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Remarks on Surjectivity of Gradient Operators

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Abstract: Let X be a real Banach space with dual X^* and suppose that $F : X \to X^*$. We give a characterisation of the property that F is locally proper and establish its stability under compact perturbation. Modifying an recent result of ours, we prove that any gradient map that has this property and is additionally bounded, coercive and continuous is surjective. As before, the main tool for the proof is the Ekeland Variational Principle. Comparison with known surjectivity results is made; finally, as an application, we discuss a Dirichlet boundary-value problem for the p-Laplacian $(1 < p < \infty)$, completing our previous result which was limited to the case $p \geq 2$.

Keywords: coercive operator; locally proper operator; Ekeland's variational principle; operator of type (S); p-Laplacian

1. Introduction

This is partly a research paper and partly a review paper, whose main purpose is to discuss a variant of the surjectivity result that we have proved in [1]. The latter asserts the following: let X be a real Banach space with dual X^*, let $\langle x, y \rangle$ denote the pairing with $x \in X^*$ and $y \in X$ and let $F : X \to X^*$ be a continuous gradient operator. Suppose that F is strongly coercive in the sense that for some $c > 0$ and some $p > 1$,

$$\langle F(x), x \rangle \geq c\|x\|^p \tag{1}$$

for all $x \in X$, and suppose moreover that F is locally proper, by which we mean that given any closed bounded set M of X, the set $M \cap F^{-1}(K)$ is compact whenever $K \subset X^*$ is compact. Under the above conditions, $F(X) = X^*$. In Section 4 of [1], we gave an example of an application of this abstract result by considering a boundary-value problem for the p-Laplace equation in a bounded domain Ω of $\mathbb{R}^n$, as we shall better indicate at the end of this introduction.

With reference to [1], the purpose of the present paper was two-fold:

- to complete the discussion of the abstract result, Theorem 2.1 in [1], commenting both on the coercivity condition (1) and on the local properness of F;
- to extend the application of the abstract result to the existence of solutions for the cited problem for the p-Laplacian on considering the range $1 < p < 2$: indeed in [1], only the range $2 \leq p < \infty$ was considered.

To explain the restriction to the range $p \geq 2$ used in [1], let us briefly recall the concrete way followed there in order to ensure the required local properness of the operator involved. Let $\alpha(A)$ denote the measure of noncompactness (see, e.g., [2]) of a bounded subset A of either X or X^*, and assume that $F : X \to X^*$ is bounded on bounded subsets of X. For $0 < \gamma < \infty$, put

$$\omega_\gamma(F) = \inf \left\{ \frac{[\alpha(F(A))]^\gamma}{\alpha(A)} : A \subset X, A \text{ bounded}, \alpha(A) > 0 \right\}. \tag{2}$$

Then, if $\omega_\gamma(F) > 0$ for some $\gamma \in]0, \infty[$, F is locally proper. This is proved in [1] (Proposition 3.3); related results are to be found, for instance, in the papers [3–5] and in the book [2].

When we consider maps of the form $F = T + N$ (as is the case if we deal with perturbed forms of the p-Laplacian), the application of this criterion for local properness seems to be successful only when T is strongly monotone in the sense that for some $k > 0$ and some $p \in [2, \infty]$,

$$\langle T(x) - T(y), x - y \rangle \geq k\|x - y\|^p \tag{3}$$

for all $x, y \in X$. As we have remarked in [1], the restriction of p to the interval $[2, \infty]$ seems to be unavoidable, because—at least in case X is reflexive—when $1 < p < 2$ there are no maps F satisfying (3). In addition, from the direct standpoint of existence results for solutions of differential equations, it is well known (see, for instance, [6] (Exercise 6.2.13)) that for $1 < p < 2$, the operator T representing the p-Laplacian and defined by the equality

$$\langle T(u), v \rangle_X := \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla v dx \tag{4}$$

for u, v in the Sobolev space $X = \overset{0}{W}{}^1_p(\Omega)$, is not strongly monotone but merely strictly monotone, this last meaning that for all $x, y \in X$ with $x \neq y$,

$$\langle T(x) - T(y), x - y \rangle > 0.$$

This difficulty made us reconsider the local properness property itself, and seek different but equivalent forms of expressing it. We discuss this point in Section 3, in the framework of the "generalized monotonicity" properties introduced by F. Browder and illustrated for instance in his papers [7,8], where they are named conditions of type (S); to this purpose, we introduce a new one that we call $(S)_2$ and that is apparently weaker than many others in the same class. For continuous maps, condition $(S)_2$ turns out to be equivalent to local properness (see Proposition 5); however, we believe that it is worth keeping the $(S)_2$ description because it facilitates comparison with the notion of monotonicity.

