(2^{-j\cdot\frac{N-p}{p}} u_n(x))\right)\right|^p dx \cdot \left(\int_{\mathbb{R}^{N-1}} \left(\chi(2^{-j\cdot\frac{N-p}{p}} u_n(2^{-j}x))\, dy\right)^{\overline{p}}\right)^{1-\frac{p}{\overline{p}}} \\ &\le c\int_{2^{(j-1)\cdot\frac{N-p}{p}} \le |u_n(x)| \le 2^{(j+2)\cdot\frac{N-p}{p}}} |\nabla u_n|^p\, dx \cdot \sup_{j\in Z^{N-1}} \left(\int_{\mathbb{R}^{N-1}} (\chi(h_j(u_n)))^{\overline{p}}\, dy\right)^{1-\frac{p}{\overline{p}}},\end{aligned} \tag{A42}$$

choosing $j_n \in Z$ such that

$$\sup_{j\in Z^{N-1}} \int_{\mathbb{R}^{N-1}} (\chi(h_j(u_n)))^{\overline{p}}\, dy \le 2\int_{\mathbb{R}^{N-1}} (\chi(h_j(u_n)))^{\overline{p}}\, dy \tag{A43}$$

Then,

$$\begin{aligned} &\int_{2^{j\cdot\frac{N-p}{p}} \le |u_n| \le 2^{(j+1)\cdot\frac{N-p}{p}}} |u_n|^p\, dy \\ \le c &\int_{2^{(j-1)\cdot\frac{N-p}{p}} \le |u_n| \le 2^{(j+2)\cdot\frac{N-p}{p}}} |\nabla u_n|^p\, dx \cdot \left(\int_{\mathbb{R}^{N-1}} (\chi(h_j(u_n)))^{\overline{p}}\, dy\right)^{1-\frac{p}{\overline{p}}}. \end{aligned} \tag{A44}$$

Taking sum over $j \in Z^{N-1}$ and taking into account that the sets $2^{(j-1)\cdot\frac{N-p}{p}} \le |u_n| \le 2^{(j+2)\cdot\frac{N-p}{p}}$ cover R with uniformly finite multiplicity, by Step 2, we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |u_n|^p\, dy &\le c\int_{\mathbb{R}^N_+} |\nabla u_n|^p\, dx \cdot \left(\int_{\mathbb{R}^{N-1}} (\chi(h_j(u_n)))^{\overline{p}}\, dy\right)^{1-\frac{p}{\overline{p}}} \\ &\le c\left(\int_{\mathbb{R}^{N-1}} (\chi(h_{j_n}(u_n)))^{\overline{p}}\, dy\right)^{1-\frac{p}{\overline{p}}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

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