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When Are Graded Rings Graded S-Noetherian Rings

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Abstract: Let Γ be a commutative monoid, $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ a Γ-graded ring and S a multiplicative subset of R_0. We define R to be a graded S-Noetherian ring if every homogeneous ideal of R is S-finite. In this paper, we characterize when the ring R is a graded S-Noetherian ring. As a special case, we also determine when the semigroup ring is a graded S-Noetherian ring. Finally, we give an example of a graded S-Noetherian ring which is not an S-Noetherian ring.

Keywords: S-Noetherian ring; graded S-Noetherian ring; S-finite algebra; Cohen type theorem

MSC: 13A02; 13A15; 13E99; 20M25

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1. Introduction

1.1. Graded Rings and Semigroup Rings

Let R be a commutative ring with identity and let Γ be a commutative monoid written additively. Then R is called a Γ-*graded ring* if there exists a nonempty family $\{R_\alpha \mid \alpha \in \Gamma\}$ of additive abelian groups such that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ and $R_\alpha \cdot R_\beta \subseteq R_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded ring. It is obvious that for all $\alpha \in \Gamma$, R_α is an R_0-module. (In this paper, all modules are assumed to be unitary.) In particular, R_0 can be regarded as a subring of R. Also, it is easy to see that $\bigcup_{\alpha\in\Gamma} R_\alpha$ is the set of homogeneous elements of R. An ideal I of R is said to be *homogeneous* if $I = \bigoplus_{\alpha\in\Gamma}(I \cap R_\alpha)$ (or equivalently, I has a set of homogeneous generators). It is routine to show that if I and J are homogeneous ideals of R, then $(I : J) := \{x \in R \mid xJ \subseteq I\}$ is also a homogeneous ideal of R. Let $\{I_\beta \mid \beta \in \Lambda\}$ be a nonempty set of homogeneous ideals of R. Then $\sum_{\beta\in\Lambda} I_\beta$ is an ideal of R which has a set of homogeneous generators; so $\sum_{\beta\in\Lambda} I_\beta$ is a homogeneous ideal of R. In particular, if $\{I_\beta \mid \beta \in \Lambda\}$ is a chain of homogeneous ideals of R, then $\bigcup_{\beta\in\Lambda} I_\beta$ is also a homogeneous ideal of R.

For more on graded rings, the readers can refer to [1].

One of the most important examples of Γ-graded rings is the semigroup ring. Let R be a commutative ring with identity, Γ a commutative monoid written additively and $R[\Gamma]$ the set of functions f from Γ to R that are finitely nonzero with the usual addition and multiplication defined as

$$(fg)(\gamma) = \textstyle\sum_{\alpha+\beta=\gamma} f(\alpha)g(\beta).$$

Then $R[\Gamma]$ becomes a commutative ring with identity and we call $R[\Gamma]$ the semigroup ring of Γ over R.

Let $R[\Gamma]$ be the semigroup ring of Γ over R. Then each $f \in R[\Gamma]$ can be written as $f = a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n}$ for some $a_1, \ldots, a_n \in R$ and $\alpha_1, \ldots, \alpha_n \in \Gamma$. Also, $R[\Gamma] = \bigoplus_{\alpha\in\Gamma} RX^\alpha$ is a Γ-graded ring in the natural way with $(R[\Gamma])_\alpha = RX^\alpha$.

The readers can refer to [2] for semigroup rings and semigroups.

1.2. *S*-Noetherian Rings

The concept of Noetherian rings is one of the most important tools in the arsenal of algebraists. Because of its significance, there have been many attempts to generalize the notion of Noetherian rings. One of several generalizations is an S-Noetherian ring. Let R be a commutative ring with identity, S a (not necessarily saturated) multiplicative subset of R and M a unitary R-module. In [3], the authors introduced the concept of "almost finitely generated" to study Querre's characterization of divisorial ideals in integrally closed polynomial rings. Later, in [4], Anderson and Dumitrescu abstracted this notion to any commutative ring and defined a general concept of Noetherian rings. Recall from (Definition 1 [4]) that an ideal I of R is *S-finite* if there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$; and R is an *S-Noetherian ring* if each ideal of R is S-finite. Also, M is said to be *S-finite* if there exist an element $s \in S$ and a finitely generated R-submodule F of M such that $sM \subseteq F$. If S consists of units of R, then the notion of S-Noetherian rings (resp., S-finite ideals, S-finite modules) is precisely the same as that of Noetherian rings (resp., finitely generated ideals, finitely generated modules).

For more on S-Noetherian rings, the readers can refer to [4–12].

Let R be a commutative ring with identity and let S be a (not necessarily saturated) multiplicative subset of R. Recall from [4] (p. 4411) that S is an *anti-Archimedean subset* of R if $\bigcap_{n=1}^{\infty} s^n R \cap S \neq \emptyset$ for all $s \in S$. This concept originally came from that of anti-Archimedean rings [13] (p. 3223). As an example, every multiplicative subset consisting of units is anti-Archimedean. Also, if V is a valuation domain with no height-one prime ideals, then $V \setminus \{0\}$ is an anti-Archimedean subset of V (Proposition 2.1 [13]).

