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Ivanov's Theorem for Admissible Pairs Applicable to Impulsive Differential Equations and Inclusions on Tori

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Abstract: The main aim of this article is two-fold: (i) to generalize into a multivalued setting the classical Ivanov theorem about the lower estimate of a topological entropy in terms of the asymptotic Nielsen numbers, and (ii) to apply the related inequality for admissible pairs to impulsive differential equations and inclusions on tori. In case of a positive topological entropy, the obtained result can be regarded as a nontrivial contribution to deterministic chaos for multivalued impulsive dynamics.

Keywords: topological entropy; asymptotic Nielsen number; admissible pairs; coincidences; impulsive differential equations and inclusions; poincaré operators; coexistence of periodic solutions

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1. Introduction

In 1982, Ivanov formulated in [1] his remarkable inequality in the form of the following (slightly more precise) theorem.

Theorem 1. (cf. [1] Theorem) *Let f be a (single-valued) continuous self-map of a compact polyhedron. Then the inequality*

$$h(f) \geq \log N^{\infty}(f) := \log \max \left\{1, \limsup_{m\to\infty} \left(N(f^m)\right)^{\frac{1}{m}}\right\}$$

holds for the topological entropy $h(f)$ of f, where $N(f^m)$ denotes the Nielsen number of the m-th iterate of f.

The notion of a topological entropy can be understood in the sense of both [2,3]. For the definition and properties of the Nielsen number, see e.g., [4], where some single-valued generalizations of Theorem 1 for more abstract manifolds can also be found in ([4] Chapter VII.2).

For a positive topological entropy, Theorem 1 was effectively applied to discrete chaotic dynamics in [5–8] and to chaotic impulsive differential equations on tori in [9]. Let us note that the results dealing with the topological entropy for dynamic processes and differential equations from another perspective are rather rare (see e.g., [10–14]), and those for multivalued dynamics are even more delicate (see e.g., [15]).

On the other hand, although there exist several various definitions of a topological entropy for multivalued maps (see e.g., [16–22]), as far as we know, none have been applied to differential equations without uniqueness or, more generally to differential inclusions. Moreover, no analogy of Theorem 1 exists for multivalued maps.

Since the appropriate multivalued Nielsen theory, applicable to (impulsive) differential equations without uniqueness or differential inclusions on tori, does not possess (in difference to the standard single-valued Nielsen theory) the lower estimate of the number of fixed points, but "only" of coincidences of the associated admissible pairs (see [23–26] and cf. [27] Chapter I.10, [4] Chapter VII.4), no possible extension of Theorem 1 for quoted multivalued generalizations of a topological entropy would apply there via the associated Poincaré translation operators along the trajectories.

Hence, we need a new definition of a topological entropy for the class of (multivalued) admissible maps. On this basis, we will be able to generalize Theorem 1 in a desired way, i.e., in order to be applied to impulsive differential inclusions on tori.

Our paper is therefore organized as follows. At first, we will recall some preliminaries about ANR-spaces and admissible maps in the sense of Górniewicz, for which we will newly define the topological entropy. Then, in Section 3, we will make more precise the technicalities concerning the Nielsen theory for multivalued maps. In Section 4, a multivalued version of Ivanov's Theorem 1 will be formulated on compact polyhedra, jointly with its particular consequences on tori. The application to impulsive differential inclusions on tori, supplied by an illustrative example, will be given in Section 5. In the concluding Section 6, besides the comments, the relationship between a positive topological entropy for multivalued dynamics and the coexistence of (subharmonic) periodic solutions with various periods will be clarified.

2. Preliminaries (Including the Definition of a Topological Entropy for Admissible Pairs)

In the entire text, all topological spaces will be metric. A space X is an absolute neighborhood retract (written $X \in \mathrm{ANR}$) if, for every space Y and every closed subset $A \subset Y$, each continuous map $f\colon A \to X$ is extendable over some open neighborhood U of A in Y. A space X is an absolute retract (written $X \in \mathrm{AR}$) if each $f\colon A \to X$ is extendable over Y. Evidently, if $X \in \mathrm{AR}$, then $X \in \mathrm{ANR}$.

