mathematics

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General Local Convergence Theorems about the Picard Iteration in Arbitrary Normed Fields with Applications to Super–Halley Method for Multiple Polynomial Zeros

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Received: 23 June 2020; Accepted: 14 September 2020; Published: 17 September 2020

Abstract: In this paper, we prove two general convergence theorems with error estimates that give sufficient conditions to guarantee the local convergence of the Picard iteration in arbitrary normed fields. Thus, we provide a unified approach for investigating the local convergence of Picard-type iterative methods for simple and multiple roots of nonlinear equations. As an application, we prove two new convergence theorems with a priori and a posteriori error estimates about the Super-Halley method for multiple polynomial zeros.

Keywords: iterative methods; local convergence; error estimates; normed fields; Super-Halley method; polynomial zeros; multiple zeros

1. Introduction and Preliminaries

Let $(\mathbb{K}, |\cdot|)$ be a normed field and $f\colon \mathbb{K} \to \mathbb{K}$ be an arbitrary function. It is well known that one of the most commonly used tools for finding the zeros of f is the Picard iteration

$$x_{k+1} = Tx_k, \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $T\colon D \subset \mathbb{K} \to \mathbb{K}$ is some iteration function (see, e.g., Traub [1]). Although the Picard iteration is an old method, it still draws strong interest as a powerful tool for solving a wide range of mathematical problems (see e.g., [2–4] and references therein). Nowadays, there are a plethora of convergence results about the iterative methods of the type (1) but the proofs are often based on Taylor's expansions or other methods that require the existence of higher-order derivatives of f and ensure only asymptotic error constants where available. Moreover, the most of the existing convergence theorems do not provide exact information about the sets of initial approximations that guarantee the convergence of the iteration (1) (see e.g., [5–7] and the references therein).

In the last decade, a general convergence theory of the Picard iteration (1) in cone metric spaces and in n-dimensional vector spaces has been developed by Proinov [8–11]. The main role in this theory is played by a real-valued function called function of initial conditions of the iteration function T. Following the ideas of Proinov [11], in this paper we make two contributions. First, in Section 2, using two different functions of initial conditions we prove two general local convergence theorems (Theorem 1 and Theorem 2) that provide domains of initial approximations (convergence domains) to guarantee the convergence of the Picard iteration (1) with a priori and a posteriori error estimates right from the first step. In this manner, we reduce the convergence analysis of (1) up to studying of some simple properties of the iteration function T. Second, applying this general approach, in Section 3 we obtain two new local convergence theorems with error estimates (Theorem 3 and Theorem 4) about the

Super-Halley method for approximation of polynomial zeros with known multiplicity m. The obtained theorems are new even in the case of simple zeros ($m = 1$).

In the remainder of this section, we recall some definitions and theorems of Proinov [9,12] which are crucial for the proof of the upcoming results.

Furthermore, J always denotes an interval on $\mathbb{R}_+$ containing 0 and we assume that $0^0 \equiv 1$. Also, we denote by $S_k(r)$ the sum of the first k terms of geometric sequence $1, r, r^2, \cdots$, i.e., for all $k \in \mathbb{N}$, we have

$$S_k(r) = 1 + r + \cdots + r^{k-1}.$$

In the case $k = 0$, we set $S_0(r) = 0$.

Definition 1 ([9] Definition 2.1)**.** *A function $\varphi \colon J \subset \mathbb{R} \to \mathbb{R}_+$ is called quasi-homogeneous of degree $p \geq 0$ if it is such that $\varphi(\lambda t) \leq \lambda^p \varphi(t)$ for all $\lambda \in [0, 1)$ and $t \in J$.*

Recall some simple but useful properties of the quasi-homogeneous functions.

(P1) A function f is quasi-homogeneous of degree $p = 0$ on some interval J if and only if f is nondecreasing on J;

(P2) If a function f is quasi-homogeneous of degree p on J, then there exists a nondecreasing function $F \colon J \to \mathbb{R}_+$ such that $f(t) = t^p F(t)$;

(P3) If g and h are quasi-homogeneous of degree $p \geq 0$ and $q \geq 0$ on J, then $g\,h$ is quasi-homogeneous of degree $p + q$ on J.