Recall that for a commutative monoid Γ, a Γ-graded ring $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded Noetherian ring* if every homogeneous ideal of R is finitely generated. In [14,15], the authors studied graded Noetherian rings. More precisely, Goto and Yamagishi showed that if Γ is a finitely generated abelian group and $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a Γ-graded ring, then R is a graded Noetherian ring if and only if R is a Noetherian ring, if and only if R_0 is a Noetherian ring and R is a finitely generated R_0-algebra (Theorem 1.1 [14]). Also, Rush proved that if R is a commutative ring with identity and Γ is a torsion-free cancellative monoid, then $R[\Gamma]$ is a graded Noetherian ring if and only if R is a Noetherian ring and Γ is finitely generated (Theorem 2.4 [15]). In [11], the authors dealt with semigroup rings as S-Noetherian rings. For a commutative ring R with identity, an anti-Archimedean subset S of R and a commutative monoid Γ, it was shown that if R is an S-Noetherian ring and Γ is finitely generated, then $R[\Gamma]$ is an S-Noetherian ring (Proposition 3.1 [11]); and if $R[\Gamma]$ is an S-Noetherian ring and Γ is cancellative with $G(\Gamma) = \{0\}$, R is an S-Noetherian ring and Γ is finitely generated, where $G(\Gamma)$ is the largest subgroup of Γ (Lemma 3.2 and Proposition 3.3 [11]).

Motivated by the results in the previous paragraph, in this paper, we introduce the concept of graded S-Noetherian rings and determine when the both a graded ring and the semigroup ring are graded S-Noetherian rings. In Section 2, we introduce the concepts of graded S-Noetherian rings and S-finite algebras and show that if Γ is a finitely generated abelian group, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a Γ-graded ring and S is an anti-Archimedean subset of R_0, then R is an S-Noetherian ring if and only if R is a graded S-Noetherian ring, if and only if R_0 is an S-Noetherian ring and R is an S-finite R_0-algebra. In Section 3, we investigate to study when the semigroup ring is a graded S-Noetherian ring. More precisely, we prove that if R is a commutative ring with identity, S is a multiplicative subset of R and Γ is a torsion-free cancellative monoid, then $R[\Gamma]$ is a graded S-Noetherian ring if and only if R is an S-Noetherian ring and every ideal of Γ is finitely generated. We also give an example of a graded S-Noetherian ring which is not an S-Noetherian ring.

2. Graded Rings as Graded *S*-Noetherian Rings

Let R be a commutative ring with identity and let T be a unitary R-module. Then we say that R is a *direct summand* of T as an R-module if there exists an R-module A such that $T = R \oplus A$.

Lemma 1. *Let $R \subseteq T$ be an extension of commutative rings with identity, S a multiplicative subset of R and I an ideal of R. Suppose that R is a direct summand of T as an R-module. Then I is an S-finite ideal of R if and only if IT is an S-finite ideal of T.*

Proof. ($\Rightarrow$) Suppose that I is an S-finite ideal of R. Then there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. Hence $sIT \subseteq JT \subseteq IT$. Note that JT is a finitely generated ideal of T. Thus IT is an S-finite ideal of T.

($\Leftarrow$) Suppose that IT is an S-finite ideal of T. Then there exist an element $s \in S$ and a finitely generated subideal J of I such that $sIT \subseteq JT$. Since R is a direct summand of T as an R-module, we can define an R-module epimorphism $\phi : T \to R$ such that $\phi(r) = r$ for all $r \in R$. Let $a \in I$. Then there exist $j_1, \dots, j_n \in J$ and $t_1, \dots, t_n \in T$ such that $sa = \sum_{k=1}^{n} j_k t_k$. Therefore we obtain

$$sa = \phi(sa) = \phi(\textstyle\sum_{k=1}^{n} j_k t_k) = \sum_{k=1}^{n} j_k \phi(t_k) \in J.$$

Hence $sI \subseteq J \subseteq I$. Thus I is an S-finite ideal of R. □

Let Γ be a commutative monoid and let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded ring. Then for each $\alpha \in \Gamma$, $R_\alpha \cdot R$ denotes the ideal of R generated by the set R_α.

Lemma 2. *Suppose that Γ is a cancellative monoid. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded ring and let S be a multiplicative subset of R_0. For an element $\alpha \in \Gamma$, if $R_\alpha \cdot R$ is an S-finite ideal of R, then R_α is an S-finite R_0-module.*

Proof. Suppose that $R_\alpha \cdot R$ is an S-finite ideal of R. Then we can find $s \in S$ and $a_1, \dots, a_n \in R_\alpha$ such that $sR_\alpha \cdot R \subseteq (a_1, \dots, a_n)$. Let $a \in R_\alpha$. Then there exist $r_1, \dots, r_n \in R$ such that $sa = a_1 r_1 + \cdots + a_n r_n$. Since Γ is cancellative, we may assume that $r_i \in R_0$ for all $i \in \{1, \dots, n\}$. Hence $sR_\alpha \subseteq a_1 R_0 + \cdots + a_n R_0 \subseteq R_\alpha$. Thus R_α is an S-finite R_0-module. □

We denote by $\mathbb{Z}$ the (additive) group of integers.

Lemma 3. *Let $R = \bigoplus_{\alpha \in \mathbb{Z}} R_\alpha$ be a $\mathbb{Z}$-graded ring and let S be a multiplicative subset of R_0. If every ideal of R generated by elements of R_0 is S-finite, then R_0 is an S-Noetherian ring.*

Proof. Let I be an ideal of R_0. Then IR is an ideal of R generated by elements of R_0. By the assumption, IR is an S-finite ideal of R. Hence by Lemma 1, I is an S-finite ideal of R_0. Thus R_0 is an S-Noetherian ring. □

Let R be a commutative ring with identity, A an R-algebra and S a (not necessarily saturated) multiplicative subset of R. We say that A is an *S-finite R-algebra* if there exist $s \in S$ and $a_1, \dots, a_n \in A$ such that $sA \subseteq R[a_1, \dots, a_n]$.