By a polyhedron, we understand usually a triangulable space. It is well known that every polyhedron is an ANR-space. An important example of a compact polyhedron will be for us a torus. By the n-torus $\mathbb{T}^n$, $n \geq 1$, we will mean here either the factor space $\mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$ or the Cartesian product $\underbrace{S^1 \times \ldots \times S^1}_{n\text{-times}}$, where $\mathbb{R}$ denotes the set of reals, $\mathbb{Z}$ denotes the set of integers, and

$$S^1 := \{x \in \mathbb{R}^2 \mid |x| = 1\} \quad \text{or} \quad S^1 := \{z = \mathrm{e}^{2\pi s\,\mathrm{i}} \mid s \in [0,1]\}.$$

In particular, for $n = 1$, $\mathbb{T}^1 = S^1$ becomes a circle.

If not explicitly specified, we will not distinguish between the additive and multiplicative notations, because the logarithm map $\mathrm{e}^{2\pi s\,\mathrm{i}} \to s$, $s \in [0,1]$, establishes an isomorphism between these two representations.

Let us also note that the relation between the Euclidean space $\mathbb{R}^n$ and its factorization $\mathbb{R}^n/\mathbb{Z}^n$ can be realized by means of the natural projection, sometimes also called a canonical mapping, $\tau\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$, $x \to [x]$, where the symbol $[x] := \{y \in \mathbb{R}^n \mid (y - x) \in \mathbb{Z}^n\}$ stands for the equivalent class of elements with x in $\mathbb{R}^n/\mathbb{Z}^n$, i.e., $\mathbb{R}^n/\mathbb{Z}^n := \{[x] \mid x \in \mathbb{R}^n\}$, where $[x] = x + \mathbb{Z}^n$, $x \in [0,1)^n$.

A nonempty set X is called acyclic (or, more precisely, $\mathbb{Q}$-acyclic), provided

$$H_n(X) = \begin{cases} 0, & \text{for } n > 0, \\ \mathbb{Q}, & \text{for } n = 0, \end{cases}$$

where $\mathbb{Q}$ stands for the field of rationals and H_n denotes the n-dimensional Čech homology functor with rational coefficients. When $\mathbb{Q}$ is replaced by $\mathbb{Z}$, then we speak about the $\mathbb{Z}$-acyclicity.

A continuous map $p\colon \Gamma \Rightarrow X$ is called the Vietoris map, provided

(i) p is onto (i.e., $p(\Gamma) = X$),
(ii) p is proper, i.e., $p^{-1}(K)$ is a nonempty compact set, for every compact $K \subset X$,
(iii) the given set $p^{-1}(x)$ is acyclic, for every $x \in X$.

By a multivalued map $\varphi\colon X \multimap Y$, we understand $\varphi\colon X \to 2^Y \setminus \{\emptyset\}$. In the entire text, we will still assume that φ has closed values.

Definition 1. *A map $\varphi\colon X \multimap Y$ is said to be upper semicontinuous (written u.s.c.) if, for every open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X, or equivalently, if for every closed $V \subset Y$, the set $\{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$ is closed in X.*

If, in particular, a single-valued map $f\colon X \to Y$ is u.s.c., then it is continuous. Furthermore, every u.s.c. map $\varphi\colon X \multimap Y$ has a closed graph $\{(x,y) \in X \times Y \mid y \in \varphi(x)\}$, but not vice versa. Nevertheless, if the graph of a compact map $\varphi\colon X \multimap Y$ (i.e., when the set $\varphi(X) = \bigcup_{x\in X} \varphi(x)$ is contained in a compact subset of Y) is closed, then φ is u.s.c. If $\varphi\colon X \multimap Y$ is u.s.c. with compact values and $A \subset X$ is compact, then $\varphi(A)$ is compact.

Definition 2. *Assume that we have a diagram $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$, where $p\colon \Gamma \Rightarrow X$ is a (single-valued) Vietoris map and $q\colon \Gamma \to Y$ is a continuous mapping. Then the map $\varphi\colon X \multimap Y$ is called admissible (in the sense of Górniewicz) if it is induced by $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$. Thus, we determine the admissible map φ by the pair (p,q) called an admissible (selected) pair.*

One can readily check that every admissible map is u.s.c. with nonempty compact connected values, but not vice versa. The class of admissible maps is closed with respect to finite compositions of admissible maps, i.e., a finite composition of admissible maps is also admissible. In fact, a map is admissible if and only if it is a composition of acyclic maps, i.e., u.s.c. maps with compact acyclic values. On the other hand, the composition of two acyclic maps need not be acyclic. For more details, see e.g., [27] Chapter I.4).