The remainder of this paper is organized as follows. In Section 2, we state and prove our main result, Theorem 1, in which the strong coercivity assumption used in [1] is replaced by simple coercivity (see below for the precise definition) at the expense of adding the requirement of boundedness on F. For the proof of the desired surjectivity of F, this replacement of assumptions does not seem to be trivial, and we put evidence on the necessary changes by demonstrating two new "ad hoc" results, Lemma 1 and Proposition 2. However, the core of the proof of Theorem 1 remains—as in [1,9]—the Ekeland variational principle (see, e.g., [10]), employed after converting the existence problem

$$F(x) = y$$

with y given in X^*, into a minimization problem for the potential f of F, or more precisely for the perturbed functional f_1 defined on X by putting

$$f_1(x) = f(x) - \langle y, x \rangle, \qquad x \in X. \tag{5}$$

As a whole, the overall demonstration of the surjectivity property for gradient mappings is here reorganized with respect to that given in [1], and we believe that it gains in clarity, facilitating among others a comparison with companion results for non-gradient operators that abound in the literature (see for instance the books [6,11] and the recent review [12] by H. Brézis) and that have their prototype in the famous Minty–Browder Theorem (see, e.g., Theorem 2 of [6]) stating that if X is a reflexive Banach space and if $T : X \to X^*$ is monotone, continuous and coercive, then $T(X) = X^*$. Another statement of surjectivity, proved by Browder in the same paper (Theorem 5 of [6]) under somewhat different assumptions, is reported here as Theorem 3 in Section 3. It is important to notice that while the proofs

of these and similar results are based on the convergence of finite-dimensional approximations to solutions of the equation $T(x) = y$, in the case of gradient operators we can rely upon the full strength of the Ekeland principle as a general result for functionals defined on complete metric spaces that makes no reference to finite-dimensional approximation. To give some more details about this point, we devote part of Section 2 to the statements of the Ekeland principle itself (see Theorem 2) and to the special version for C^1 functionals on Banach spaces that it implies (see Corollary 1).

In the final Section 4, we point out the modifications with respect to [1] necessary to deal with the problem

$$ -\Delta_p u - \lambda_1 |u|^{p-2} u + f(x, u) = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{6} $$

for the whole range $1 < p < \infty$. Here, $\Delta_p u = \text{div} \left(|\nabla u|^{p-2} \nabla u \right)$ is the p-Laplacian, Ω is a bounded open subset of $\mathbb{R}^n$, $h \in L_{p'}(\Omega)$ $(p' = p/(p-1))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies adequate assumptions; moreover, λ_1 is the first eigenvalue of the Dirichlet p-Laplacian in Ω [13]. We see (6) as a perturbation of the same problem where $f = 0$, namely

$$ -\Delta_p u - \lambda_1 |u|^{p-2} u = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{7} $$

As recalled in [1], (7) is not solvable for every $h \in L_{p'}(\Omega)$, and we ask if perturbing (7) with an appropriate additional term f we can restore surjectivity for the original problem (6). Indeed, using the results of Section 2 we complete the discussion of this point made in [1] for the case $p \geq 2$ by allowing any p in the range $(1, \infty)$. Precisely, we prove that (6) has a solution $u \in \overset{0}{W}{}^1_p(\Omega)$ for any $h \in L_{p'}(\Omega)$ provided that f satisfies, besides the standard regularity and growth assumptions, a coercivity condition of the form $sf(x, s) \geq m |s|^p$, for some $m > 0$ and all $(x, s) \in \Omega \times \mathbb{R}$.

2. A Variant of a Surjectivity Theorem

Let X be a real Banach space with norm $\|.\|$ and dual X^*. We denote by $\langle x, y \rangle$ the pairing between $x \in X^*$ and $y \in X$. Recall (see e.g., [14], Definition 2.5.1) that a map $F : X \to X^*$ is said to be a gradient (or potential) operator if there exists a differentiable functional $f : X \to \mathbb{R}$ such that

$$ F(x) = f'(x) \qquad \text{for all} \quad x \in X \tag{8} $$

where $f'(x) \in X^*$ denotes the (Fréchet) derivative of f at the point $x \in X$. In this case, the functional f—the potential of F—is defined up to an additive constant, and is made unique by requiring—as we do—that $f(0) = 0$. Assuming in addition that F is continuous, f is explicitly related to F via the equation

$$ f(x) = \int_0^1 \langle F(tx), x \rangle \, dt. \tag{9} $$

We recall moreover that given a differentiable functional $f : X \to \mathbb{R}$, a point $x \in X$ is said to be a critical point of f if $f'(x) = 0$. Therefore, the zeroes of a gradient operator are precisely the critical points of its potential.

A map $F : X \to X^*$ that is bounded on bounded subsets of X will be merely said to be bounded.

In the following, the coercivity of operators and functionals will play an essential role. To avoid ambiguities and/or repeated distinctions, we define this property in a formal way.

