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Intuitionistic Fuzzy Normed Subrings and Intuitionistic Fuzzy Normed Ideals

Nour Abed Alhaleem * and Abd Ghafur Ahmad

Department of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia; ghafur@ukm.edu.my

* Correspondence: p102361@siswa.ukm.edu.my; Tel.: +60-1169784519

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Abstract: The main goal of this paper is to introduce the notion of intuitionistic fuzzy normed rings and to establish basic properties related to it. We extend normed rings by incorporating the idea of intuitionistic fuzzy to normed rings, we develop a new structure of fuzzy rings which will be called an intuitionistic fuzzy normed ring. As an extension of intuitionistic fuzzy normed rings, we define the concept of intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals. Some essential operations specially subset, complement, union, intersection and several properties relating to the notion of generalized intuitionistic fuzzy normed rings are identified. Homomorphism and isomorphism of intuitionistic fuzzy normed subrings are characterized. We identify the image and the inverse image of intuitionistic fuzzy normed subrings under ring homomorphism and study their elementary properties. Some properties of intuitionistic fuzzy normed rings and relevant examples are presented.

Keywords: normed space; fuzzy normed ring; intuitionistic fuzzy normed ring; intuitionistic fuzzy normed subring; intuitionistic fuzzy normed ideal

1. Introduction

After the inception of the notion of fuzzy set as a generalization of a crisp set by Zadeh [1], which laid to the establishment of the fuzzy set theory. Rosenfeld in 1971 [2] applied the concept of fuzzy sets to the group theory, extended and formulated the notion of fuzzy subgroup of a group. Later, Liu in [3] introduced the definition of fuzzy ring and discussed fuzzy subrings and fuzzy ideals and presented some basic concepts of fuzzy algebra, as fuzzy invariant subgroups, fuzzy ideals and proved some related properties. Swamy and Swamy [4], Mukherjee and Sen [5] and Zhang Yue [6] introduced new concepts of fuzzy rings and fuzzy ideals, they studied fuzzy ideals and fuzzy prime ideals, characterised regular rings and Noetherian rings by fuzzy ideals and determined all fuzzy prime ideals of the ring Z of integers. Later, in the year 1999 Atanassov [7] presented the idea of the intuitionistic fuzzy set a way to handle uncertainty, while fuzzy set deals with the degree of membership of an element in a given set, intuitionistic fuzzy sets assign both degrees of membership and non-membership with an addition that sum of these degrees should not exceeds 1. In 1992 [8], Dixit et al. introduced the notion of intuitionistic fuzzy groups, followed by a study of the properties of intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings conducted by Hur et al. in [9]. In Reference [10], Marashdeh introduced the concept of intuitionistic fuzzy group and intuitionistic fuzzy ring based on intuitionistic fuzzy spaces. Marashdeh and Salleh generalized Dib's notion for fuzzy rings based on the notion of fuzzy space in [11] by applying the notion of fuzzy space to intuitionistic fuzzy space and to introduce related concepts to present a new formulation of intuitionistic fuzzy group and intuitionistic fuzzy ring. In [12] a generalisation of the notion of fuzzy homomorphism and fuzzy normal subgroup based on fuzzy spaces to intuitionistic fuzzy hyperhomomorphism based on intuitionistic fuzzy spaces to introduce the concept of an intuitionistic fuzzy quotient hypergroup induced by an intuitionistic fuzzy normal subhypergroup. Complex intuitionistic fuzzy ideals were defined and its characteristics were investigated by Alsarahead and Ahmad in [13] and some of its applications were studied in [14] by combining complex intuitionistic fuzzy sets and graph theory which can be used in the utilization of the complex intuitionistic fuzzy graphs in decision making problems. After studies by Naimark [15] of normed rings, Arens presented [16] the generalization of normed rings and Gel'fand defined commutative normed rings in [17] and presented the notion of commutative normed rings. In [18], Jarden characterized the ultrametric absolute value and investigated the properties of normed rings. Gupta and Qi [19] studied T-norms and T-conorm, T-norm, T-conorm and negation function was described as a set of T-operators, some typical T-operators and their mathematical properties were given. Properties of fuzzy normed ring theory were established by Emniyet and Sahin [20] in 2018, they defined fuzzy normed homomorphism, fuzzy normed subring and fuzzy normed ideal and showed some algebraic properties of normed ring theory on fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy to normed rings. We present and study the notions of intuitionistic fuzzy normed subring and intuitionistic fuzzy normed ideal. Further, we define homomorphism and isomorphism of intuitionistic fuzzy normed subrings of any two rings. The paper is divided into the following sections. In Section 2, we provide several definitions and preliminary results. In Section 3, we present the concept of intuitionistic fuzzy normed subring and intuitionistic fuzzy normed ideal. Homomorphism and isomorphism of intuitionistic fuzzy normed subrings of any two rings are introduced in Section 4. In Section 5, the conclusion is summarized.