Let $\varphi\colon X \multimap X$ be an admissible self-map determined by the admissible pair (p,q), i.e., $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X$, where $\varphi(\cdot) = q(p^{-1}(\cdot))$.

The point $x \in X$ such that $x \in \varphi(x)$ is called a fixed point of φ, while $z \in \Gamma$ such that $p(z) = q(z)$ is called a coincidence point of φ, shortly a coincidence of φ. Observe that, letting the fixed point set of (p,q) as

$$\mathrm{Fix}(p,q) := \{x \in X \mid x \in q(p^{-1}(x))\},$$

and the coincidence point set of (p,q) as

$$\mathrm{Coin}(p,q) := \{z \in \Gamma \mid p(z) = q(z)\},$$

we obviously have the equalities

$$p(\mathrm{Coin}(p,q)) = q(\mathrm{Coin}(p,q)) = \mathrm{Fix}(p,q).$$

Definition 3. *The sequence of points $z_1, \ldots, z_n \in \Gamma$ is called an n-orbit of the pair (p,q) if $q(z_i) = p(z_{i+1})$, $i = 1, \ldots, n-1$. If, additionally, $p(z_1) = q(z_n)$, then we call it a periodic n-orbit, written $z_1 \in \mathrm{Coin}((p,q)^n)$. Schematically:*

$$\begin{tikzcd}
z_1 \arrow[dr, "q"] & & z_2 \arrow[dl, "p"'] \arrow[dr, "q"] & & & & z_n \arrow[dl, "p"'] \arrow[dr, "q"] & \\
& q(z_1) = p(z_2) & & q(z_2) = p(z_3) & \cdots & q(z_{n-1}) = p(z_n) & & q(z_n) = p(z_1)\,.
\end{tikzcd}$$

Observe that $z_1 \in \mathrm{Coin}((p,q)^n)$ implies $z_k \in \mathrm{Coin}((p,q)^n)$, for all $k = 1, \ldots, n$.

Now, we are in position to define the topological entropy for admissible pairs (p,q), when following the definition of Bowen [3] for single-valued maps.

Definition 4. *We say that the n-orbits $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n)$ of (p, q) are (n, ε)-separable if $d_1(z_k, z'_k) > \varepsilon$ holds for at least one $k = 1, \ldots, n$, where*

$$d_1(z_k, z'_k) := \max\{d(p(z_k), p(z'_k)), d(q(z_k), q(z'_k))\}$$

stands for the semi-metric in Γ and d is the metric in X.

Furthermore, denoting by $s(n, \varepsilon)$ the maximal number of n-orbits such that any two of them are (n, ε)-separable, we put

$$h_{\mathrm{se}}(p, q; \varepsilon) := \log \max \left\{1, \limsup_{m \to \infty} (s(m, \varepsilon))^{\frac{1}{m}}\right\}.$$

Since $0 < \varepsilon' < \varepsilon$ implies that $h_{\mathrm{se}}(p, q; \varepsilon) \leq h_{\mathrm{se}}(p, q; \varepsilon')$, we define the topological entropy $h(p, q)$ for admissible pairs (p, q) as

$$h(p, q) := \lim_{\varepsilon \to 0} h_{\mathrm{se}}(p, q; \varepsilon).$$

Remark 1. *On compact spaces, the topological entropy $h(p, q)$ does not depend on the metric. Let d, d' be two metrics on the compact metric space X. Since they are equivalent,*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, x' \in X\colon\ d(x, x') < \delta \Rightarrow d'(x, x') < \varepsilon.$$

Hence, if two m-orbits are (m, ε)-separable with respect to d, then they are (m, δ)-separable with respect to d'. This implies $s^{(d)}(p, q; m, \varepsilon) \leq s^{(d')}(p, q; m, \delta)$, $h^{(d)}_{\mathrm{se}}(p, q; \varepsilon) \leq h^{(d')}_{\mathrm{se}}(p, q; \delta)$, and finally $h^{(d)}(p, q) \leq h^{(d')}(p, q)$. The symmetry gives the opposite inequality.