For a $\mathbb{Z}$-graded ring $R = \bigoplus_{\alpha \in \mathbb{Z}} R_\alpha$, let $R^+ = \bigoplus_{\alpha > 0} R_\alpha$ and $R^- = \bigoplus_{\alpha < 0} R_\alpha$. Then $R^+ \cdot R$ denotes the ideal of R generated by the set R^+ and $R^- \cdot R$ stands for the ideal of R generated by the set R^-. It is easy to see that $R^+ \cdot R$ (resp., $R^- \cdot R$) is the ideal of R generated by all homogeneous elements of R^+ (resp., R^-). For $a, b \in \mathbb{Z}$ with $b \leq a$, we define $R_0[R_b, \dots, R_a]$ to be the R_0-algebra generated by the set $\bigcup_{b \leq i \leq a} R_i$.

Lemma 4. *Let $R = \bigoplus_{\alpha \in \mathbb{Z}} R_\alpha$ be a $\mathbb{Z}$-graded ring and let S be an anti-Archimedean subset of R_0. If $R^+ \cdot R$, $R^- \cdot R$ and $R_\alpha \cdot R$ for all $\alpha \in \mathbb{Z}$ are S-finite ideals of R, then R is an S-finite R_0-algebra.*

Proof. Suppose that $R^+ \cdot R$ and $R^- \cdot R$ are S-finite ideals of R. Then there exist $s, t \in S$, $f_1, \dots, f_n \in R^+$ and $g_1, \dots, g_m \in R^-$ such that

$$s(R^+ \cdot R) \subseteq (f_1, \dots, f_n) \text{ and } t(R^- \cdot R) \subseteq (g_1, \dots, g_m).$$

By an easy calculation, we may assume that $f_1, \ldots, f_n, g_1, \ldots, g_m$ are homogeneous. Let $a = \text{Max}\{\deg(f_i) \mid 1 \leq i \leq n\}$ and $b = \text{Min}\{\deg(g_i) \mid 1 \leq i \leq m\}$.

Claim: There exists an element $u \in S$ such that $uR^+ \subseteq R_0[R_b, \ldots, R_a]$.

Let $x \in R_{a+1}$. Then $stx \in (f_1, \ldots, f_n)$; so there exist $c_1, \ldots, c_n \in R$ such that $stx = f_1c_1 + \cdots + f_nc_n$. Since stx is a homogeneous element of R with $\deg(stx) = a + 1$, we may assume that for each $i \in \{1, \ldots, n\}$, either $c_i = 0$ or $1 \leq \deg(c_i) \leq a$. Therefore $stx \in R_0[R_1, \ldots, R_a] \subseteq R_0[R_b, \ldots, R_a]$. Hence $stR_{a+1} \subseteq R_0[R_b, \ldots, R_a]$.

Fix an integer $k \geq 2$ and suppose that $(st)^i R_{a+i} \subseteq R_0[R_b, \ldots, R_a]$ for all $i = 1, \ldots, k-1$. Then we have

$$(st)^{k-1} R_{a+i} \subseteq R_0[R_b, \ldots, R_a]$$

for all $i = 1, \ldots, k-1$. Let $y \in R_{a+k}$. Then $sty \in (f_1, \ldots, f_n)$; so there exist $d_1, \ldots, d_n \in R$ such that $sty = f_1d_1 + \cdots + f_nd_n$. This implies that $(st)^k y = (st)^{k-1}(f_1d_1 + \cdots + f_nd_n)$. Since $(st)^k y$ is a homogeneous element of R with $\deg((st)^k y) = a + k$, we may assume that for each $i \in \{1, \ldots, n\}$, either $d_i = 0$ or $k \leq \deg(d_i) \leq a + k - 1$. Note that by the induction hypothesis, $(st)^{k-1} d_i \in R_0[R_b, \ldots, R_a]$ for all $i \in \{1, \ldots, n\}$; so we have

$$\begin{aligned} (st)^k y &= f_1(st)^{k-1}d_1 + \cdots + f_n(st)^{k-1}d_n \\ &\in R_0[R_b, \ldots, R_a]. \end{aligned}$$

Hence $(st)^k R_{a+k} \subseteq R_0[R_b, \ldots, R_a]$.

By the induction, $(st)^j R_{a+j} \subseteq R_0[R_b, \ldots, R_a]$ for all $j \geq 1$. Since S is an anti-Archimedean subset of R_0, there exists an element $u \in \bigcap_{j=1}^{\infty} (st)^j R_0 \cap S$; so we have

$$uR_{a+j} \subseteq R_0[R_b, \ldots, R_a]$$

for all $j \geq 1$. Thus $uR^+ \subseteq R_0[R_b, \ldots, R_a]$.

Also, a similar argument as in Claim shows that $uR^- \subseteq R_0[R_b, \ldots, R_a]$, where u is as in the proof of Claim. Hence $uR \subseteq R_0[R_b, \ldots, R_a] \subseteq R$.

Let $\alpha \in \mathbb{Z}$ with $b \leq \alpha \leq a$. Then by the assumption, $R_\alpha \cdot R$ is an S-finite ideal of R; so there exist $s_\alpha \in S$ and $w_{\alpha 1}, \ldots, w_{\alpha p_\alpha} \in R_\alpha$ such that

$$s_\alpha(R_\alpha \cdot R) \subseteq (w_{\alpha 1}, \ldots, w_{\alpha p_\alpha})R \subseteq R_\alpha \cdot R.$$