2. Preliminaries

Within this section, we outline the most significant definitions and results of fuzzy sets, intuitionistic fuzzy sets and fuzzy normed rings needed for the following sections:

2.1. Intuitionistic Fuzzy Sets

We review some basic ideas of intuitionistic fuzzy set, intuitionistic fuzzy subring and intuitionistic fuzzy ideal that are related to the study in this work.

Definition 1. [21] *The fuzzy set A in a fixed universe X is a function $A : X \to [0,1]$, this function is referred to the membership function and denoted by $\mu_A(z)$. A fuzzy set A is symbolized in the form:*

$$A = \{(z, \mu_A(z)) : z \in X\}.$$

For every $z \in X$, $0 \leq \mu_A(z) \leq 1$, if $z \notin A$ then $\mu_A(z) = 0$, and if z is fully contained in A then $\mu_A(z) = 1$.

Definition 2. [8] *Let A be a fuzzy subset of a set X. The set $A_\alpha = \{\mu_A(z) \geq \alpha : z \in X\}$ is the level subset of $\mu_A(z)$, where $\alpha \in [0,1]$.*

Definition 3. [22] *An intuitionistic fuzzy set (shortly, IFS) A in a nonempty set of X in the form IFS $A = \{(z, \mu_A(z), \gamma_A(z) : z \in X\}$, where the function of the degree of membership is $\mu_A(z) : X \to [0,1]$ and the function of the degree of nonmembership is $\gamma_A(z) : X \to [0,1]$ where $0 \leq \mu_A(z) + \gamma_A(z) \leq 1$ for every $z \in X$. We will use the notation $A = (\mu_A, \gamma_A)$ for the IFS.*

Remark 1. [23] *For an intuitionistic fuzzy set $A = (\mu_A(z), \gamma_A(z))$, the support set of A is the subset of X which denoted by:*

$$A_\circ = \{\mu_A(z) \neq 0, \gamma_A(z) \neq 1 : z \in X\}.$$

Definition 4. [24] *Let $(R, +, .)$ be a ring and $A = (\mu_A, \gamma_A)$ be an IFS of R. Then A is an intuitionistic fuzzy subring (in short IFSR) of R if the following are satisfied:*

i. $\mu_A(z-q) \geq min\{\mu_A(z), \mu_A(q)\}$,
ii. $\mu_A(zq) \geq min\{\mu_A(z), \mu_A(q)\}$,
iii. $\gamma_A(z-q) \leq max\{\gamma_A(z), \gamma_A(q)\}$,
iv. $\gamma_A(zq)) \leq max\{\gamma_A(z), \gamma_A(q)\}$.

Definition 5. [24] *Let $(R, +, .)$ be a ring. Then an IFS $A = (\mu_A, \gamma_A)$ of R is an intuitionistic fuzzy ideal (shortly, IFI) in R if:*

i. $\mu_A(z-q) \geq min\{\mu_A(z), \mu_A(q)\}$,
ii. $\mu_A(zq) \geq max\{\mu_A(z), \mu_A(q)\}$,
iii. $\gamma_A(z-q) \leq max\{\gamma_A(z), \gamma_A(q)\}$,
iv. $\gamma_A(zq)) \leq min\{\gamma_A(z), \gamma_A(q)\}$.