3. Elements of the Nielsen Theory for Multivalued Maps

In the classical (single-valued) Nielsen fixed point theory, we consider a continuous self-map $f\colon X \to X$ of a space X, divide the fixed point set $\mathrm{Fix}(f) := \{x \in X \mid x = f(x)\}$ into the Nielsen classes (using the paths linking the fixed points), and finally we define the Nielsen number $N(f)$ of f as the number of essential Nielsen classes, i.e., those with nonzero fixed point index $\mathrm{ind}(f; A) \neq 0$. For the related definitions and more details, see e.g., [4].

It is natural to extend this approach to admissible maps. In [25] (cf. also [27] Example I.10.1 and [4] Example 7.4.7), we have given an example of an admissible self-map $\varphi\colon S^1 \multimap S^1$, determined by the admissible pair (p, q), with the following properties:

(i) the values $p^{-1}(x)$ are points or compact arcs,
(ii) φ is admissibly homotopic to a single-valued map f of degree d (for any prescribed integer d),
(iii) the fixed point set $\mathrm{Fix}(\varphi) := \{x \in S^1 \mid x \in q(p^{-1}(x))\}$ consists of two points.

Since $d \in \mathbb{Z}$ can be arbitrary and the fixed point set $\mathrm{Fix}(\varphi)$ has only two elements, this example demonstrates that there is no natural extension of the ordinary Nielsen fixed point theory to the class of multivalued admissible maps.

Instead of it, we should ask about the number of coincidences, i.e.,

$$\#\mathrm{Coin}(p, q) := \#\{z \in \Gamma \mid p(z) = q(z)\}.$$

For this goal, we will recall and modify the Nielsen theory for admissible self-maps, developed by ourselves in [25,26] (cf. also [27] Chapter I.10 and [4] Chapter VII.4). This theory is nontrivial, because the space Γ need not be path-connected and the splitting into the Nielsen classes involves some extra problems. Moreover, we cannot apply the classical index theory to maps on such an arbitrary space as Γ.

Hence, let us proceed to the definition of the Nielsen classes. Let us consider an admissible pair (p,q), i.e., $X \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} X$, where X is a compact ANR-space, Γ is a compact space, p is the Vietoris map and q is continuous. We are willing to split the coincidence set $\mathrm{Coin}(p,q)$ into the Nielsen classes. Since the space Γ need not be locally connected, we cannot traditionally use the paths. Instead of it, we will use the universal covering spaces approach.

We fix a universal covering $p_X\colon \widetilde{X} \to X$. Letting $\widetilde{\Gamma} := \{(\widetilde{x},z) \in \widetilde{X} \times \Gamma \mid p_X(\widetilde{x}) = p(z)\}$, denote the pullback giving the commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[d, "p_\Gamma"] \\ X & \Gamma, \arrow[l, Rightarrow, "p"'] \end{tikzcd}$$

where $\widetilde{p}(\widetilde{x},z) := \widetilde{x}$, $p_\Gamma(\widetilde{x},z) := z$. In order to split $\mathrm{Coin}(p,q)$ into classes, which we will also call the Nielsen classes, we will need a lift $\widetilde{q}\colon \widetilde{\Gamma} \to \widetilde{X}$ satisfying $p_X(\widetilde{q}) = q(p_\Gamma)$. This gives the commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"] & \widetilde{X} \arrow[d, "p_X"] \\ X & \Gamma \arrow[l, Rightarrow, "p"'] \arrow[r, "q"] & X. \end{tikzcd}$$

Let us emphasize that $\widetilde{p}$ is the pullback projection, while $\widetilde{q}$ is a fixed, but arbitrarily chosen, lift of q.

We will use the above diagram to split $\mathrm{Coin}(p,q)$ into the Nielsen classes, but before it, we would like to explain the existence of the lift $\widetilde{q}$ by means of the following lemmas.