Let $F = \{w_{\alpha j} \mid b \leq \alpha \leq a \text{ and } 1 \leq j \leq p_\alpha\}$ and let $\bar{s} = \prod_{b \leq \alpha \leq a} s_\alpha$. Then for each $\alpha \in \{b, \ldots, a\}$, $\bar{s}(R_\alpha \cdot R) \subseteq (w_{\alpha 1}, \ldots, w_{\alpha p_\alpha})R \subseteq R_\alpha \cdot R$. Since $\mathbb{Z}$ is a group, we have

$$\bar{s}R_\alpha \subseteq (w_{\alpha 1}, \ldots, w_{\alpha p_\alpha})R_0 \subseteq R_0[F] \subseteq R_\alpha$$

for each $\alpha \in \{b, \ldots, a\}$. Hence we have

$$\bar{s}uR \subseteq \bar{s}R_0[R_b, \ldots, R_a] \subseteq R_0[F] \subseteq R.$$

Note that $R_0[F]$ is a finitely generated R_0-algebra. Thus R is an S-finite R_0-algebra. □

Lemma 5. *Let $A \subseteq B \subseteq C$ be extensions of commutative rings with identity and let S be a multiplicative subset of A. If B is an S-finite A-algebra and C is an S-finite B-algebra, then C is an S-finite A-algebra.*

Proof. Suppose that B is an S-finite A-algebra and C is an S-finite B-algebra. Then there exist $s, t \in S$, $b_1, \ldots, b_n \in B$ and $c_1, \ldots, c_m \in C$ such that $sB \subseteq A[b_1, \ldots, b_n]$ and $tC \subseteq B[c_1, \ldots, c_m]$. Hence we have

$$stC \subseteq sB[c_1, \ldots, c_m] \subseteq A[b_1, \ldots, b_n, c_1, \ldots, c_m].$$

Thus C is an S-finite A-algebra. □

In (Corollary 2.1 [16]) or (Theorem 3.1 [6]), the authors showed the Eakin-Nagata theorem for S-Noetherian rings which states that for an extension $R \subseteq T$ of commutative rings with identity and a multiplicative subset S of R, if R is an S-Noetherian ring and T is an S-finite R-module, then T is an S-Noetherian ring. If S is anti-Archimedean, then we have the following result.

Lemma 6. *Let $R \subseteq T$ be an extension of commutative rings with identity and let S be an anti-Archimedean subset of R. If R is an S-Noetherian ring and T is an S-finite R-algebra, then T is an S-Noetherian ring.*

Proof. Suppose that T is an S-finite R-algebra. Then there exist $s_1 \in S$ and $t_1, \ldots, t_n \in T$ such that $s_1 T \subseteq R[t_1, \ldots, t_n]$. Let $\{X_1, \ldots, X_n\}$ be a set of indeterminates over R and let $\phi : R[X_1, \ldots, X_n] \to R[t_1, \ldots, t_n]$ be the evaluation homomorphism such that $\phi|_R$ is the identity map on R and ϕ sends X_i to t_i for all $i \in \{1, \ldots, n\}$. Then $R[t_1, \ldots, t_n]$ is isomorphic to $R[X_1, \ldots, X_n]/\mathrm{Ker}(\phi)$. Since R is an S-Noetherian ring and S is an anti-Archimedean subset of R, $R[X_1, \ldots, X_n]$ is an S-Noetherian ring (Proposition 9 [4]) or (Corollary 3.3 [5]). Hence $R[X_1, \ldots, X_n]/\mathrm{Ker}(\phi)$ is an S-Noetherian $R[X_1, \ldots, X_n]$-module (Lemma 2.14(1) [5]).

Let $S/\mathrm{Ker}(\phi) = \{s + \mathrm{Ker}(\phi) \,|\, s \in S\}$. Then $S/\mathrm{Ker}(\phi)$ is a multiplicative subset of $R[X_1, \ldots, X_n]/\mathrm{Ker}(\phi)$. Since $R[X_1, \ldots, X_n]/\mathrm{Ker}(\phi)$ is an S-Noetherian $R[X_1, \ldots, X_n]$-module, $R[X_1, \ldots, X_n]/\mathrm{Ker}(\phi)$ is an $(S/\mathrm{Ker}(\phi))$-Noetherian ring. Note that $R[X_1, \ldots, X_n]/\mathrm{Ker}(\phi)$ is isomorphic to $R[t_1, \ldots, t_n]$ and $S/\mathrm{Ker}(\phi)$ is isomorphic to S. Hence $R[t_1, \ldots, t_n]$ is an S-Noetherian ring.

Let I be an ideal of T. Then $s_1 I \subseteq I \cap R[t_1, \ldots, t_n]$. Since $R[t_1, \ldots, t_n]$ is an S-Noetherian ring, there exist $s_2 \in S$ and $a_1, \ldots, a_m \in I \cap R[t_1, \ldots, t_n]$ such that

$$s_2(I \cap R[t_1, \ldots, t_n]) \subseteq (a_1, \ldots, a_m)R[t_1, \ldots, t_n].$$

Therefore we obtain

$$s_1 s_2 I \subseteq s_2(I \cap R[t_1, \ldots, t_n]) \subseteq (a_1, \ldots, a_m)T \subseteq I.$$

Hence I is an S-finite ideal of T. Thus T is an S-Noetherian ring. □

Let Γ be a commutative monoid, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ a Γ-graded ring and S a multiplicative subset of R_0. Then we say that R is a *graded S-Noetherian ring* if every homogeneous ideal of R is S-finite.

We are now ready to give the main result in this section.

Theorem 1. *Suppose that Γ is a finitely generated abelian group. Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded ring and let S be an anti-Archimedean subset of R_0. Then the following statements are equivalent.*

(1) *R is an S-Noetherian ring.*
(2) *R is a graded S-Noetherian ring.*
(3) *R_0 is an S-Noetherian ring and R is an S-finite R_0-algebra.*

Proof. (1) ⇒ (2) This implication follows from definitions of S-Noetherian rings and graded S-Noetherian rings.

(2) ⇒ (3) Suppose that R is a graded S-Noetherian ring.