Definition 2 ([9] Definition 2.4)**.** *A function $\varphi \colon J \to \mathbb{R}_+$ is said to be a* gauge function *of order $p \geq 1$ on J if it satisfies the following conditions:*

(i) *φ is a quasi-homogeneous function of degree p on J;*

(ii) *$\varphi(t) \leq t$ for all $t \in J$.*

A gauge function φ of order p on J is said to be a strict gauge function *if the inequality (ii) holds strictly whenever $t \in J \setminus \{0\}$.*

Definition 3 ([9] Definition 3.1)**.** *Let $T \colon D \subset X \to X$ be a map on an arbitrary set X. A function $E \colon D \to \mathbb{R}_+$ is said to be a function of initial conditions of T (with gauge function φ on an interval J) if there exists a function $\varphi \colon J \to J$ such that*

$$E(Tx) \leq \varphi(E(x)) \quad \textit{for all } x \in D \textit{ with } Tx \in D \textit{ and } E(x) \in J.$$

Definition 4 ([9] Definition 3.2)**.** *Let $T \colon D \subset X \to X$ be a map on an arbitrary set X. Suppose $E \colon D \to \mathbb{R}_+$ is a function of initial conditions of T (with gauge function on an interval J). Then a point $x \in D$ is said to be an* initial point *of T if $E(x) \in J$ and all of the iterates $T^n(x)$ $(n = 0, 1, \ldots)$ are well defined and belong to D.*

The next theorem provides a sufficient condition for initial points.

Proposition 1 ([9] Proposition 4.1)**.** *Let $T \colon D \subset X \to X$ be a map on a set X, and let $E \colon D \to \mathbb{R}_+$ be a function of initial conditions of T with a gauge function φ on J. Suppose that*

$$x \in D \quad \textit{with} \quad E(x) \in J \quad \textit{implies} \quad Tx \in D.$$

Then every point $x_0 \in D$ such that $E(x_0) \in J$ is an initial point of T.

Proposition 2 ([12] Lemma 3.6)**.** *For every initial point $x_0 \in D$ of T and every $k \geq 0$, we have*

$$E(x_{k+1}) \leq \varphi(E(x_k)) \quad \textit{and} \quad E(x_k) \leq \varphi^k(E(x_0)).$$

In addition, if φ is a gauge function of order $r \geq 1$, then

$$E(x_k) \leq E(x_0)\, \lambda^{S_k(r)} \quad \text{and} \quad \phi(E(x_k)) \leq \lambda^{r^k},$$

where $\lambda = \phi(E(x_0))$ and ϕ is a non-negative and nondecreasing function on J such that $\varphi(t) = t\,\phi(t)$ for all $t \in J$.

2. General Local Convergence Theorems

Let $T \colon D \subset \mathbb{K} \to \mathbb{K}$ be an arbitrary iteration function. In this section, we present two general theorems about the local convergence of the Picard iteration (1). The first one (Theorem 1) is a case of Theorem 3.1 of Proinov [11] but provides an extra error estimate that ensures the Q-order of convergence of the Picard iteration (1) (see, e.g., [13]). The second theorem (Theorem 2) is of bit more practical importance than Theorem 1 since there is less unknown information involved into the initial conditions and error estimates.

Let $f \colon \mathscr{D} \subset \mathbb{K} \to \mathbb{K}$ be an arbitrary function which has at least one zero in $\mathscr{D}$. For two points $x, \xi \in \mathbb{K}$, we define the functions $E \colon \mathbb{K} \to \mathbb{R}_+$ and $\mathcal{E} \colon \mathcal{D} \subset \mathbb{K} \to \mathbb{R}_+$ by

$$E(x) = \frac{|x - \xi|}{d} \quad \text{and} \quad \mathcal{E}(x) = \frac{|x - \xi|}{\rho(x)}, \tag{2}$$

where d denotes the distance between ξ and the nearest zero of f and $\rho(x)$ is the radius of a disk with center x that not contains zeros of f. If ξ is a zero of f and it is unique, then we set $E(x) \equiv 0$. In this case, we assume that $d = \infty$. Also, we observe that the domain of $\mathcal{E}$ is the set

$$\mathcal{D} = \{x \in \mathbb{K} \colon \rho(x) > 0\}.$$

We have to note that usually ξ is a zero of f but we do not assume this in Theorem 1 and Theorem 2.

The following theorem is our first main result.