Lemma 1. *Let $p\colon \widetilde{X} \to X$ be a universal covering of a connected locally simply-connected space X. Let Γ be a paracompact space and Γ_0 be its compact subset. Let $f_0\colon \Gamma_0 \to X$ have a lift $\widetilde{f_0}\colon \Gamma_0 \to \widetilde{X}$ such that $p(\widetilde{f_0}) = f_0$. Then there exists an extension of the lift $\widetilde{f_0}$ to $\widetilde{f_1}\colon \Gamma_1 \to \widetilde{X}$ onto an open neighbourhood $\Gamma_1 \supset \Gamma_0$.*

Proof. For each $x \in \Gamma_0$, we fix an open neighbourhood $U_x \subset \Gamma$ such that $x \in U_x$ and $\widetilde{f_x}$ admits an extension $\widetilde{f_x}\colon \Gamma_0 \cup U_x \to \widetilde{X}$. We may assume, when taking each U_x smaller, that the intersection of each fibre $\widetilde{f_x}(U_x) \cap p^{-1}(y)$ is a singleton or an empty set, for each $y \in X$. We choose a finite covering $\{V_i\}_{i\in I}$ of Γ_0, which is star-subwritten to $\{U_x\}$. For each $i \in I$, we furthermore choose an $x_i \in \Gamma_0$ satisfying $V_i \subset U_{x_i}$, and we define the extension $\widetilde{f_i}$ as the restriction of $\widetilde{f_{x_i}}$. We will show that the system $\{\widetilde{f_i}\}$ defines correctly the extension $\widetilde{f_1}\colon \bigcup_{i\in I} V_i \to \widetilde{X}$, which gives the desired extension on the neighbourhood $\Gamma_1 := \bigcup_{i\in I} V_i$.

It remains to show that the extensions $\widetilde{f_i}$ are consistent. Letting $z \in V_i \cap V_{i'}$, then $z \in V_i \cup V_{i'} \subset U_j$ for some $j \in I$. Since $\widetilde{f_i}(z)$ and $\widetilde{f_{i'}}(z)$ belong to the same fibre of p and a small U_j, $\widetilde{f_i}(z)$ and $\widetilde{f_{i'}}(z)$ must be equal, which completes the proof. $\square$

Lemma 2. ([27] Lemma I.10.6) *Let $p\colon \Gamma \Rightarrow X$, $q\colon \Gamma \to Y$ be continuous maps. where X, Y are compact connected ANR-spaces, Γ is a compact connected space, and p is the Vietoris map. Assume, furthermore, that X is simply-connected and that the restriction $q|_{p^{-1}(x)}\colon p^{-1}(x) \to Y$ admits a lift $\widetilde{q}|_{p^{-1}(x)}\colon p^{-1}(x) \to \widetilde{Y}$, for each $x \in X$. Then there exists a map $\widetilde{q}\colon \Gamma \to \widetilde{Y}$ making the diagram commutative:*

$$\begin{tikzcd} & \widetilde{Y} \arrow[d, "p_Y"] \\ \Gamma \arrow[ur, "\widetilde{q}"] \arrow[r, "q"'] & Y. \end{tikzcd}$$

Proof. For the proof, it is enough to follow the arguments in ([27] Lemma I.10.6). $\square$

Lemma 3. *If the restriction $q|_{p^{-1}(x)}\colon p^{-1}(x) \to X$ lifts, for each $x \in X$, to a map $\widetilde{q}|_{p^{-1}(x)}\colon p^{-1}(x) \to \widetilde{X}$, then $q\colon \Gamma \to X$ admits a lift $\widetilde{q}\colon \widetilde{\Gamma} \to \widetilde{X}$.*

Proof. It is enough to apply Lemma 2 for $X = Y$ and $q = q(p_\Gamma)$. $\square$

Let $\theta_X := \{\alpha : \widetilde{X} \to \widetilde{X} \mid p_X\alpha = p_X\}$ denote the set of natural transformations of $\widetilde{X}$. Recall that then $\theta_X = \pi_1 X$.

Similarly, we define $\theta_\Gamma = \{\alpha : \widetilde{\Gamma} \to \widetilde{\Gamma} \mid p_\Gamma\alpha = p_\Gamma\}$. Since $\widetilde{\Gamma}$ is the pullback, there is a natural bijection given by $\theta_X = \theta_\Gamma$, i.e., $\alpha \in \theta_X$ induces $\theta_\Gamma \ni (\widetilde{x}, z) \to (\alpha(\widetilde{x}), z) \in \theta_\Gamma$.