Case 1. $\Gamma = \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$. We use the induction on n. If $n = 0$, then there is nothing to prove. If $n = 1$, then the result comes directly from Lemmas 3 and 4.

Fix an integer $n \geq 2$ and suppose that the result is true for $\Gamma = \mathbb{Z}^{n-1}$. For each $\beta \in \mathbb{Z}$, let $A_\beta = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i = \beta\}$ and let $T_\beta = \bigoplus_{\alpha \in A_\beta} R_\alpha$. Let $T = \bigoplus_{\beta \in \mathbb{Z}} T_\beta$. Then it is routine to see that T is a $\mathbb{Z}$-graded ring. Note that $R = T$ as sets; so $T^+ \cdot T$, $T^- \cdot T$ and $T_\beta \cdot T$ for all $\beta \in \mathbb{Z}$ are homogeneous ideals of R. Since R is a graded S-Noetherian ring, $T^+ \cdot T$, $T^- \cdot T$ and $T_\beta \cdot T$ for all $\beta \in \mathbb{Z}$ are S-finite ideals of T. Note that S is an anti-Archimedean subset of T_0; so by Lemma 4, T is an S-finite T_0-algebra.

Let $\phi : \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^n$ be the group homomorphism given by $\phi(r_1, \ldots, r_{n-1}) = (r_1, \ldots, r_{n-1}, -\sum_{i=1}^{n-1} r_i)$ for all $(r_1, \ldots, r_{n-1}) \in \mathbb{Z}^{n-1}$. For each $\gamma \in \mathbb{Z}^{n-1}$, let $C_\gamma = R_{\phi(\gamma)}$ and consider $C := \bigoplus_{\gamma \in \mathbb{Z}^{n-1}} C_\gamma$. Then it is easy to check that C is both a $\mathbb{Z}^{n-1}$-graded ring and a subring of R. Note that $C = T_0$ as sets and every homogeneous element of C is homogeneous in R. Let I be a homogeneous ideal of C. Then IR is a homogeneous ideal of R. Since R is a graded S-Noetherian ring, IR is an S-finite ideal of R; so by Lemma 1, I is an S-finite ideal of C. Therefore C is a graded S-Noetherian ring. By the induction hypothesis, C_0 is an S-Noetherian ring and C is an S-finite C_0-algebra. Since T is an S-finite T_0-algebra and T_0 is an S-finite C_0-algebra, T is an S-finite C_0-algebra by Lemma 5. Note that $R = T$ and $C_0 = R_{\phi(0)} = R_0$. Hence R_0 is an S-Noetherian ring and R is an S-finite R_0-algebra.

Thus by the induction, the result holds for all $n \in \mathbb{N}_0$.

Case 2. We consider the general case. Let G denote the torsion part of Γ. Then there exists an integer $n \geq 0$ such that $\Gamma = \mathbb{Z}^n \oplus G$. For each $g \in G$, let $A_g = \bigoplus_{\alpha \in \mathbb{Z}^n} R_{\alpha+g}$, and let $A = \bigoplus_{g \in G} A_g$. Then it is easy to see that A is a G-graded ring. Note that $A = R$ as sets. Let $g \in G$ be fixed. Then $A_g \cdot A$ is a homogeneous ideal of R. Since R is a graded S-Noetherian ring, $A_g \cdot A$ is an S-finite ideal of R. Hence by Lemma 2, A_g is an S-finite A_0-module. Since G is a finite set, A is an S-finite A_0-module, and hence R is an S-finite A_0-module.

Let I be a homogeneous ideal of A_0. Then IR is a homogeneous ideal of R. Since R is a graded S-Noetherian ring, IR is an S-finite ideal of R; so by Lemma 1, I is an S-finite ideal of A_0. Hence A_0 is a graded S-Noetherian ring as a $\mathbb{Z}^n$-graded ring. Note that $(A_0)_0 = R_0$; so by Case 1, R_0 is an S-Noetherian ring and A_0 is an S-finite R_0-algebra. Since R is an S-finite A_0-module and A_0 is an S-finite R_0-algebra, Lemma 5 forces R to be an S-finite R_0-algebra.

(3) $\Rightarrow$ (1) This implication follows directly from Lemma 6. $\square$

Corollary 1. *Let Γ_0 be a submonoid of a finitely generated abelian group, $R = \bigoplus_{\alpha \in \Gamma_0} R_\alpha$ a Γ_0-graded ring and S an anti-Archimedean subset of R_0. Then the following assertions are equivalent.*

(1) *R is an S-Noetherian ring.*
(2) *R is a graded S-Noetherian ring.*
(3) *R_0 is an S-Noetherian ring and R is an S-finite R_0-algebra.*

Proof. Let Γ_0 be a submonoid of a finitely generated abelian group Γ and let

$$D_\alpha = \begin{cases} R_\alpha & \text{if } \alpha \in \Gamma_0 \\ \{0\} & \text{otherwise.} \end{cases}$$

Let $D = \bigoplus_{\alpha \in \Gamma} D_\alpha$. Then $D = R$; so R can be regarded as a Γ-graded ring. Thus the equivalences follow directly from Theorem 1. $\square$

Let Γ be a commutative monoid and let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a Γ-graded ring with identity. Then we say that R is a *graded Noetherian ring* if every homogeneous ideal of R is finitely generated. Let S be a multiplicative subset of R_0. If S is the set of units of R_0, then S is an anti-Archimedean subset of R_0 and the concept of graded S-Noetherian rings (resp., S-Noetherian rings, S-finite algebras) is precisely the same as that of graded Noetherian rings (resp., Noetherian rings, finitely generated algebras). Hence by Corollary 1, we obtain

Corollary 2. (cf. *(Theorem 1.1. [14])) Let Γ_0 be a submonoid of a finitely generated abelian group and let $R = \bigoplus_{\alpha \in \Gamma_0} R_\alpha$ be a Γ_0-graded ring. Then the following conditions are equivalent.*

(1) *R is a Noetherian ring.*
(2) *R is a graded Noetherian ring.*
(3) *R_0 is a Noetherian ring and R is a finitely generated R_0-algebra.*

We end this section with some examples which show that some conditions in Lemma 6 and Theorem 1 are not superfluous.