2.2. Fuzzy Normed Rings

In the following, we recall the definitions of normed linear spaces, T-norm, S-norm, normed rings and some examples of normed rings are presented.

Definition 6. [20] *A functional $\|\|$ defined on a linear space L is a norm if it satisfies the following:*
N1: $\|z\| \geq 0$ *for every* $z \in L$, *where* $\|z\| = 0$ *if and only if* $z = 0$;
N2: $\|\delta.z\| = |\delta|.\|z\|$; *(and hence* $\|z\| = \|-z\|$*), for all* δ *and for all* $z \in L$;
N3: $\|z + q\| \leq \|z\| + \|q\|$ *for all* $z, q \in L$ *[Triangle inequality].*
A normed linear space is linear space L equipped with a norm.

Definition 7. [19] *A norm is a non-negative real-valued function $\|\| : A \to R$. A ring A is a normed ring (NR) if it satisfy the following properties for any $z, q \in A$:*

1. $\|z\| = 0 \Leftrightarrow z = 0$,
2. $\|z + q\| \leq \|z\| + \|q\|$,
3. $\|z\| = \|-z\|$, *(and thus* $\|1_A\| = 1 = \|-1\|$ *if identity exists), and*
4. $\|zq\| \leq \|z\|\|q\|$.

Definition 8. [19] *A function $* : [0,1] \times [0,1] \to [0,1]$ is a t-norm if it satisfies the following for all $z, q, h \in [0,1]$:*

1. $*(z, 1) = z$,
2. $*(z, q) \leq *(z, h)$ *if* $q \leq h$; *($*$ is monotone),*
3. $*(z, z) < z$ *for* $z \in (0, 1)$, *and*
4. *If* $z < h$ *and* $q < t$ *then* $*(z, q) < *(h, t)$ *for all* $z, q, h, t \in (0, 1)$.

We will write $z * q$ instead of $*(z, q)$.

Definition 9. [19] *A function $\diamond : [0,1] \times [0,1] \to [0,1]$ is a s-norm if it satisfies the following for all $z, q, h \in [0,1]$:*

1. $\diamond(z, 0) = z$,
2. $\diamond(z, q) \leq \diamond(z, h)$ *if* $q \leq h$; *($\diamond$ is monotone),*
3. $\diamond(z, z) > z$ *for* $z \in (0, 1)$, *and*
4. *If* $z < h$ *and* $q < t$ *then* $\diamond(h, t) \leq \diamond(z, q)$ *for all* $z, q, h, t \in (0, 1)$.

We will write $z \diamond q$ instead of $\diamond(z, q)$.

Example 1. [15] *The field of real numbers $\mathbb{R}$ is a normed ring with respect to the absolute value and the field of complex numbers $\mathbb{C}$ is a normed ring with respect to the modulus. More general examples are the ring of real square matrices with the matrix norm and the ring of real polynomials with a polynomial norm.*

3. Intuitionistic Fuzzy Normed Rings and Intuitionistic Fuzzy Normed Ideals

We define the intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals. We also provide basic properties, prove theorems and present related examples. In this paper, R is an associative ring with identity, NR is a normed ring and $\mathbb{R}$ is the set of real numbers.

Definition 10. *Let $*$ to be a continuous t-norm and $\diamond$ to be a continuous s-norm, NR is a normed ring. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normed subring (IFNSR) over a normed ring $(NR, +, \cdot)$ if it satisfies the following conditions for every $z, q \in NR$:*

i. $\mu_A(z - q) \geq \mu_A(z) * \mu_A(q)$,
ii. $\mu_A(zq) \geq \mu_A(z) * \mu_A(q)$,
iii. $\gamma_A(z - q) \leq \gamma_A(z) \diamond \gamma_A(q)$,
iv. $\gamma_A(zq) \leq \gamma_A(z) \diamond \gamma_A(q)$.

Example 2. *Let $R = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$ be the normed ring of all 2×2 matrices.*
Define the IFS $A = (\mu_A, \gamma_A)$ with

$$\mu_A(z) = \begin{cases} 1, & z = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, c \in \mathbb{R} \\ v, & otherwise \end{cases}, \gamma_A(z) = \begin{cases} 0, & z = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, c \in \mathbb{R} \\ \omega, & otherwise \end{cases}$$

For $v, \omega \in [0, 1]$. Then A is called an intuitionistic fuzzy normed subring over the ring $M_2(\mathbb{R})$.