Theorem 1. *Let $T \colon D \subset \mathbb{K} \to \mathbb{K}$ be an iteration function, $\xi \in \mathbb{K}$ and $E \colon \mathbb{K} \to \mathbb{R}_+$ be defined by (2). Suppose $\phi \colon J \to \mathbb{R}_+$ is a quasi-homogeneous function of degree $p \geq 0$ such that for each $x \in \mathbb{K}$ with $E(x) \in J$, the following two conditions are satisfied:*

(a) $x \in D$
(b) $|\,Tx - \xi\,| \leq \phi(E(x))\, |x - \xi\,|$.

Also, let $x_0 \in \mathbb{K}$ be an initial approximation such that

$$E(x_0) \in J \quad \text{and} \quad \phi(E(x_0)) < 1, \tag{3}$$

then the following statements hold:

(i) *The Picard iteration (1) is well defined and converges to ξ with order $r = p + 1$.*
(ii) *For all $k \geq 0$, we have the following error estimates:*

$$|x_{k+1} - \xi\,| \leq \lambda^{r^k}\, |x_k - \xi\,| \quad \text{and} \quad |x_k - \xi\,| \leq \lambda^{S_k(r)}\, |x_0 - \xi\,|,$$

where $\lambda = \phi(E(x_0))$.

(iii) *The Picard iteration (1) converges to ξ with Q-order $r = p + 1$ and with the following error estimate:*

$$|x_{k+1} - \xi| \leq (R\,d)^{1-r}\, |x_k - \xi|^r \quad \text{for all} \quad k \geq 0,$$

where R is the minimal solution of the equation $\phi(t) = 1$ in the interval $J \setminus \{0\}$.

Proof. The claims (i) and (ii) follow from Theorem 3.1 of Proinov [11] in the one dimensional case. So, it remains to prove the estimate (iii).

By (i) and (ii), for all $k \geq 0$ we have $x_k \in U$, where $U = \{x \in \mathbb{K} : E(x) \in J\}$. Since, ϕ is a quasi-homogeneous function of degree $p \geq 0$, then by (P2) it follows that there exists a nondecreasing function $\Phi \colon J \to \mathbb{R}_+$ such that $\phi(t) = t^p\, \Phi(t)$. This together with $\phi(R) = 1$ implies that $\Phi(R) = 1/R^p$. Hence, from conditions (a) and (b), for all $x \in U$ we get

$$|Tx - \xi| \leq \phi(E(x))\, |x - \xi| = \frac{\Phi(E(x))}{d^p}\, |x - \xi|^{p+1} \leq \frac{1}{(Rd)^p}\, |x - \xi|^{p+1}.$$

Now, setting $x = x_k$ and $p = r - 1$ in the last inequality, we get the estimate of (iii) which implies that the Picard iteration (1) converges to ξ with Q-order $r = p + 1$ and completes the proof. □

In what follows, for a nondecreasing function $\beta \colon J \to \mathbb{R}_+$, we define the function $\psi \colon J \to \mathbb{R}$ by

$$\psi(t) = 1 - t(1 + \beta(t)). \tag{4}$$

If the function ψ is positive, then we can define the function $\phi \colon J \to \mathbb{R}_+$ by

$$\phi(t) = \beta(t)/\psi(t). \tag{5}$$

The second main result of this paper is represented by the following theorem:

Theorem 2. *Let $T \colon D \subset \mathbb{K} \to \mathbb{K}$ be an iteration function, $\xi \in \mathbb{K}$ and $\mathcal{E} \colon \mathcal{D} \subset \mathbb{K} \to \mathbb{R}_+$ be defined by (2). Suppose $\beta \colon J \to \mathbb{R}_+$ is a nonzero quasi-homogeneous function of degree $p \geq 0$ and for each $x \in \mathbb{K}$ with $\mathcal{E}(x) \in J$, the following two conditions are satisfied:*

(a) $x \in D$;
(b) $|Tx - \xi| \leq \beta(\mathcal{E}(x))\, |x - \xi|$.