Example 1.

(1) *Let F be a field and let Y and Z be indeterminates over F. Let $\mathbf{T} = \{Y\} \cup \{YZ^{2^n} \mid n \in \mathbb{N}_0\}$, $D = F[\mathbf{T}]$, $S = D \setminus \{0\}$ and X an indeterminate over D. Then it is easy to see that D is an S-Noetherian ring. However, D is not agreeable [17] (p. 73); so $D[X]$ is not an S-Noetherian ring (Remark 2.1 [3]). (Recall from [3] (p. 4862) that an integral domain D with quotient field K is* agreeable *if for each fractional ideal I of $D[X]$ with $I \subseteq K[X]$, there exists a nonzero element $d \in D$ such that $sI \subseteq D[X]$.) Note that $D[X]$ is an S-finite D-algebra and S is not an anti-Archimedean subset of D because $\bigcap_{n=1}^{\infty} Y^n D \cap S = \emptyset$. Hence the anti-Archimedean condition in Lemma 6 is essential.*

For each $\alpha \in \mathbb{Z}$, let

$$R_\alpha = \begin{cases} DX^\alpha & \text{if } \alpha \in \mathbb{N}_0 \\ \{0\} & \text{otherwise.} \end{cases}$$

Then $R_0 = D$ and $D[X] = \bigoplus_{\alpha \in \mathbb{Z}} R_\alpha$ is a $\mathbb{Z}$-graded ring. Hence the anti-Archimedean condition in (3) $\Rightarrow$ (1) in Theorem 1 is also essential.

(2) *Let p be a prime integer and let $\Gamma = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ be a nonfinitely generated additive abelian group. In (Proposition 3.1 [14]), the authors found an example of Γ-graded integral domains $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ such that R and R_0 are fields but R is not a finitely generated R_0-algebra. Thus the finitely generated condition on Γ in (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) in Theorem 1 is essential. This is the case when S consists of units of $R_{0+\mathbb{Z}}$.*

3. Semigroup Rings as Graded S-Noetherian Rings

In this section, we study the graded S-Noetherian property via the semigroup ring which is a special example of graded rings. To do this, we require the next lemma.

Lemma 7. *Let R be a commutative ring with identity, S a multiplicative subset of R and Γ a commutative monoid. Then the following assertions are equivalent.*

(1) *$R[\Gamma]$ is an S-finite R-algebra.*
(2) *$R[\Gamma]$ is a finitely generated R-algebra.*
(3) *Γ is finitely generated.*

Proof. (1) $\Rightarrow$ (3) Suppose that $R[\Gamma]$ is an S-finite R-algebra. Then there exist $s \in S$ and $f_1, \dots, f_n \in R[\Gamma]$ such that $sR[\Gamma] \subseteq R[f_1, \dots, f_n] \subseteq R[\Gamma]$. Note that $f_1, \dots, f_n \in R[X^{\alpha_1}, \dots, X^{\alpha_m}]$ for some $\alpha_1, \dots, \alpha_m \in \Gamma$; so we obtain

$$sR[\Gamma] \subseteq R[X^{\alpha_1}, \dots, X^{\alpha_m}] \subseteq R[\Gamma].$$

Let Γ_0 be the submonoid of Γ generated by the set $\{\alpha_1, \dots, \alpha_m\}$ and let $\alpha \in \Gamma$. Then $sX^\alpha \in R[X^{\alpha_1}, \dots, X^{\alpha_m}]$. Hence there exist nonnegative integers $k_1, \dots, k_m$ such that $\alpha = \sum_{i=1}^m k_i \alpha_i$, which shows that $\alpha \in \Gamma_0$. Thus $\Gamma = \Gamma_0$, which indicates that Γ is finitely generated.

(3) $\Rightarrow$ (2) Suppose that Γ is generated by the set $\{\alpha_1, \dots, \alpha_m\}$ and let $\alpha \in \Gamma$. Then $\alpha = \sum_{i=1}^m k_i \alpha_i$ for some nonnegative integers $k_1, \dots, k_m$; so $X^\alpha = (X^{\alpha_1})^{k_1} \cdots (X^{\alpha_m})^{k_m} \in R[X^{\alpha_1}, \dots, X^{\alpha_m}]$. Hence $R[\Gamma] = R[X^{\alpha_1}, \dots, X^{\alpha_m}]$. Thus $R[\Gamma]$ is a finitely generated R-algebra.