We denote by $p^!\colon \theta_X \to \theta_\Gamma$ the natural isomorphism given by the formula $\widetilde{p}^!(\alpha)(\widetilde{x}, z) = (\alpha\widetilde{x}, z)$. On the other hand, the right hand side of the diagram gives a homomorphism $\widetilde{q}_!\colon \theta_\Gamma \to \theta_X$ given by $\widetilde{q}\alpha = (\widetilde{q}_!\alpha)\widetilde{q}$. Finally, we denote $\widetilde{\rho} = \widetilde{q}_!\widetilde{p}^!$. Now, we are in position to define the Reidemeister relation.

We consider the action of θ_X on itself, $\alpha \circ \omega = \alpha\omega\widetilde{\rho}(\alpha^{-1})$. The quotient set is called the set of Reidemeister classes and is denoted by $\mathcal{R}(\widetilde{p}, \widetilde{q})$. In [25], we have shown that the definition does not depend on the choice of the lift $\widetilde{q}$.

To split the set $\mathrm{Coin}(p, q)$ to the Nielsen classes, we need the above theory modulo a subgroup.

We consider the pair (p, q) as above and assume that $H \triangleleft \theta_X$ is a normal subgroup preserved by $\widetilde{\rho}$, i.e., $\widetilde{\rho}(H) \subset H$. Then we can follow the above construction to get the commutative diagram:

$$\begin{tikzcd}
\widetilde{X}_H \arrow[d, "p_{X_H}"'] & \widetilde{\Gamma}_H \arrow[l, "\widetilde{p}_H"'] \arrow[r, "\widetilde{q}_H"] \arrow[d, "p_{\Gamma_H}"'] & \widetilde{X}_H \arrow[d, "p_{X_H}"'] \\
X & \Gamma \arrow[l, Rightarrow, "p"'] \arrow[r, "q"] & X\,,
\end{tikzcd}$$

where $\widetilde{X}_H$ is the covering corresponding to the subgroup H and $\widetilde{\Gamma}_H$ is the corresponding pullback. This gives the homomorphism $\widetilde{q}_{H!}\widetilde{p}^!_H\colon \theta_{XH} \to \theta_{XH}$.

Lemma 4. (cf. [25] Lemma 4.1)

(i) $\mathrm{Coin}(p, q) = \bigcup_{\alpha\in\theta_{X_H}} p_{\Gamma_H} \mathrm{Coin}(\widetilde{p}_H, \alpha\widetilde{q}_H)$;

(ii) *if*

$$p_{\Gamma_H} \mathrm{Coin}(\widetilde{p}_H, \alpha\widetilde{q}_H) \cap p_{\Gamma_H} \mathrm{Coin}(\widetilde{p}_H, \beta\widetilde{q}_H)$$

is not empty, then there exists a $\gamma \in \theta_{X_H}$ such that

$$\beta = \gamma \cdot \alpha \cdot (\widetilde{q}_{H!}\widetilde{p}^!_H\gamma)^{-1};$$

(iii) *the sets $p_{\Gamma_H} \mathrm{Coin}(\widetilde{p}_H, \alpha\widetilde{q}_H)$ are either disjoint or equal.*

Now, we get the splitting of $\mathrm{Coin}(p, q)$ into the H-Nielsen classes and the natural injection from the set of H-Nielsen classes into the set of the Reidemeister classes modulo H, namely $\mathcal{R}_H(p, q)$.

At last, we can proceed to the essential Nielsen classes, i.e., the coincidence classes which do not disappear under any admissible homotopy. We recall the notion of essential classes given in (Section 5 [25]). However, besides the assumptions that $p, q\colon \Gamma \to X$ is an admissible pair (i.e., X is a compact ANR-space, Γ is a compact space, p is the Vietoris map and q is continuous), we will still assume that there exists a normal subgroup of finite index $H \triangleleft \pi_1 X$ such that $\widetilde{\rho}(H) \subset H$ (which will enable us to use finite covering spaces).

Since X is a compact ANR-space and $\mathrm{cl}(q(\Gamma)) \subset X$ is compact, the Lefschetz number $L(p, q)$ is defined (see e.g., [27] Chapter I.6).

Let us recall that, according to Lemma 4, $\mathrm{Coin}(p, q) = \bigcup_{\alpha\in\theta_{X_H}} p_{\Gamma_H} \mathrm{Coin}(\widetilde{p}_H, \alpha\widetilde{q}_H)$, where any two summands are either equal or empty. This gives the splitting into the Nielsen classes modulo H. On the other hand, since $\widetilde{X}_H$ is compact, by the second property $L(\widetilde{p}, \alpha\widetilde{q})$ is defined. If $L(\widetilde{p}, \alpha\widetilde{q}) \neq 0$, then we call the corresponding H-Nielsen class essential. As in the single-valued case, each essential Nielsen class is nonempty.