Proposition 1. *Let A be an intuitionistic fuzzy normed ring and 0_{NR} is the zero of the normed ring. Then, for every $z \in NR$:*

i. $\mu_A(z) \leq \mu_A(0)$, $\gamma_A(0) \leq \gamma_A(z)$,
ii. $\mu_A(z) = \mu_A(-z)$, $\gamma_A(z) = \gamma_A(-z)$.

Definition 11. *If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two intuitionistic fuzzy normed rings over NR. Then, A is an intuitionistic fuzzy normed subring of B if:*

i. $\mu_A(z) \leq \mu_B(z)$,
ii. $\gamma_A(z) \geq \gamma_B(z)$.

Definition 12. *let A be a nonempty intuitionistic fuzzy normed ring. For every $z, q \in NR$, A is an intuitionistic fuzzy normed left (right) ideal if:*

i. $\mu_A(z - q) \geq \mu_A(z) * \mu_A(q)$,
ii. $\mu_A(zq) \geq \mu_A(q)$ $(\mu_A(zq) \geq \mu_A(z))$,
iii. $\gamma_A(z - q) \leq \gamma_A(z) \diamond \gamma_A(q)$,
iv. $\gamma_A(zq) \leq \gamma_A(q)$ $(\gamma_A(zq) \leq \gamma_A(z))$.

Definition 13. *If the fuzzy set $A = (\mu_A, \gamma_A)$ is both right and left intuitionistic fuzzy normed ideal of NR. Then A is called intuitionistic fuzzy normed ideal, if for every $z, q \in NR$:*

i. $\mu_A(z - q) \geq \mu_A(z) * \mu_A(q)$,
ii. $\mu_A(zq) \geq \mu_A(z) \diamond \mu_A(q)$,
iii. $\gamma_A(z - q) \leq \gamma_A(z) \diamond \gamma_A(q)$,
iv. $\gamma_A(zq) \leq \gamma_A(z) * \gamma_A(q)$.

Lemma 1. *Let 1_{NR} be the multiplicative identity in NR then for all $z \in NR$:*

1. $\mu_A(z) \geq \mu_A(1_{NR})$,

2. $\gamma_A(z) \leq \gamma_A(1_{NR})$.

Proof. Let $z \in NR$. Then

1. $\mu_A(z.1_{NR}) \geq \mu_A(z) \diamond \mu_A(1_{NR})$ implies $\mu_A(z) \geq \mu_A(1_{NR})$,
2. $\gamma_A(z.1_{NR}) \leq \gamma_A(z) * \gamma_A(1_{NR})$ implies $\gamma_A(z) \leq \gamma_A(1_{NR})$.

□

Definition 14. *If A and B are two intuitionistic fuzzy normed subrings. Then we characterize:*

i. $A + B = (\mu_{A+B}, \gamma_{A+B})$ *where* $\mu_{A+B}(z) = \{(\overset{\diamond}{_{z=q+h}}(\mu_A(q) * \mu_B(h)) : q, h \in NR, z = q + h\}$ *and* $\gamma_{A+B}(z) = (\overset{*}{_{z=q+h}}(\gamma_A(q) \diamond \gamma_B(h)) : q, h \in NR, z = q + h\}$

ii. $A - B = (\mu_{A-B}, \gamma_{A-B})$ *where* $\mu_{A-B}(z) = \{(\overset{\diamond}{_{z=q-h}}(\mu_A(q) * \mu_B(h)) : q, h \in NR, z = q - h\}$ *and* $\gamma_{A-B}(z) = (\overset{*}{_{z=q-h}}(\gamma_A(q) \diamond \gamma_B(h)) : q, h \in NR, z = q - h\}$