(2) $\Rightarrow$ (1) This implication is obvious. □

Theorem 2. *Let R be a commutative ring with identity, S an anti-Archimedean subset of R and Γ a submonoid of a finitely generated abelian group. Then the following statements are equivalent.*

(1) *$R[\Gamma]$ is an S-Noetherian ring.*

(2) $R[\Gamma]$ *is a graded* S*-Noetherian ring.*

(3) R *is an* S*-Noetherian ring and* Γ *is finitely generated.*

Proof. Suppose that Γ is a submonoid of a finitely generated abelian group G. Then $R[\Gamma] = \bigoplus_{\alpha \in G} R_\alpha$ is a G-graded ring, where

$$R_\alpha = \begin{cases} \{aX^\alpha \mid a \in R\} & \text{if } \alpha \in \Gamma \\ \{0\} & \text{otherwise.} \end{cases}$$

In particular, $R_0 = R$. Note that by Theorem 1, $R[\Gamma]$ is an S-Noetherian ring if and only if $R[\Gamma]$ is a graded S-Noetherian ring, if and only if R is an S-Noetherian ring and $R[\Gamma]$ is an S-finite R-algebra; and by Lemma 7, $R[\Gamma]$ is an S-finite R-algebra if and only if Γ is finitely generated. Thus (1), (2) and (3) are equivalent. □

In Theorem 2, if S consists of units in R, then we recover

Corollary 3. (cf. *(Corollary 1.2 [14])) Let* R *be a commutative ring with identity and let* Γ *be a submonoid of a finitely generated abelian group. Then the following conditions are equivalent.*

(1) $R[\Gamma]$ *is a Noetherian ring.*

(2) $R[\Gamma]$ *is a graded Noetherian ring.*

(3) R *is a Noetherian ring and* Γ *is finitely generated.*

Let R be a commutative ring with identity and let S be a multiplicative subset of R. In (Corollary 5 [4]) or (Corollary 2.3 [6]), it was shown that R is an S-Noetherian ring if and only if every prime ideal of R (disjoint from S) is S-finite. This result is known as the Cohen type theorem for S-Noetherian rings.

We next give the Cohen type theorem for graded S-Noetherian rings.

Proposition 1. *Suppose that* Γ *is a torsion-free cancellative monoid. Let* $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ *be a* Γ*-graded ring and let* S *be a multiplicative subset of* R_0*. Then the following assertions are equivalent.*

(1) R *is a graded* S*-Noetherian ring.*

(2) *Every homogeneous prime ideal of* R *is* S*-finite.*

Proof. (1) ⇒ (2) This implication is obvious.

(2) ⇒ (1) Suppose to the contrary that R is not a graded S-Noetherian ring and let $\mathcal{F}$ be the set of homogeneous ideals of R which is not S-finite. Then by the assumption, $\mathcal{F}$ is nonempty. Let $\{I_\beta \mid \beta \in \Lambda\}$ be a chain of elements in $\mathcal{F}$ and let $I = \bigcup_{\beta \in \Lambda} I_\beta$. Then I is a homogeneous ideal of R. Suppose that I is not S-finite. Then there exist $s \in S$ and $a_1, \ldots, a_n \in R$ such that $sI \subseteq (a_1, \ldots, a_n) \subseteq I$; so for some $\beta \in \Lambda$, $sI_\beta \subseteq (a_1, \ldots, a_n) \subseteq I_\beta$. This shows that I_β is an S-finite ideal of R, which is a contradiction. Consequently, I is not an S-finite ideal of R. Also, it is obvious that I is an upper bound of the chain $\{I_\beta \mid \beta \in \Lambda\}$. By Zorn's lemma, there exists a maximal element in $\mathcal{F}$, say P. Suppose that P is not a prime ideal of R. Then there exist homogeneous elements $a, b \in R \setminus P$ such that $ab \in P$ (p. 124, Lemma 13 [1]). Since $P + (a)$ is a homogeneous ideal of R properly containing P, $P + (a)$ is an S-finite ideal of R by the maximality of P. Hence there exist $s \in S$, $p_1, \ldots, p_n \in P$ and $r_1, \ldots, r_n \in R$ such that

$$s(P + (a)) \subseteq (p_1 + ar_1, \ldots, p_n + ar_n) \subseteq P + (a).$$

Since $(P : a)$ is a homogeneous ideal of R containing P and b, $(P : a)$ is an S-finite ideal of R by the maximality of P; so there exist $t \in S$ and $q_1, \ldots, q_m \in R$ such that

$$t(P : a) \subseteq (q_1, \ldots, q_m) \subseteq (P : a).$$

Let $x \in P$. Then $sx = \sum_{i=1}^{n} u_i(p_i + ar_i)$ for some $u_1, \ldots, u_n \in R$. Since $a(\sum_{i=1}^{n} u_i r_i) = sx - \sum_{i=1}^{n} u_i p_i \in P$, we have $\sum_{i=1}^{n} u_i r_i \in (P : a)$. Therefore $t(\sum_{i=1}^{n} u_i r_i) \in (q_1, \ldots, q_m)$. Hence we obtain

$$\begin{aligned} stx &= \sum_{i=1}^{n} tu_i p_i + \sum_{i=1}^{n} tu_i r_i a \\ &\in (tp_1, \ldots, tp_n, q_1 a, \ldots, q_m a). \end{aligned}$$

Since $q_i \in (P : a)$ for all $i \in \{1, \ldots, m\}$, $q_i a \in P$ for all $i \in \{1, \ldots, m\}$; so we have

$$stP \subseteq (tp_1, \ldots, tp_n, q_1 a, \ldots, q_m a) \subseteq P.$$

This means that P is an S-finite ideal of R, which is a contradiction to the choice of P. Thus R is a graded S-Noetherian ring. □

Let $R[\Gamma]$ be the semigroup ring of Γ over R. For an element $f = \sum_{i=1}^{n} a_i X^{\alpha_i} \in R[\Gamma]$, $c(f)$ denotes the ideal of R generated by the set $\{a_1, \ldots, a_n\}$.

We next give the main result in this section.