Let also, $x_0 \in \mathbb{K}$ be an initial guess such that

$$\mathcal{E}(x_0) \in J \quad \text{and} \quad \beta(\mathcal{E}(x_0)) \leq \psi(\mathcal{E}(x_0)), \tag{6}$$

where the function ψ is defined by (4). Then the Picard iteration (1) is well defined and converges to ξ with the following error estimates:

$$|x_{k+1} - \xi| \leq \theta \lambda^{r^k}\, |x_k - \xi| \ \text{and} \ |x_k - \xi| \leq \theta^k \lambda^{S_k(r)}\, |x_0 - \xi| \ \text{for all} \ k \geq 0, \tag{7}$$

where $\lambda = \phi(\mathcal{E}(x_0))$ and $\theta = \psi(\mathcal{E}(x_0))$ with ϕ and ψ defined by (5) and (4), respectively. In addition, if the second inequality in (6) is strict, then the order of convergence of Picard iteration (1) is at least $r = p + 1$.

Proof. The initial condition (6) can be written in the form $\mathcal{E}(x_0) \in \Delta$, where $\Delta = \{t \in J \colon \beta(t) \leq \psi(t)\}$. It is not hard to verify that for all $t \in \Delta$, the following inequalities hold (see, e.g., ([11] Lemma 7.3)):

$$0 < \psi(t) \leq 1, \quad 0 \leq \phi(t) \leq 1 \quad \text{and} \quad 0 \leq \beta(t) < 1. \tag{8}$$

According to the first of these inequalities and the fact that $\rho(x) > 0$, we get

$$\rho(Tx) \geq \psi(\mathcal{E}(x))\, \rho(x) > 0. \tag{9}$$

Indeed, if $\eta \in \mathscr{D}$ is a zero of $f\colon \mathscr{D} \subset \mathbb{K} \to \mathbb{K}$, then by the triangle inequality, the definition of $\rho(x)$ and the condition (b), we obtain

$$\begin{aligned} |Tx - \eta| &\geq |x - \eta| - |x - \xi| - |Tx - \xi| \geq \rho(x) - (1 + \beta(\mathcal{E}(x))|x - \xi| \\ &= [1 - \mathcal{E}(x)\,(1 + \beta(\mathcal{E}(x)))]\rho(x) = \psi(\mathcal{E}(x))\rho(x). \end{aligned}$$

This implies (9) which in turn means that $Tx \in \mathcal{D}$. So, dividing both sides of (b) by $\rho(Tx)$, we reach the inequality

$$\mathcal{E}(Tx) \leq \varphi(\mathcal{E}(x)), \tag{10}$$

where $\varphi(t) = t\,\phi(t)$ with ϕ defined by (5). Since the inequality (10) implies that $\mathcal{E}(Tx) \in J$, then we have both

$$Tx \in \mathcal{D} \quad \text{and} \quad \mathcal{E}(Tx) \in J \tag{11}$$

which according to (a) means that $Tx \in D$. This, together with (10) purports that $\mathcal{E}$ is a function of initial conditions of T with gauge function φ on the interval J (see Definition 3).

Now let U be the set $U = \{x \in \mathcal{D}\colon \mathcal{E}(x) \in \Delta\}$. Then the condition (6) implies that $x_0 \in U$. On the other hand (10) and (11) imply that $T(U) \subset U$ which means that starting from x_0 the Picard iteration (1) is well defined and remains in the set U. The convergence to ξ follows from the second estimate of (7). So, it remains to prove the estimates (7). Since x_0 is an initial point of T with respect to $\mathcal{E}$, and φ is a gauge function of order $r = p + 1$, then from Proposition 2 it follows that

$$\mathcal{E}(x_k) \leq \lambda^{S_k(r)}\,\mathcal{E}(x_0),$$

where $\lambda = \phi(\mathcal{E}(x_0))$. From this, taking into account that β is quasi-homogeneous function of degree $p \geq 0$, and that $\beta = \phi\,\psi$, we get

$$\beta(\mathcal{E}(x_k)) \leq \beta(\lambda^{S_k(r)}\,\mathcal{E}(x_0)) \leq \lambda^{p\,S_k(r)}\,\beta(\mathcal{E}(x_0)) = \lambda^{p\,S_k(r)+1}\,\psi(\mathcal{E}(x_0)) = \theta\,\lambda^{r^k}.$$

From this and condition (b) applied to x_k, we obtain the first estimate in (7). From the first estimate in (7) one can easily obtain the second one which in turn implies the convergence of the Picard iteration (1) due to $S_k(r) \geq k$ and $0 \leq \theta\,\lambda < 1$. $\square$