iii. $A.B = (\mu_{A.B}, \gamma_{A.B})$ *where* $\mu_{A.B}(z) = \{(\overset{\diamond}{_{z=qh}}(\mu_A(q) * \mu_B(h)) : q, h \in NR, z = qh\}$ *and* $\gamma_{A.B}(z) = (\overset{*}{_{z=qh}}(\gamma_A(q) \diamond \gamma_B(h)) : q, h \in NR, z = qh\}$

iv. $A \cap B = (\mu_{A\cap B}, \gamma_{A\cap B})$ *where* $\mu_{A\cap B}(z) = \{(z, min\{\mu_A(z), \mu_B(z)\}) : z \in NR\}$ *and* $\gamma_{A\cap B}(z) = \{(z, max\{\gamma_A(z), \gamma_B(z)\}) : z \in NR\}$

v. $A \cup B = (\mu_{A\cup B}, \gamma_{A\cup B})$ *where* $\mu_{A\cup B}(z) = \{(z, max\{\mu_A(z), \mu_B(z)\}) : z \in NR\}$ *and* $\gamma_{A\cup B}(z) = \{(z, min\{\gamma_A(z), \gamma_B(z)\}) : z \in NR\}$

Theorem 1. *If A and B are two intuitionistic fuzzy subrings of NR. Then the intersection of A and B is an intuitionistic fuzzy normed subring of NR.*

Proof. Let $z, q \in NR$.
Since

$$\begin{aligned} A \cap B &= \{(z, \mu_{A\cap B}(z), \gamma_{A\cap B}(z)) : z \in NR\} \\ &= \{(z, min\{\mu_A(z), \mu_B(z)\}, max\{\gamma_A(z), \gamma_B(z)\}) : z \in NR\} \end{aligned}$$

then

(i)

$$\begin{aligned} \mu_{A\cap B}(z-q) &= min\{\mu_A(z-q), \mu_B(z-q)\} \\ &\geq min\{\mu_A(z) * \mu_A(q), \mu_B(z) * \mu_B(q)\} \\ &\geq min\{(\mu_A(z) * \mu_B(z)), (\mu_A(q) * \mu_B(q))\} \\ &\geq min\{\mu_{A\cap B}(z), \mu_{A\cap B}(q)\} \\ &\geq \mu_{A\cap B}(z) * \mu_{A\cap B}(q) \end{aligned}$$

(ii)

$$\begin{aligned} \mu_{A\cap B}(zq) &= min\{\mu_A(zq), \mu_B(zq)\} \\ &\geq min\{\mu_A(z) * \mu_A(q), \mu_B(z) * \mu_B(q)\} \\ &\geq min\{(\mu_A(z) * \mu_B(z)), (\mu_A(q) * \mu_B(q))\} \\ &\geq min\{\mu_{A\cap B}(z), \mu_{A\cap B}(q)\} \\ &\geq \mu_{A\cap B}(z) * \mu_{A\cap B}(q) \end{aligned}$$

(iii)

$$\begin{aligned} \gamma_{A\cap B}(z-q) &= max\{\gamma_A(z-q), \gamma_B(z-q)\} \\ &\leq max\{\gamma_A(z) \diamond \gamma_A(q), \gamma_B(z) \diamond \gamma_B(q)\} \\ &\leq max\{(\gamma_A(z) \diamond \gamma_B(z)), (\gamma_A(q) \diamond \gamma_B(q))\} \\ &\leq max\{\gamma_{A\cap B}(z), \gamma_{A\cap B}(q)\} \\ &\leq \gamma_{A\cap B}(z) \diamond \gamma_{A\cap B}(q) \end{aligned}$$

(iv)

$$\begin{aligned}\gamma_{A\cap B}(zq) &= max\{\gamma_A(zq), \gamma_B(zq)\}\\ &\leq max\{\gamma_A(z)\diamond\gamma_A(q), \gamma_B(z)\diamond\gamma_B(q)\}\\ &\leq max\{(\gamma_A(z)\diamond\gamma_B(z)), (\gamma_A(q)\diamond\gamma_B(q))\}\\ &\leq max\{\gamma_{A\cap B}(z), \gamma_{A\cap B}(q)\}\\ &\leq \gamma_{A\cap B}(z)\diamond\gamma_{A\cap B}(q)\end{aligned}$$