Definition 1. *A map $F : X \to X^*$ is said to be coercive if*

$$ \frac{\langle F(x), x \rangle}{\|x\|} \to +\infty \quad as \quad \|x\| \to \infty. \tag{10} $$

A functional $f : X \to \mathbb{R}$ is said to be coercive if

$$f(x) \to +\infty \quad as \quad ||x|| \to \infty. \tag{11}$$

Remark 1. *Other notions of coercivity of a map $F : X \to X^*$ are present in the literature. In particular, such a map F is said to be strongly coercive if, for some $c > 0$ and some $p > 1$,*

$$\langle F(x), x \rangle \geq c||x||^p \tag{12}$$

for all $x \in X$, and weakly coercive if

$$||F(x)|| \to \infty \quad as \quad ||x|| \to \infty. \tag{13}$$

Evidently, we have the implications

$$F \text{ strongly coercive} \Rightarrow F \text{ coercive} \Rightarrow F \text{ weakly coercive.}$$

The Lemma that follows establishes a useful relation between the two coercivity properties reported in Definition 1.

Lemma 1. *Let X be a real Banach space and let $F : X \to X^*$ be a gradient operator. Suppose that F is continuous, bounded and coercive. Then its potential f is bounded on bounded subsets of X and is also coercive: more precisely, given any $M > 0$ there exists $R > 0$ such that, for all $x \in X$ with $||x|| > R$ we have*

$$f(x) \geq M||x|| - (M + K_R)R, \qquad K_R = \sup_{||x|| \leq R} ||F(x)||. \tag{14}$$

Proof. The boundedness property of f follows immediately from that of F using the Formula (9). As to coercivity, for $x \neq 0$ write $x = ||x||u$, $u = x/||x||$; then using again (9) we have

$$f(x) = \int_0^1 \langle F(t||x||u, ||x||u \rangle \, dt = \int_0^{||x||} \langle F(su), u \rangle \, ds.$$

Given any $M > 0$, by (10) there is an $R > 0$ such that

$$\langle F(x), x \rangle \geq M||x|| \qquad (||x|| \geq R). \tag{15}$$

Then for $||x|| > R$, write

$$f(x) = \int_0^R \langle F(su), u \rangle \, ds + \int_R^{||x||} \langle F(su), u \rangle \, ds. \tag{16}$$

Using the definition of K_R given in (14), we have $|\langle F(su), u \rangle| \leq ||F(su)|| \leq K_R$ for $0 \leq s \leq R$ and $u \in X$ with $||u|| = 1$, whence

$$\left| \int_0^R \langle F(su), u \rangle \, ds \right| \leq K_R R. \tag{17}$$

On the other hand, using (15), we have

$$\int_R^{||x||} \langle F(su), u \rangle \, ds = \int_R^{||x||} \frac{\langle F(su), su \rangle}{s} \rangle \, ds \geq \int_R^{||x||} M \, ds = M(||x|| - R) \tag{18}$$

The desired inequality for f now follows from (16), using (17) and (18). $\square$

Before stating and proving our main result, we recall that a map $F : X \to Y$ (X, Y metric spaces) is said to be proper if the preimage $F^{-1}(K)$ is a compact subset of X whenever $K \subset Y$ is compact, and is

said to be proper on closed bounded sets (or locally proper) if given any closed bounded set M of X, the set $M \cap F^{-1}(K)$ is compact whenever $K \subset Y$ is compact.

Theorem 1. *Let X be a real Banach space with dual X^*, and let $F : X \to X^*$ be a bounded continuous gradient operator. Suppose that F satisfies the following assumptions:*

(i) *F is coercive;*
(ii) *F is proper on closed bounded sets.*

Then F is surjective.

For expository convenience, we organize the proof of Theorem 1 in four essential steps. Each of them—save the first—has some degree of complexity, that we choose to illustrate separately, giving appropriate references in the text.

Proof. (a) Given $y \in X^*$, the map $G_y : X \to X^*$ defined by putting

$$G_y(x) = F(x) - y$$

satisfies (i) and (ii). It is therefore enough to show that under the conditions (i) and (ii), F has a zero. As F is a gradient, this is equivalent to showing that the potential f of F has a critical point, and in turn this will be ensured by verifying the condition that f is bounded below on X and attains its infimum on X.

(b) By our assumptions and by virtue of Lemma 1, f is coercive and is bounded on bounded subsets of X. In turn, by virtue of Proposition 2 below, that we state and prove independently, this implies that f is bounded below on X and that any minimizing sequence is bounded.

(c) As f is of class C^1, the Ekeland Variational Principle (see, for instance, [10]) implies the existence of a minimizing sequence along which the derivative of f tends to 0, that is, a sequence $(x_n) \subset X$ such that

$$f(x_n) \to c \equiv \inf_{x \in X} f(x) \quad \text{and} \quad F(x_n) = f'(x_n) \to 0.$$

Some details explaining how Ekeland's principle leads to the statement just made are given at the end of this section.