3. Local Convergence of the Super-Halley Method for Multiple Polynomial Zeros

In 2008, Osada [5] derived a modification of the Chebyshev–Halley family of iterative methods for computation of multiple zeros of known multiplicity m. The most famous iterative methods, namely Newton's, Halley's and Chebyshev's methods for multiple zeros are members of this family. A detailed local convergence analysis of these illustrious methods applied to multiple polynomial zeros can be found in the papers [8,14–16]. Another member of the Chebyshev–Halley family that has been rarely studied is the so called Super-Halley method for multiple zeros, which can be defined by the following iteration (see, e.g., [17,18] and references therein):

$$x_{k+1} = Sx_k, \tag{12}$$

where the iteration function $S\colon D \subset \mathbb{K} \to \mathbb{K}$ is defined by

$$Sx = \begin{cases} x - \dfrac{N(x)}{2}\left(m + \dfrac{1}{1 - L(x)}\right) & \text{if } f'(x) \neq 0, \\ x & \text{if } f'(x) = 0 \end{cases} \tag{13}$$

with $N(x)$ and $L(x)$ defined as follows

$$N(x) = \frac{f(x)}{f'(x)} \quad \text{and} \quad L(x) = N(x)\,\frac{f''(x)}{f'(x)}\,. \tag{14}$$

It is seen that the domain of the function (13) is the set

$$D = \left\{x \in \mathbb{K} : f'(x) \neq 0 \;\Rightarrow\; 1 - L(x) \neq 0\right\}. \tag{15}$$

In this section, we implement Theorem 1 and Theorem 2 to the Super-Halley method (12) applied for computation of polynomial zeros of known multiplicity m. Thus, we obtain two new local convergence theorems (Theorem 3 and Theorem 4) that provide sufficient conditions to guarantee the cubic convergence of the method (12) right from the first step. A priory and a posteriori error estimates at any iteration are also provided. The obtained results are new even in the case of simple zeros. Note that in this case, i.e., when $m = 1$ the Super-Halley method (12) was presented in 1992 by Hernández-Verón [19] and later investigated in [20,21], etc.

Before proceeding further, we give two useful technical lemmas.

Lemma 1. *Let K be an arbitrary field, and let $f \in K[x]$ be polynomial of degree $n \geq 2$ which splits over K. Suppose that $\xi_1, \ldots, \xi_s$ are all distinct zeros of f with multiplicities $m_1, \ldots, m_s$ $\left(\sum_{j=1}^{s} m_j = n\right)$, respectively.*

(i) If $x \in K$ is not a zero of f, then for any $i = 1, \ldots, s$ we have

$$\frac{f'(x)}{f(x)} = \frac{m_i + a_i}{x - \xi_i}\,, \quad \text{where} \quad a_i = (x - \xi_i) \sum_{j \neq i} \frac{m_j}{x - \xi_j}\,.$$

(ii) If $x \in K$ is not a zero of both f and f', then for any $i = 1, \ldots, s$ we have

$$\frac{f''(x)}{f'(x)} = \frac{(m_i + a_i)^2 - m_i - b_i}{(x - \xi_i)(m_i + a_i)}\,, \quad \text{where } a_i \text{ is defined in (i) and } b_i = (x - \xi_i)^2 \sum_{j \neq i} \frac{m_j}{(x - \xi_j)^2}\,.$$

Proof. (i) Using a well known identity (see, e.g., ([15] Lemma 4.2(i))), we obtain

$$\frac{f'(x)}{f(x)} = \sum_{j=1}^{s} \frac{m_j}{x - \xi_j} = \frac{m_i}{x - \xi_i} + \sum_{j \neq i} \frac{m_j}{x - \xi_j} = \frac{m_i + a_i}{x - \xi_i}\,, \quad \text{where} \quad a_i = (x - \xi_i) \sum_{j \neq i} \frac{m_j}{x - \xi_j}$$

which proves the first claim.

The second claim follows from (i) and the following identities (see, e.g., ([15] Lemma 4.2(ii)) and ([14] Equation (15))):

$$\frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)} - \frac{f(x)}{f'(x)} \sum_{j=1}^{s} \frac{m_j}{(x - \xi_j)^2} \quad \text{and} \quad \sum_{j=1}^{s} \frac{m_j}{(x - \xi_j)^2} = \frac{m_i + b_i}{(x - \xi_i)^2}\,.$$

□

Lemma 2. *Let $x, \xi \in \mathbb{K}$ and $\xi_1, \ldots, \xi_s \in \mathbb{K}$ be all zeros of f which are not equal to ξ, then for any $j = 1, \ldots, s$ the following inequality holds:*

$$|x - \xi_j| \geq (1 - E(x))\, d, \tag{16}$$

where $E \colon \mathbb{K} \to \mathbb{R}_+$ is defined by (2).