Theorem 3. (cf. *Propositions 3.1 and 3.3 [11]) Let R be a commutative ring with identity, S a multiplicative subset of R and Γ a torsion-free cancellative monoid. Then the following statements are equivalent.*

(1) *$R[\Gamma]$ is a graded S-Noetherian ring.*
(2) *R is an S-Noetherian ring and every ideal of Γ is finitely generated.*

Proof. (1) ⇒ (2) Let I be an ideal of R. Then $IR[\Gamma]$ is a homogeneous ideal of $R[\Gamma]$. Since $R[\Gamma]$ is a graded S-Noetherian ring, there exist $s \in S$ and $f_1, \ldots, f_n \in R[\Gamma]$ such that

$$sIR[\Gamma] \subseteq (f_1, \ldots, f_n) \subseteq IR[\Gamma].$$

Therefore $sI \subseteq c(f_1) + \cdots + c(f_n) \subseteq I$. Hence I is an S-finite ideal of R. Thus R is an S-Noetherian ring.

Let J be an ideal of Γ and let A be the ideal of $R[\Gamma]$ generated by the set $\{X^\alpha \mid \alpha \in J\}$. Then A is a homogeneous ideal of $R[\Gamma]$. Since $R[\Gamma]$ is a graded S-Noetherian ring, there exist $t \in S$ and $g_1, \ldots, g_m \in R[\Gamma]$ such that

$$tA \subseteq (g_1, \ldots, g_m) \subseteq A.$$

Note that $g_1, \ldots, g_m \in (X^{\alpha_1}, \ldots, X^{\alpha_n})$ for some $\alpha_1, \ldots, \alpha_n \in J$; so we obtain

$$tA \subseteq (X^{\alpha_1}, \ldots, X^{\alpha_n}) \subseteq A.$$

Let F be the ideal of Γ generated by the set $\{\alpha_1, \ldots, \alpha_n\}$ and let $\alpha \in J$. Then $tX^\alpha \in (X^{\alpha_1}, \ldots, X^{\alpha_n})$; so there exists an element $k \in \{1, \ldots, n\}$ such that $\alpha \in \alpha_k + \Gamma$. Therefore $\alpha \in F$. Hence $J = F$, which implies that J is a finitely generated ideal of Γ. Thus every ideal of Γ is finitely generated.

(2) ⇒ (1) Let P be a homogeneous prime ideal of $R[\Gamma]$ and let f be a nonzero homogeneous element of P. Then $f = aX^\alpha$ for some $a \in R \setminus \{0\}$ and $\alpha \in \Gamma$. Since P is a prime ideal of $R[\Gamma]$, we obtain that $a \in P \cap R$ or $X^\alpha \in P$. Therefore every homogeneous generator of P can be chosen in $(P \cap R) \cup \{X^\alpha \mid X^\alpha \in P\}$. Since R is an S-Noetherian ring, there exist $s \in S$ and $a_1, \ldots, a_n \in R$ such that

$$s(P \cap R) \subseteq (a_1, \ldots, a_n) \subseteq P \cap R.$$

Let $A = \{\alpha \in \Gamma \mid X^\alpha \in P\}$ and let J be the ideal of Γ generated by the set A. Then by the assumption, there exist $\alpha_1, \ldots, \alpha_m \in A$ such that

$$J = \bigcup_{i=1}^{m} (\alpha_i + \Gamma).$$

Hence we obtain

$$sP \subseteq (a_1, \dots, a_n, X^{\alpha_1}, \dots, X^{\alpha_m}) \subseteq P,$$

which shows that P is an S-finite ideal of $R[\Gamma]$. Thus by Proposition 1, $R[\Gamma]$ is a graded S-Noetherian ring. □

When $S = \{1\}$ in Theorem 3, we recover

Corollary 4. (*Theorem 2.4 [15]*) *Let R be a commutative ring with identity and Γ a torsion-free cancellative monoid. Then the following conditions are equivalent.*

(1) *$R[\Gamma]$ is a graded Noetherian ring.*
(2) *R is a Noetherian ring and every ideal of Γ is finitely generated.*

We are closing this paper with an example of graded S-Noetherian rings which are not S-Noetherian rings.

Example 2. Let D and S be as in Example 1(1).

(1) Note that every ideal of $\mathbb{N}_0$ is of the form $\{n, n+1, n+2, \dots\}$ for some $n \in \mathbb{N}_0$; so every ideal of $\mathbb{N}_0$ is finitely generated. Thus by Theorem 3, $D[X]$ is a graded S-Noetherian ring.
(2) Note that by Example 1(1), $D[X]$ is not an S-Noetherian ring. Also, note that $D[X]$ is regarded as a $\mathbb{Z}$-graded ring as in Example 1(1). Hence the anti-Archimedean condition in (2) $\Rightarrow$ (1) in Theorem 1 is essential.

4. Conclusions

In this paper, we introduce the concept of graded S-Noetherian rings and determine when both the graded ring and the semigroup ring are graded S-Noetherian rings. More precisely, we show that if Γ is a finitely generated abelian group and S is an anti-Archimedean subset of R_0, then a Γ-graded ring $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded S-Noetherian ring if and only if R is an S-Noetherian ring, if and only if R_0 is an S-Noetherian ring and R is an S-finite R_0-algebra. We also prove that if Γ is a torsion-free cancellative monoid, then the semigroup ring $R[\Gamma]$ is a graded S-Noetherian ring if and only if R is an S-Noetherian ring and every ideal of Γ is finitely generated. By constructing an example from our results, we find out that the concept of graded S-Noetherian rings is different from that of S-Noetherian rings. Furthermore, we discover the existence of polynomial type rings of graded S-Noetherian rings without any condition on S. This is a big difference from S-Noetherian rings because the polynomial extension of S-Noetherian rings is possible under some condition on S (Proposition 9 [4]).

In ensuing work, we are going to study another properties of graded S-Noetherian rings including the generalized power series ring extension, the Nagata's idealization and the amalgamated algebra. As one of the referees suggested, we will also try to find more applications of our results to several areas of mathematics including algebraic geometry.

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