(d) The last sentence in point b) ensures that the sequence (x_n) is bounded. As $F(x_n) \to 0$ and as F is proper on closed bounded sets by assumption, it follows that (x_n) contains a convergent subsequence: this is made clear looking at Definition 5 and Proposition 5 of the next section. Letting (x_{n_k}) denote this subsequence and putting $x = \lim_{k \to \infty} x_{n_k}$, we then see immediately by the continuity of f and F that $f(x) = c$ and $F(x) = 0$.

□

Wishing to give adequate space for a full discussion of point (d) in the proof above, we place it separately in the next section. The rest of this section is thus devoted to illustrations of the steps (b) and (c) of the proof. As to the former, we note that the same conclusion of Theorem 1 was reached in [1] under a different set of assumptions, namely with no boundedness condition on F but with (i) replaced by the condition that F should be strongly coercive. This condition, namely (12), immediately implies—via (9)—that

$$f(x) \geq c' ||x||^{p+1} \quad (x \in X)$$

with $c' = c/p$, and this in turn led towards the conclusion on using the following result (Proposition 2.1 of [1]):

Proposition 1. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be such that*

$$f(x) \geq \phi(||x||) \tag{19}$$

where $\phi : [0, +\infty[\rightarrow [0, +\infty[$ is bounded on bounded sets and such that $\phi(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Then f is bounded below on X. Moreover, any minimizing sequence is necessarily bounded.

For the proof of the present Theorem 1, we have available the weaker coercivity condition (10), accompanied however by the boundedness of F. The technical Lemma 1 above shows that under this different set of hypotheses the potential f maintains quite good properties in view of its minimization, and in fact we can replace Proposition 1 with the following more general result.

Proposition 2. *Let X be a real Banach space and let $f : X \rightarrow \mathbb{R}$ be coercive and bounded below on bounded subsets of X. Then, f is bounded below on X and any minimizing sequence is necessarily bounded.*

Proof. We first claim that f is bounded below on X. Suppose on the contrary that $\inf_{x \in X} f(x) = -\infty$, and let $(x_n) \subset X$ be such that

$$f(x_n) \rightarrow -\infty. \tag{20}$$

The sequence (x_n) is necessarily bounded (otherwise there would exists a subsequence (x_{n_k}) with $||x_{n_k}|| \rightarrow \infty$, and thus $f(x_{n_k}) \rightarrow +\infty$ by (11), contradicting (20)). Then, by assumption, $f(x_{n_k})$ must be bounded below, contradicting again (20).

Therefore, f is bounded below on X. Using again (11) and reasoning as before, we also see that any minimizing sequence is bounded. $\square$

As for step (c) of the proof of Theorem 1, this presents no novelty with respect to [1], but we find it desirable both for completeness and for the reader's convenience to give here some more details. The original variational principle of Ekeland, in its "weak form" (see, e.g., [10] (Theorem 4.1)), states the following:

Theorem 2. *Let (X, d) be a complete metric space. Let $f : X \rightarrow \mathbb{R}$ be lower semicontinuous and bounded below. Put $c = \inf_{x \in X} f(x)$; then given any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that*

$$\begin{cases} f(x_\epsilon) < c + \epsilon, \\ f(x_\epsilon) < f(x) + \epsilon d(x, x_\epsilon), \quad \forall x \in X, \ x \neq x_\epsilon. \end{cases} \tag{21}$$

We use Ekeland's principle in the form given by the following statement, which is in fact a special form of Theorem 4.4 of [10].

Corollary 1. *Let f be a C^1 functional defined on the Banach space X and suppose that f is bounded below on X. Let $c = \inf_{x \in X} f(x)$. Then, given any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that*

$$\begin{cases} f(x_\epsilon) < c + \epsilon, \\ ||f'(x_\epsilon)|| \leq \epsilon. \end{cases} \tag{22}$$

Let us see how Corollary 1 follows from Theorem 2. First recall that the derivative $f'(x_0)$ of f at a given point $x_0 \in X$ is a bounded linear form on X, that satisfies the equality

$$f'(x_0)y = \lim_{t \rightarrow 0} \frac{f(x_0 + ty) - f(x_0)}{t} \tag{23}$$

for every $y \in X$, and whose norm in the dual X^* of X is by definition

$$||f'(x_0)|| = \sup \{|f'(x_0)v| : v \in X, ||v|| = 1\}. \tag{24}$$

Now given $\epsilon > 0$, let $x_\epsilon \in X$ be as in the statement of Theorem 2, so that the first inequality in (22) is satisfied. Additionally, given $v \in X$ with $||v|| = 1$, take $x = x_\epsilon + tv, t \neq 0$ in the second inequality in (21): this yields

$$f(x_\epsilon) < f(x_\epsilon + tv) + \epsilon|t|. \tag{25}$$

Thus, taking $t > 0$, we obtain

$$\frac{f(x_\epsilon + tv) - f(x_\epsilon)}{t} > -\epsilon$$

and therefore, letting $t \to 0^+$,

$$f'(x_\epsilon)v \geq -\epsilon. \tag{26}$$

Similarly, taking $t < 0$ in (25) yields

$$\frac{f(x_\epsilon + tv) - f(x_\epsilon)}{t} < \epsilon$$

whence, letting $t \to 0^-$,

$$f'(x_\epsilon)v \leq \epsilon. \tag{27}$$

Using (24), (26) and (27) then yields the second inequality in (22).