Proof. Since $d \leq |\xi - \xi_j|$ for all $j = 1, \ldots, s$, then by the triangle inequality in $\mathbb{K}$, we get

$$|x - \xi_j| = |\xi - \xi_j + x - \xi| \geq |\xi - \xi_j| - |x - \xi| \geq (1 - E(x))\, d.$$

□

3.1. Local Convergence of the First Type

From now on $\mathbb{K}[x]$ denotes the ring of polynomials over the normed field $\mathbb{K}$. Let $f \in \mathbb{K}[x]$ be a polynomial of degree $n \geq 2$. We shall study the convergence of the Super-Halley method (12) with respect to the function of initial conditions $E\colon \mathbb{K} \to \mathbb{R}_+$ defined by (2).

Onwards, for $n \geq m \geq 1$, we define the real functions g and h by

$$g(t) = (n - m)\left(\frac{2(n - m)\, t}{1 - t} + n\right) t^2 \quad \text{and} \quad h(t) = \frac{2(m - nt)\left((2m - n)\, t^2 - 2mt + m\right)}{1 - t}. \tag{17}$$

Obviously, the function g is positive and quasi-homogeneous of second degree on the interval $[0, 1)$ and the function h is decreasing and positive on the interval $[0, \tau)$, where

$$\tau = \begin{cases} m/n & \text{if} \quad n \geq 2m, \\ m/(m + \sqrt{m(n - m)}) & \text{if} \quad n < 2m \end{cases}. \tag{18}$$

Hence, we can define the function $\phi\colon [0, \tau) \to \mathbb{R}_+$ by

$$\phi(t) = \frac{g(t)}{h(t)} = \frac{(n - m)(n + (n - 2m)t)\, t^2}{2(m - nt)\left((2m - n)\, t^2 - 2mt + m\right)}. \tag{19}$$

Note that the function ϕ is quasi-homogeneous of second degree on the interval $[0, \tau)$ owing to the properties (P1) and (P3).

The next lemma shows that the function ϕ defined by (19) satisfies the conditions (a) and (b) of Theorem 1 with the Super-Halley iteration function and the function E defined by (2).

Lemma 3. *Let $f \in \mathbb{K}[x]$ be a polynomial of degree $n \geq 2$ which splits over $\mathbb{K}$, and let $\xi \in \mathbb{K}$ be a zero of f with multiplicity m. Let $x \in \mathbb{K}$ be such that*

$$E(x) < \tau, \tag{20}$$

where the function E is defined by (2) and τ is the number (18). Then:

(i) *x belongs to the set D defined by (15).*
(ii) *$|\, Sx - \xi\,| \leq \phi(E(x))\, |x - \xi\,|$, where the function ϕ is defined by (19).*

Proof. Let $x \in \mathbb{K}$ satisfy (20). If $m = n$ or $x = \xi$, then $Sx = \xi$ and the statements of the lemma hold. Let $m < n$ and $x \neq \xi$. Let also $\xi_1, \ldots, \xi_s$ be all distinct zeros of f with respective multiplicities $m_1, \ldots, m_s$ and the quantities a_i and b_i be defined as in Lemma 1. Put $\xi = \xi_i$, $m = m_i$, $a = a_i$ and $b = b_i$ for some $1 \leq i \leq s$.

(i) According to (15), we have to prove that $f'(x) \neq 0$ implies $1 - L(x) \neq 0$, where $L(x)$ is defined by (14). From $E(x) < \tau < 1$ and Lemma 2, we get

$$|x - \xi_j| \geq (1 - E(x))\, d > 0 \tag{21}$$

for each $j \neq i$ which means that $f(x) \neq 0$. Then, from (14) and Lemma 1 (i), we get

$$\frac{1}{N(x)} = \frac{f'(x)}{f(x)} = \frac{m + a}{x - \xi}, \quad \text{where} \quad a = (x - \xi) \sum_{j \neq i} \frac{m_j}{x - \xi_j}. \tag{22}$$