Therefore, $A \cap B$ is an intuitionistic fuzzy normed subring of NR. □

Lemma 2. *$A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy normed subring of NR if:*

1. *$\mu_A \supseteq \mu_{A-A}$ and $\mu_A \supseteq \mu_{A.A}$, and*
2. *$\gamma_A \subseteq \gamma_{A-A}$ and $\gamma_A \subseteq \gamma_{A.A}$.*

Proof. Let $z, q \in A$, then

1.

$$\begin{aligned}\mu_A(z-q) &\geq \mu_{A-A}(z-q)\\ &= \overset{\diamond}{_{a-b=z-q}}(\mu_A(a) * \mu_A(b))\\ &\geq \mu_A(z) * \mu_A(q)\end{aligned}$$

and

$$\begin{aligned}\mu_A(zq) &\geq \mu_{A.A}(zq)\\ &= \overset{\diamond}{_{ab=zq}}(\mu_A(a) * \mu_A(b))\\ &\geq \mu_A(z) * \mu_A(q)\end{aligned}$$

2.

$$\begin{aligned}\gamma_A(z-q) &\leq \gamma_{A-A}(z-q)\\ &= \overset{*}{_{a-b=z-q}}(\gamma_A(a)\diamond\gamma_A(b))\\ &\leq \gamma_A(z)\diamond\gamma_A(q)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(zq) &\leq \gamma_{A-A}(zq)\\ &= \overset{*}{_{ab=zq}}(\gamma_A(a)\diamond\gamma_A(b))\\ &\leq \gamma_A(z)\diamond\gamma_A(q)\end{aligned}$$

So, A is an intuitionistic fuzzy normed subring of NR. □

Proposition 2. *If A is an intuitionistic fuzzy ideal of NR, then the intuitionistic fuzzy subring $A_* = \{z \in NR : \mu_A(z) = \mu_A(0_{NR})$ and $\gamma_A(z) = \gamma_A(0_{NR})\}$ is an ideal of NR.*

Proof. Let $z, q \in A_*$, then $\mu_A(z) = \mu_A(q) = \mu_A(0)$ and $\gamma_A(z) = \gamma_A(q) = \gamma_A(0)$. Since A is an intuitionistic fuzzy ideal of NR, then
$\mu_A(z-q) \geq \mu_A(z) * \mu_A(q) = \mu_A(0) * \mu_A(0) = \mu_A(0)$,
and
$\gamma_A(z-q) \leq \gamma_A(z)\diamond\gamma_A(q) = \gamma_A(0)\diamond\gamma_A(0) = \gamma_A(0)$,
Hence, $\mu_A(z-q) = \mu_A(0)$ and $\gamma_A(z-q) = \gamma_A(0)$. Thus, $z - q \in A_*$.
Let $r \in NR$ and $z \in A_*$. We have,
$\mu_A(rz) \geq \mu_A(r)\diamond\mu_A(z) = \mu_A(0)$, and $\gamma_A(rz) \leq \gamma_A(r) * \gamma_A(z) = \gamma_A(0)$.
Therefore by Proposition 1, $\mu_A(rz) = \mu_A(0)$ and $\gamma_A(rz) = \gamma_A(0)$. This implies that $rz \in A_*$. Thus, A_* is an ideal of NR. □

Theorem 2. *Let A and B be two intuitionistic fuzzy ideals of a normed ring NR. Then $A \cap B$ is an intuitionistic fuzzy normed ideal of NR.*

Proof. Let $z, q \in NR$. Then,

(i)
$$\begin{aligned} \mu_{A\cap B}(z-q) &= min\{\mu_A(z-q), \mu_B(z-q)\} \\ &\geq min\{\mu_A(z) * \mu_A(q), \mu_B(z) * \mu_B(q)\} \\ &\geq min\{(\mu_A(z) * \mu_B(z)), (\mu_A(q) * \mu_B(q))\} \\ &\geq min\{\mu_{A\cap B}(z), \mu_{A\cap B}(q)\} \\ &\geq \mu_{A\cap B}(z) * \mu_{A\cap B}(q) \end{aligned}$$