3. Operators of Type (S)

In this section, except for the two final statements (Proposition 5 and Theorem 4), we suppose that X is a reflexive Banach space. The definition and the theorem that follow are due to F. Browder: see, for instance [7] (Theorem 5).

Definition 2. *A map $T : X \to X^*$ is said to be of type (S) (or to satisfy condition (S)) if for every sequence $(x_n) \subset X$ which converges weakly to some $x \in X$ and is such that*

$$\langle T(x_n) - T(x), x_n - x \rangle \to 0 \tag{28}$$

we have that (x_n) converges strongly to x.

Theorem 3. *Let X be a real, reflexive, separable Banach space, and let $T : X \to X^*$ be of type (S), and be bounded, coercive and continuous. Then $T(X) = X^*$.*

The condition (S) is called in [7] a generalized monotonicity condition. To understand this, let us generalize—following, for instance, Amann's paper [15]—the definition presented in the introduction and say that a mapping $T : X \to X^*$ is called strongly monotone if there exists a continuous function $\alpha : [0, \infty) \to [0, \infty)$ which is positive on $(0, \infty)$ and satisfies $\alpha(r) \to \infty$ as $r \to \infty$ such that, for all $u, v \in X$,

$$\langle T(u) - T(v), u - v \rangle \geq \alpha(||u - v||)||u - v||.$$

Clearly, every strongly monotone operator satisfies condition (S).

Condition (S) was subsequently modified in various directions. Browder himself in [8] strengthened it as follows:

Definition 3. *A map $T : X \to X^*$ is said to be of type $(S)_+$ if whenever (x_n) converges weakly to $x \in X$ and*

$$\limsup \langle T(x_n), x_n - x \rangle \leq 0 \tag{29}$$

we have that $x_n \to x$.

That $(S)_+$ is a strengthening of (S) is clear since condition (28) is equivalent to require that $\langle T(x_n), x_n - x \rangle \to 0$. On the other hand, H. Amann proposed in [15] to weaken condition (S) as follows.

Definition 4. *A mapping $T : X \to X^*$ is said to satisfy condition $(S)_1$ if for every sequence (u_j) in X with $u_j \to u$ weakly and $T(u_j) \to v$ we have $u_j \to u$.*

To prove that condition $(S)_1$ generalizes condition (S), let $(u_j) \subset X$ with $u_j \to u$ weakly and $T(u_j) \to v$. Writing

$$\langle T(u_j) - T(u), u_j - u \rangle = \langle T(u_j) - v, u_j - u \rangle + \langle v - T(u), u_j - u \rangle$$

we see immediately that the left-hand side tends to zero as $j \to \infty$, and therefore that $u_j \to u$ by the assumption that T satisfies (S).

The following definition seems to be new, and we add it to the list of those concerning maps "of type (S)" to the aim of facilitating the comparison of our Theorem 1 with others existing in the literature.

Definition 5. *A map $T : X \to X^*$ is said to be of type $(S)_2$ if whenever $(x_n) \subset X$ is bounded and $(T(x_n))$ converges to some $y \in X$, (x_n) contains a convergent subsequence.*

Proposition 3. *Condition $(S)_1$ implies condition $(S)_2$.*

Indeed, let $F : X \to X^*$ satisfy $(S)_1$. Suppose that $(x_n) \subset X$ is bounded and that $F(x_n) \to y$, say. Let (x_{n_k}) be a subsequence of (x_n) such that (x_{n_k}) converges weakly to x_0, say. As $F(x_{n_k}) \to y$, by the assumption on F it follows that (x_{n_k}) converges strongly to x_0.

In summary, in the framework of reflexive spaces we have the following hierarchy for the definitions given above:

$$(S)_+ \Rightarrow (S) \Rightarrow (S)_1 \Rightarrow (S)_2. \tag{30}$$

Our next result proves that the newly defined operators share with most of those of type (S) the property of being stable under a compact perturbation: for instance, Lemma 5.8.33 of [6] shows this for operators of type $(S)_+$. We recall that $G : X \to X^*$ is said to be compact if $G(B)$ is a relatively compact subset of X^* whenever B is a bounded subset of X.

Proposition 4. *Suppose that $F : X \to X^*$ is of the form $F = T + G$, with T of type $(S)_2$ and G compact. Then, F is of type $(S)_2$.*

Indeed, let $(x_n) \subset X$ be bounded and suppose that $(F(x_n))$ converges to y, say. As G is compact, there exists a subsequence (x_{n_k}) of (x_n) such that $G(x_{n_k}) \to z$, say. Therefore, $T(x_{n_k}) = F(x_{n_k}) - G(x_{n_k}) \to y - z$; and as T is assumed to be of type (S_2), it follows that (x_{n_k}) contains a convergent subsequence, whence the result.