Now, from the triangle inequality and (21), we obtain the following estimate:

$$|a| \leq |x - \xi| \sum_{j \neq i} \frac{m_j}{|x - \xi_j|} \leq \frac{|x - \xi|}{(1 - E(x))\, d} \sum_{j \neq i} m_j = \frac{(n - m)E(x)}{1 - E(x)}. \tag{23}$$

From this, using the triangle inequality and $E(x) < \tau \leq m/n$, we get

$$|m + a| \geq m - |a| \geq m - \frac{(n - m)E(x)}{1 - E(x)} = \frac{m - nE(x)}{1 - E(x)} > 0. \tag{24}$$

Hence, $m + a \neq 0$ which implies $f'(x) \neq 0$. Then, from Lemma 1 (ii), we have

$$\frac{f''(x)}{f'(x)} = \frac{(m + a)^2 - m - b}{(x - \xi)(m + a)}, \quad \text{where} \quad b = (x - \xi)^2 \sum_{j \neq i} \frac{m_j}{(x - \xi_j)^2}. \tag{25}$$

Now, from (14), (22) and (25), we obtain

$$1 - L(x) = 1 - N(x)\frac{f''(x)}{f'(x)} = 1 - \frac{(m + a)^2 - m - b}{(m + a)^2} = \frac{m + b}{(m + a)^2}.$$

Further, by means of the triangle inequality, (21) and $E(x) < \tau$, we reach the estimates (see e.g., ([14] Equation (11)))

$$|b| \leq \frac{(n - m)E(x)^2}{(1 - E(x))^2} \quad \text{and} \quad |m + b| \geq m - |b| \geq \frac{h(E(x))}{2(m - nE(x))\,(1 - E(x))} > 0. \tag{26}$$

The last estimate implies that $|1 - L(x)| > 0$ and so $x \in D$.

(ii) From the definition of the Super-Halley iteration function (13), (22) and (25), we easily get

$$Sx - \xi = x - \xi - \frac{x - \xi}{2(m + a)}\left(m + \frac{(m + a)^2}{m + b}\right) = \sigma(x - \xi),$$

where

$$\sigma = 1 - \frac{2m^2 + mb + 2ma + a^2}{2(m + a)(m + b)} = \frac{2ab + mb - a^2}{2(m + a)(m + b)}. \tag{27}$$

Hence, to complete the proof it is sufficient to estimate $|\sigma|$ from above. With this aim, using the estimates (23), (24) and (26), we get

$$\begin{aligned} |\sigma| &\leq \frac{2|a|\,|b| + m|b| + |a|^2}{2|m + a|\,|m + b|} \leq \frac{2\dfrac{(n - m)E(x)}{1 - E(x)}\dfrac{(n - m)E(x)^2}{(1 - E(x))^2} + \dfrac{m(n - m)E(x)^2}{(1 - E(x))^2} + \dfrac{(n - m)^2E(x)^2}{(1 - E(x))^2}}{2\dfrac{m - nE(x)}{1 - E(x)}\dfrac{h(E(x))}{2(m - nE(x))\,(1 - E(x))}} \\ &= \frac{g(E(x))}{h(E(x))} = \phi(E(x)). \end{aligned} \tag{28}$$

This completes the proof of the lemma. □

The following theorem is the first convergence result about the Super-Halley method (12).

Theorem 3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which splits over $\mathbb{K}$, and let $\xi \in \mathbb{K}$ be a zero of f with multiplicity m. Suppose $x_0 \in \mathbb{K}^n$ is an initial approximation satisfying*

$$E(x_0) < R = \frac{2m}{n + m + \sqrt{3(n - m)(n + m)}}, \tag{29}$$

where the function E is defined by (2). Then the Super-Halley iteration (12) is well defined and converges cubically to ξ with the following error estimates for all $k \geq 0$:

$$|x_{k+1} - \xi| \leq \lambda^{3^k} |x_k - \xi| \quad \text{and} \quad |x_k - \xi| \leq \lambda^{(3^k-1)/2} |x_0 - \xi|, \tag{30}$$

where $\lambda = \phi(E(x_0))$ and the function ϕ is defined by (19). Besides, the iteration (12) converges Q-cubically to ξ with the following error estimate

$$|x_{k+1} - \xi| \leq \frac{1}{(Rd)^2} |x_k - \xi|^3 \quad \text{for all} \quad k \geq 0. \tag{31}$$