(ii)
$$\begin{aligned} \mu_{A\cap B}(zq) &= min\{\mu_A(zq), \mu_B(zq)\} \\ &\geq min\{\mu_A(q), \mu_B(q)\} \\ &\geq \mu_{A\cap B}(q) \end{aligned}$$

(iii)
$$\begin{aligned} \gamma_{A\cap B}(z-q) &= max\{\gamma_A(z-q), \gamma_B(z-q)\} \\ &\leq max\{\gamma_A(z) \diamond \gamma_A(q), \gamma_B(z) \diamond \gamma_B(q)\} \\ &\leq max\{(\gamma_A(z) \diamond \gamma_B(z)), (\gamma_A(q) \diamond \gamma_B(q))\} \\ &\leq max\{\gamma_{A\cap B}(z), \gamma_{A\cap B}(q)\} \\ &\leq \gamma_{A\cap B}(z) \diamond \gamma_{A\cap B}(q) \end{aligned}$$

(iv)
$$\begin{aligned} \gamma_{A\cap B}(zq) &= max\{\gamma_A(zq), \gamma_B(zq)\} \\ &\leq max\{\gamma_A(q), \gamma_B(q)\} \\ &\leq \gamma_{A\cap B}(q) \end{aligned}$$

Therefore, $A \cap B$ is an intuitionistic fuzzy normed left ideal. Correspondingly it can be proven that $A \cap B$ is an intuitionistic right ideal. So, $A \cap B$ is a intuitionistic fuzzy normed ideal of NR. □

4. Homomorphism and Isomorphism

We define the image and the inverse image of intuitionistic fuzzy normed subrings and study their elementary properties. We also characterize homomorphism and isomorphism of an intuitionistic fuzzy normed rings.

Definition 15. *Let NR and NR' be normed rings and let $f : NR \to NR'$ be a ring homomorphism mapping. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be an intuitionistic fuzzy normed subrings of NR and NR', respectively. Then $f(A) = \{(q, \mu_{f(A)}(q), \gamma_{f(A)}(q)) : q \in NR'\}$ is the intuitionistic image of A, where $\mu_{f(A)}(q)$ and $\gamma_{f(A)}(q)$ is defined for all $q \in NR'$ as:*

$$\mu_{f(A)}(q) = \begin{cases} \overset{\diamond}{\underset{f(z)=q}{}} \quad \mu_A(z) & , when\ f^{-1}(q) \neq \phi \\ 0 & , otherwise \end{cases}$$

and

$$\gamma_{f(A)}(q) = \begin{cases} \overset{*}{\underset{f(z)=q}{}} \quad \gamma_A(z) & , when\ f^{-1}(q) \neq \phi \\ 1 & , otherwise \end{cases}$$

and $f^{-1}(B) = \{(z, \mu_{f^{-1}(B)}(z), \gamma_{f^{-1}(B)}(z)) : z \in NR\}$ is called intuitionistic inverse image of B, where $\mu_{f^{-1}(B)}(z) = \mu_B(f(z))$ and $\gamma_{f^{-1}(B)}(z) = \gamma_B(f(z))$ for every $z \in NR$.

Theorem 3. *Define $f : NR \to NR'$ to be an onto ring homomorphism. If A is an intuitionistic fuzzy normed subring of the normed ring NR, then $f(A)$ is an intuitionistic fuzzy normed subring of NR'.*

Proof. Let $A = \{(z, \mu_A(z), \gamma_A(z)) : z \in NR\}$ and let
$f(A) = \{(q, \overset{\diamond}{_{f(z)=q}} \quad \mu_A(z), \overset{*}{_{f(z)=q}} \quad \gamma_A(z) : z \in NR, q \in NR'\}$.
Take $q_1, q_2 \in NR'$. Since, f is onto, there exists $z_1, z_2 \in NR$ such that $f(z_1) = q_1$ and $f(z_2) = q_2$. Hence,

$$\begin{aligned}
\mu_{f(A)}(q_1 - q_2) &= \overset{\diamond}{_{f(z_1-z_2)=q_1-q_2}} \quad \mu_A(z_1 - z_2) \\
&\geq \overset{\diamond}{_{f(z_1)=q_1, f(z_2)=q_2}} \quad (\mu_A(z_1) * \mu_A(z_2)) \\
&\geq (\overset{\diamond}{_{f(z_1)=q_1}} \quad \mu_A(z_1)) * (\overset{\diamond}{_{f(z_2)=q_2}} \quad \mu_A(z_2)) \\
&\geq \mu_{f(A)}(q_1) * \mu_{f(A)}(q_2).
\end{aligned}$$