We define the coincidence Nielsen number modulo the subgroup H as the number of the essential Nielsen classes, and we denote it by $N_H(p,q)$.

The following two important theorems will be stated in the form of propositions.

Proposition 1. (cf. [25] Theorem 5.8) *$N_H(p,q)$ is a homotopy invariant (with respect to the admissible homotopies $X \times [0,1] \xLeftarrow{p} \Gamma \xrightarrow{q} X$). Moreover, (p,q) has at least $N_H(p,q)$ coincidences.*

The following statement shows that the above definition is consistent with the classical Nielsen number for single-valued maps.

Proposition 2. (cf. [25] Theorem 5.9) *If an admissible map (p,q) is admissibly homotopic to a pair (p',q') (written $(p,q) \sim (p',q')$), representing a single-valued map $\rho = (p',q')$ (i.e., $q' = \rho p'$), then $N_H(p,q) = N_H(\rho)$, i.e., the classical Nielsen number modulo H.*

For the calculation in applications, it will be also convenient to give the following statements on tori. Let us recall that by an R_δ-set, we mean an intersection of a decreasing sequence of compact absolute retracts or, equivalently, of compact contractible (i.e., homotopically equivalent to one point) sets. For more details, see e.g., ([27] Chapters I.2 and I.4).

Proposition 3. *Any admissible self-map φ on the torus $\mathbb{T}^n$, determined by the admissible pairs (p,q), where $p^{-1}(x)$ is an R_δ-set for every $x \in \mathbb{T}^n$, is admissibly homotopic to a pair (p',q'), representing a single-valued (continuous) map $\rho\colon \mathbb{T}^n \to \mathbb{T}^n$ (written $\varphi \sim \rho$).*

Proof. For the proof, we can proceed in the same way as in ([25] Theorem 6.2) (cf. also [27] Theorem I.10.24), where instead of the lemma in ([25] Lemma 6.1), we apply the fact that since an R_δ-set is $\mathbb{Z}$-acyclic, we have that $\widetilde{H}(p^{-1}(x);\mathbb{Z}) = 0$, and so any restriction $q|_{p^{-1}(x)}\colon p^{-1}(x) \to S^1$ is contractible.

The fact that any R_δ-set is $\mathbb{Z}$-acyclic follows from the following continuity arguments due to Górniewicz (N. Copernicus Univ. Toruń, Poland). By the definition, the R_δ-set A can be written as the intersection $A = \bigcap_{n\geq 1} A_n$ of a decreasing sequence $\{A_n\}$ of compact AR-spaces (or, in particular, compact contractible sets) A_n, which are $\mathbb{Z}$-acyclic, for every $n \in \mathbb{N}$. The claim is then implied by the theorem in ([28] Theorem 3.1 in Chapter X.3, p. 261), saying that the Čech homology theory based on a coefficient group which is in $\mathcal{G}_{\mathfrak{R}}$ (the category of modulus over a ring $\mathfrak{R}$) or $\mathcal{G}_C$ (the category of compact Abelian groups) is continuous on the category of compact pairs. Moreover, when $\mathfrak{R}$ is the ring of integers, $\mathcal{G}_{\mathfrak{R}}$ is just the category of ordinary Abelian groups and their homomorphisms (see e.g., [28] Chapter IV.3, p. 110). Thus, for our needs, we can put $\mathfrak{R} = \mathbb{Z}$, which completes the proof. □

Remark 2. *In fact, in the proof of the foregoing Proposition 3, we needed to have the commutative diagram:*

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"'] & \widetilde{X} \arrow[d, "p_X"'] \\
X & \Gamma \arrow[l, Rightarrow, "p"'] \arrow[r, "q"] & X\,,
\end{tikzcd}$$

in order to get the splitting of the coincidence set $\mathrm{Coin}(p,q)$ *into the Nielsen classes. More precisely, we needed the lift $\widetilde{q}$ as above.*

To get such a lift, we should assume that the restrictions $q|_{p^{-1}(x)}\colon p^{-1}(x) \to X$ admit lifts $\widetilde{q}|_{p^{-1}(x)}\colon p^{-1}(x) \to \widetilde{X}$ and $p_X(\widetilde{q}) = q$. In the case of the torus, i.e., for $X = \mathbb{T}^n = S^1 \times \cdots \times S^1$, this condition takes the form that each map $p^{-1}(x) \to S^1$ is homotopic to a constant map. It is true, provided $p^{-1}(x)$ is $\mathbb{Z}$-acyclic, for every $x \in S^1$.