We note that our Definition 5 can be given without change for a map F acting between any two metric spaces X, Y. It is desirable to observe that in this quite general context, for continuous maps the property under discussion is nothing else than local properness.

Proposition 5. *Let X, Y be metric spaces and let $F : X \to Y$ be continuous. Then, F satisfies condition $(S)_2$ if and only if it is proper on closed bounded sets.*

Proof. Suppose first that F satisfies condition $(S)_2$. Let $M \subset X$ be closed and bounded and let $K \subset Y$ be compact. To show that $M \cap F^{-1}(K)$ is compact, take a sequence $(x_n) \subset M \cap F^{-1}(K)$: then (x_n) is bounded and, by the compactness of K, a subsequence $F(x_{n_k})$ of $F(x_n)$ will converge to some $y_0 \in K$.

By the assumption on F, a further subsequence of (x_{n_k}) will converge to a point $x_0 \in M$ (M is closed) with $F(x_0) = y_0$ by the continuity of F. Thus, $x_0 \in M \cap F^{-1}(K)$, proving that $M \cap F^{-1}(K)$ is compact.

Suppose on the other hand that given any closed bounded $M \subset X$, the set $M \cap F^{-1}(K)$ is compact whenever $K \subset Y$ is compact. Let $(x_n) \subset X$ be bounded and suppose that $F(x_n) \to y_0 \in Y$. Then, the set

$$K = \{F(x_n) : n \in \mathbb{N}\} \cup \{y_0\}$$

is compact, so that if $M \subset X$ is bounded and contains (x_n), it follows that the set

$$\{x_n : n \in \mathbb{N}\} \subset M \cap F^{-1}(K)$$

is relatively compact, and therefore (x_n) contains a convergent subsequence. This shows that F satisfies $(S)_2$. $\square$

By virtue of Proposition 5, our Theorem 1 can be restated in the following form, that allows for a direct comparison with Browder's Theorem 3:

Theorem 4. *Let X be a real Banach space, and let $T : X \to X^*$ be a gradient operator. Suppose that T is of type $(S)_2$, and is bounded, coercive and continuous. Then, $T(X) = X^*$.*

Obvious differences between these two results are that while Theorem 4 applies only to gradient maps T, for such maps the conditions imposed are less demanding than those of Theorem 3: the space X does not have to be reflexive and separable, and $(S)_2$ is weaker than (S). However, these differences in assumptions are, in our opinion, secondary to those of proof: the use of the notion of local properness and the involvement of the measure of noncompactness offer a line of attack to establish surjectivity for gradient operators that warrants further examination.

4. An Application to the *p*-Laplacian

Let $p \in (1, \infty)$, let Ω be a bounded open subset of $\mathbb{R}^n$, denote by $\|\cdot\|_p$ the usual norm on the Lebesgue space $L_p(\Omega)$ and put $X = \overset{0}{W}{}^1_p(\Omega)$, the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_X$ given by

$$\|u\|_X := \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}.$$

When given this norm, X is a reflexive Banach space that is compactly embedded in $L_p(\Omega)$. The p-Laplacian Δ_p is defined on appropriate functions u by

$$\Delta_p u = \text{div} \left(|\nabla u|^{p-2} \nabla u \right). \tag{31}$$

Associated with this is the map $T : X \to X^*$ given by

$$\langle T(u), v \rangle_X := \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \quad (u, v \in X), \tag{32}$$

where $\langle w, v \rangle_X$ denotes the value of $w \in X^*$ at $v \in X$. We detail various interesting properties of T.

First, we have

$$\langle T(u), u \rangle_X = \int_\Omega |\nabla u|^p \, dx = \|u\|_X^p \quad (u \in X). \tag{33}$$

Moreover, for $p \geq 2$, T is strongly monotone (see, e.g., Section 4 of [1]), while for any $p \in (1, \infty)$ T is of type (S); in fact, it satisfies the stronger condition $(S)_+$, as shown for instance in Lemma 5.9.14 of [6]. Additionally, use of appropriate inequalities in $\mathbb{R}^n$ [16], of Hölder's inequality and of standard procedures as in Section 4 of [1] show that for every $p \in (1, \infty)$, T is bounded and continuous. Finally,

we recall that T is a gradient operator since $\langle T(u), v \rangle_X$—given by (32)—is the directional derivative in the direction $v \in X$ of the C^1-functional

$$E(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx \qquad (u \in X).$$

These results are summarised in the following

Proposition 6. *The map $T : X \to X^*$ defined by (32) is a bounded continuous gradient operator that satisfies condition $(S)_+$.*

By an eigenvalue of the (Dirichlet) p-Laplacian is meant a real number λ such that there is a function $u \neq 0$ (an eigenfunction) such that

$$-\Delta_p u = \lambda \, |u|^{p-2} \, u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{34}$$