Proof. It is easy to verify that $R \leq \tau$, where τ is defined by (18) and $\phi(t) < 1$ for all $t \in [0, R)$. Hence, the initial condition (29) is equivalent to (3) with $J = [0, \tau)$ and ϕ defined by (19). Therefore, the proof follows from Lemma 3 and Theorem 1. □

Remark 1. *It is interesting to note that in the case $n > 2m$, Theorem 3 gives larger convergence domain and better error estimates for the Super-Halley method (12) than Theorem 4.5 of [15] for the Halley's method.*

3.2. Local Convergence of the Second Type

Here, we study the convergence of the Super-Halley method (12) with respect to the function of initial conditions $\mathcal{E} : \mathcal{D} \subset \mathbb{K} \to \mathbb{R}_+$ defined by (2).

For $n > m \geq 1$, we define the real function $\beta : [0, \sqrt{m/(n-m)}) \to \mathbb{R}_+$ by

$$\beta(t) = \frac{(n-m)(n + 2(n-m)t)t^2}{2(m - (n-m)t)(m - (n-m)t^2)} . \tag{32}$$

Apparently, β is a quasi-homogeneous function of second degree on $[0, \sqrt{m/(n-m)})$.

The role played by the following lemma is the same as the one of Lemma 3.

Lemma 4. *Let $f \in \mathbb{K}[x]$ be a polynomial of degree $n \geq 2$ which splits over $\mathbb{K}$, and let $\xi \in \mathbb{K}$ be a zero of f with multiplicity m. Let $x \in \mathbb{K}$ be such that*

$$\mathcal{E}(x) < \sqrt{m/(n-m)}, \tag{33}$$

where the function $\mathcal{E}$ is defined by (2). Then:

(i) $x \in D$, *where D is the set (15).*

(ii) $|\, Sx - \xi \,| \leq \beta(\mathcal{E}(x))\, |x - \xi\,|$, *where the function β is defined by (32).*

Proof. The proof is the same as those of Lemma 3. One just should use the estimate $|x - \xi_i| \geq \rho(x) > 0$ instead of (21). □

Now we can state the second main result of this section.

Theorem 4. *Let $f \in \mathbb{K}[x]$ be a polynomial of degree $n \geq 2$ which splits over $\mathbb{K}$, and let $\xi \in \mathbb{K}$ be a zero of f with multiplicity m. Suppose an element $x_0 \in \mathbb{K}$ satisfies the following initial conditions*

$$\mathcal{E}(x_0) < \sqrt{m/(n-m)} \quad \text{and} \quad \beta(\mathcal{E}(x_0)) \leq \psi(\mathcal{E}(x_0)), \tag{34}$$

where the function $\mathcal{E}$ is defined by (2) and the real functions β and ψ are defined by (32) and (4), respectively. Then the Super-Halley iteration (12) is well defined and converges to ξ with error estimates

$$|x_{k+1} - \xi| \leq \theta \lambda^{3^k} |x_k - \xi| \text{ and } |x_k - \xi| \leq \theta^k \lambda^{(3^k-1)/2} |x_0 - \xi| \text{ for all } k \geq 0, \tag{35}$$

where $\theta = \psi(\mathcal{E}(x_0))$ *and* $\lambda = \beta(\mathcal{E}(x_0))/\psi(\mathcal{E}(x_0))$. *In addition, if the second inequality in* (34) *is strict, then the iteration* (12) *converges at least cubically to* ξ.

Proof. The proof follows immediately from Lemma 4 and Theorem 2. □

4. Conclusions

Two general theorems (Theorem 1 and Theorem 2) that give an easy algorithm for investigating the local convergence of Picard-type iterative processes in arbitrary normed fields are proven in this paper. Any of the presented theorems is supplied with a priori and a posteriori error estimates. Moreover, the initial conditions of Theorem 1 guarantee the Q-order convergence of the Picard iteration right from the first step. Furthermore, the presented general theorems are applied to study the local convergence of the Super-Halley method for multiple polynomial zeros. Thus, two types of local convergence theorems (Theorem 3 and Theorem 4) with error estimates for this method are proven. The obtained theorems are new even in the case of simple zeros.

Finally, the ideas of this paper can be further developed to obtain general convergence theorems with computationally verifiable initial conditions and error estimates that are of significant practical importance.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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