By similar techniques, we can obtain
$\mu_{f(A)}(q_1 q_2) \geq \mu_{f(A)}(q_1) * \mu_{f(A)}(q_2)$,
$\gamma_{f(A)}(q_1 - q_2) \leq \gamma_{f(A)}(q_1) \diamond \gamma_{f(A)}(q_2)$,
$\gamma_{f(A)}(q_1 q_2) \leq \gamma_{f(A)}(q_1) \diamond \gamma_{f(A)}(q_2)$.
This implies that $f(A)$ is an intuitionistic fuzzy normed subring of NR'. □

Proposition 3. *Let $f : NR \to NR'$ be a homomorphism mapping. If $B \in IFNR'$, then $f^{-1}(B) \in IFNR$.*

Proof. Let $B = \{(q, \mu_B(q), \gamma_B(q)) : q \in NR'\}$, $f^{-1}(B) = \{(z, \mu_B(f(z)), \gamma_B(f(z)) : z \in NR\}$, and let $z_1, z_2 \in NR$, then

$$\begin{aligned}
\mu_{f^{-1}(B)}(z_1 - z_2) &= \mu_B(f(z_1 - z_2)) \\
&= \mu_B(f(z_1) - f(z_2)) \\
&\geq \mu_B(f(z_1)) * \mu_B(f(z_2)) \\
&\geq \mu_{f^{-1}(B)}(z_1) * \mu_{f^{-1}(B)}(z_2).
\end{aligned}$$

By similar techniques, we can obtain
$\mu_{f^{-1}(B)}(z_1 z_2) \geq \mu_{f^{-1}(B)}(z_1) * \mu_{f^{-1}(B)}(z_2)$,
$\gamma_{f^{-1}(B)}(z_1 - z_2) \leq \gamma_{f^{-1}(B)}(z_1) \diamond \gamma_{f^{-1}(B)}(z_2)$,
$\gamma_{f^{-1}(B)}(z_1 z_2) \leq \gamma_{f^{-1}(B)}(z_1) \diamond \gamma_{f^{-1}(B)}(z_2)$.
Therefore, $f^{-1}(B)$ is an intuitionistic fuzzy normed subring of NR. □

Theorem 4. *Let $f : NR \to NR'$ be an isomorphism mapping, then*

1. *If $A \in IFNR$, then $f^{-1}(f(A)) = A$,*
2. *If $B \in IFNR'$, then $f(f^{-1}(B)) = B$*

Proof. As f is isomorphism mapping, then for each $z \in NR$ we have $f(z) = q$.

1. $\mu_{f^{-1}(f(A))}(z) = \mu_{f(A)} f(z) = \mu_{f(A)}(q) = \overset{\diamond}{_{f(z)=q}} \mu_A(z) = \mu_A(z)$.
   We also have, $\gamma_{f^{-1}(f(A))}(z) = \gamma_A(z)$ for every $z \in NR$. This implies that $f^{-1}(f(A)) = A$.

2. The proof is similar, as in 1.

□

5. Conclusions

In this study, we established the notion of intuitionistic fuzzy normed rings. We have introduced the concept of intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals. We defined and characterized some related properties of intuitionistic fuzzy normed subrings and intuitionistic fuzzy normed ideals. We investigated homomorphism and isomorphism of intuitionistic fuzzy normed rings. Besides, we studied the image and inverse image of intuitionistic fuzzy normed rings. In the future, further research could be done to study the intuitionistic fuzzy normed prime ideals and intuitionistic fuzzy normed maximal ideals, and to define their algebraic structures. We suggest some future research to investigate other properties of intuitionistic fuzzy normed rings and to expand their applications.

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