We interpret this eigenvalue problem in the weak sense, by which is meant that one asks whether there exist $\lambda \in \mathbb{R}$ and $u \in X \backslash \{0\}$ such that for all $v \in X$,

$$\int_\Omega |\nabla u|^{p-2} \, \nabla u \cdot \nabla v dx = \lambda \int_\Omega |u|^{p-2} \, uv dx. \tag{35}$$

In terms of the operator T, this means that

$$\langle T(u), v \rangle_X = \lambda \int_\Omega |u|^{p-2} \, uv dx. \tag{36}$$

It is well known (see [13]) that there is a principal eigenvalue, that is, a least such an eigenvalue, denoted by λ_1, and that it is positive, simple, isolated and characterised variationally by

$$\lambda_1 = \inf_{u \in X \backslash \{0\}} \|u\|_X^p \, / \, \|u\|_p^p. \tag{37}$$

This characterization of λ_1 and (33) show in particular that, for all $u \in X$,

$$\langle T(u), u \rangle_X = \|u\|_X^p \geq \lambda_1 \|u\|_p^p. \tag{38}$$

We consider the problem

$$-\Delta_p u - \lambda_1 \, |u|^{p-2} \, u \, + f(x, u) = h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{39}$$

where $h \in L_{p'}(\Omega) \, (p' = p/(p - 1))$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions

$$|f(x, s)| \leq a \, |s|^{p-1} + b \tag{40}$$

and

$$s f(x, s) \geq m \, |s|^p \tag{41}$$

for all $(x, s) \in \Omega \times \mathbb{R}$. Here a and b are non-negative constants and m is a positive constant; all of these are independent of (x, s). Problem (39) is also interpreted in the weak sense, so that we ask whether there exists $u \in X$ such that for all $v \in X$,

$$\int_\Omega |\nabla u|^{p-2} \, \nabla u \cdot \nabla v dx - \lambda_1 \int_\Omega |u|^{p-2} \, uv dx + \int_\Omega f(x, u) v dx = \int_\Omega hv dx. \tag{42}$$

In [17], Drábek and Holubová discussed such a problem with $f = 0$. When $1 < p < 2$, they gave conditions on h under which (4.18) has no solutions, and other conditions on h guaranteeing

the existence of at least one solution; they expect that similar results can be obtained when $2 < p < \infty$. Completing our results in [1], here we show that for any $p \in (1, \infty)$, the conditions on f given above are sufficient to ensure that there is a solution of (4.18) for all $h \in L_{p'}(\Omega)$. To this purpose, let $K, N : X \to X^*$ be defined respectively by

$$\langle K(u), v \rangle_X = \int_\Omega |u|^{p-2} uv dx, \tag{43}$$

$$\langle N(u), v \rangle_X = \int_\Omega f(x, u) v dx. \tag{44}$$

Note that, in view of (41), we have

$$\langle N(u), u \rangle_X \geq m \|u\|_p^p. \tag{45}$$

About the properties of these operators, we refer to the discussion already made in [1], that leads to the following result.

Proposition 7. *The maps $K, N : X \to X^*$ defined by (43), (44), respectively are bounded, continuous, compact gradient operators.*

The problem (42) is equivalent to the operator equation

$$T(u) - \lambda_1 K(u) + N(u) = \widehat{h}, \tag{46}$$

where $\widehat{h} \in X^*$ is defined by

$$\left\langle \widehat{h}, v \right\rangle_X = \int_\Omega hv dx \quad (v \in X).$$

Put $G = -\lambda_1 K + N$. It follows by Proposition 7 that $G : X \to X^*$ is a bounded, continuous, compact gradient operator.

Proposition 8. *For each $p \in (1, \infty)$ the map $F = T + G : X \to X^*$ is strongly coercive, that is,*

$$\langle F(x), x \rangle \geq c \|x\|^p \quad \text{for some } c > 0 \quad \text{and all } x \in X.$$

Indeed, use of (33), (43) and (45) show that, for all $u \in X$,

$$\langle T(u) + G(u), u \rangle_X \geq \|u\|_X^p + (m - \lambda_1) \|u\|_p^p. \tag{47}$$

Thus, if $m \geq \lambda_1$,

$$\langle T(u) + G(u), u \rangle_X \geq \|u\|_X^p \quad (u \in X). \tag{48}$$

If $0 < m < \lambda_1$, using (38) we see that

$$(m - \lambda_1) \|u\|_p^p \geq \left(\frac{m}{\lambda_1} - 1\right) \|u\|_X^p,$$

and using this in (47) yields

$$\langle T(u) + G(u), u \rangle_X \geq \frac{m}{\lambda_1} \|u\|_X^p \quad (u \in X).$$

Proposition 9. *For each $p \in (1, \infty)$, the map $F = T + G : X \to X^*$ is of type $(S)_2$.*

In fact, by Proposition 6 T satisfies the stronger condition $(S)_+$, and to reach the conclusion it is therefore enough to use the compactness of G and Proposition 4.

The abstract surjectivity Theorem 4 can now be applied to give the following result.

Theorem 5. *Let $1 < p < \infty$. Suppose that f is continuous and satisfies (40) and (41) with $m > 0$. Then, given any $h \in L_{p'}(\Omega)$ $(p' = p/(p-1))$, problem (42) has a solution $u \in X$.*

Remark 2. *Due to the reflexivity and separability of the Sobolev space $X = \overset{0}{W}{}^{1}_{p}(\Omega)$, Theorem 5 could be proved on the basis of Browder's Theorem 3. It would be interesting to have specific applications of our Theorem 4 allowing to measure the real strength of property $(S)_2$. In any case, we again stress the point that the proofs of the two surjectivity results just cited (as well as the tools utilized) are entirely different, and we believe that in the case of a gradient operator F the proof of the surjectivity of F via Theorem 4 is shorter and simpler.*

5. Conclusions

We prove a new surjectivity result for gradient operators, formulated in the paper as Theorem 1 and in equivalent form as Theorem 4. This abstract result is then applied to prove existence of solutions for a boundary-value problem involving the p-Laplacian $(1 < p < \infty)$. It would be desirable—and this remains as an interesting open problem—to produce examples of application (either in Operator Theory itself and/or in existence results for boundary-value problems for Partial Differential Equations) where the local properness condition is essential: namely, where our condition $(S)_2$ holds, but stronger conditions such as (S) or $(S)_+$ do not hold.

References

1. Chiappinelli, R.; Edmunds, D.E. Measure of noncompactness, surjectivity of gradient operators and an application to the p-Laplacian. *J. Math. Anal. Appl.* **2019**, *471*, 712–727. [CrossRef]
2. Appell, J.; Pascale, E.D.; Vignoli, A. *Nonlinear Spectral Theory*; De Gruyter: Berlin, Gemany, 2004.
3. Appell, J.; Giorgieri, E.; Väth, M. Nonlinear spectral theory for homogeneous operators. *Nonlinear Funct. Anal. Appl.* **2002**, *7*, 589–618.
4. Benevieri, P.; Calamai, A.; Furi, M.; Pera, M.P. On the persistence of the eigenvalues of a perturbed Fredholm operator of index zero under nonsmooth perturbations. *Z. Anal. Anwend.* **2017**, *36*, 99–128. [CrossRef]
5. Furi, M.; Martelli, M.; Vignoli, A. Contributions to the spectral theory for nonlinear operators in Banach spaces. *Ann. Mater. Pura Appl.* **1978**, *118*, 229–294. [CrossRef]
6. Drábek, P.; Milota, J. *Methods of Nonlinear Analysis, Applications to Differential Equations*, 2nd ed.; Birkhäuser: Basel, Switzerland, 2013.
7. Browder, F.E. Existence theorems for nonlinear partial differential equations. In Proceedings of the Symposia in Pure Mathematics, Berkeley, CA, USA, 1–26 July 1968; pp. 1–60.
8. Browder, F.E. Nonlinear elliptic boundary value problems and the generalized topological degree. *Bull. Am. Math. Soc.* **1970**, *76*, 999–1005. [CrossRef]
9. Chiappinelli, R. Surjectivity of coercive gradient operators in Hilbert space and Nonlinear Spectral Theory. *Ann. Funct. Anal.* **2019**, *10*, 170–179. [CrossRef]
10. Figueiredo, D.G.D. *Lectures on the Ekeland Variational Principle with Applications and Detours: Lectures delivered at the Indian Institute of Science, Bangalore under the T.I.F.R.-I.I.Sc. Programme in Applications of Mathematics*; Tata Institute of Fundamental Research: Bombay, India, 1989.
11. Fučík, S.; Kufner, A. *Nonlinear Differential Equations*; Elsevier: Amsterdam, The Netherlands; Oxford, UK; New York, NY, USA, 1980.
12. Brézis, H. Felix Browder and Monotone Operators. *Notices Amer. Math. Soc.* **2018**, *65*, 1403–1405.
13. Lindqvist, P. On the equation div $\left(|\nabla u|^{p-2}\nabla u\right) + \lambda |u|^{p-2} u = 0$. *Proc. Am. Math. Soc.* **1990**, *109*, 157–164.
14. Berger, M.S. *Nonlinearity and Functional Analysis*; Academic Press: Cambridge, MA, USA, 1977.

15. Amann, H. Lusternik-Schnirelman theory and non-linear eigenvalue problems. *Math. Ann.* **1972**, *199*, 55–72. [CrossRef]

16. Lindqvist, P. *Notes on the p-Laplace Equation*; Lecture Notes; University of Jyväskylä: Jyväskylän Yliopisto, Finland, 2006.

17. Drábek, P.; Holubovxax, G. Fredholm alternative for the *p*-Laplacian in higher dimensions. *J. Math. Anal. Appl.* **2001**, *263*, 182–194. [CrossRef]