Proposition 4. (cf. [24] equalities (4)) *Under the assumptions of Proposition 3, the equalities*

$$(N_H((p,q)^k) =)\ N((p,q)^k) = N(\rho^k) = |L(\rho^k)| \tag{1}$$

hold for every $k \in \mathbb{N}$, where N_H is the coincidence Nielsen number modulo the (concrete) subgroup H defined above, N is the ordinary Nielsen number and L is the ordinary Lefschetz number.

Proof. The equalities (1) were derived in [24], on the basis of the proposition in ([23] Proposition 2.6) and Proposition 3. For the definition of the concrete subgroup H, see the proof in ([25] Theorem 6.3) (cf. also [27] Theorem I.10.25). □

4. Ivanov's Theorem for Admissible Pairs

The crucial argument in the proof of Theorem 1 can be reformulated in the form of the following lemma (cf. [1], the proof of Theorem).

Lemma 5. *Let $f\colon X \to X$ be a continuous self-map of a compact polyhedron X endowed with the metric d. Then there exists a number $\varepsilon > 0$ such that, for any $x, y \in X$ and $n \in \mathbb{N}$: if $d(f^k(x), f^k(y)) < \varepsilon$, for $k = 1, \dots, n$, and $f^n(x) = x$, $f^n(y) = y$, then the points x, y belong to the same Nielsen class as the fixed points of f^n.*

Let us emphasize that $\varepsilon > 0$ in Lemma 5 depends only on f and it is independent of n.

We would like therefore to generalize Lemma 5 to the admissible pairs $X \xLeftarrow{p} \Gamma \xrightarrow{q} X$, where X is a compact polyhedron. Thus, we will again assume that p is the Vietoris map and q is continuous. For this generalization, we need the following technical lemma.

Lemma 6. *Consider the diagram:*

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"'] & \widetilde{X} \arrow[d, "p_X"'] \\
X & \Gamma \arrow[l, Rightarrow, "p"'] \arrow[r, "q"] & X,
\end{tikzcd}$$

with fixed $z_1, z_1' \in \Gamma$ satisfying $d_1(z_1, z_1') \leq \varepsilon$ (for the definition of the semi-metric d_1, see Definition 4). Denoting $x_1 := p(z_1)$, $x_2 := q(z_1)$, $x_1' := p(z_1')$, $x_2' := q(z_1')$, we will be able to find the points $\widetilde{z}_1, \widetilde{z}_1' \in \widetilde{\Gamma}$, $\widetilde{x}_1, \widetilde{x}_1' \in \widetilde{U}_1$, $\widetilde{x}_2, \widetilde{x}_2' \in \widetilde{U}_2$ such that:

1. $p_\Gamma(\widetilde{z}_1) = z_1$, $p_\Gamma(\widetilde{z}_1') = z_1'$,
2. $\widetilde{p}(\widetilde{z}_1) = \widetilde{x}_1$, $\widetilde{p}(\widetilde{z}_1') = \widetilde{x}_1'$, $\widetilde{q}(\widetilde{z}_1) = \widetilde{x}_2$, $\widetilde{q}(\widetilde{z}_1') = \widetilde{x}_2'$,
3. $\widetilde{x}_i, \widetilde{x}_i' \in \widetilde{U}_i$, *where $\widetilde{U}_i \subset \widetilde{X}$ is an open contractible set which is sent homeomorphically by p_X ($i = 1, 2$).*

Moreover $\widetilde{x}_1, \widetilde{x}_1' \in \widetilde{U}_1$ can be any pair of points $\widetilde{x}_1 \in p_X^{-1}(x_1)$, $\widetilde{x}_1' \in p_X^{-1}(x_1')$, satisfying 3.

Proof. Since the simplicial complex X is locally contractible and compact, there exists an $\varepsilon_0 > 0$ such that each subset $D \subset X$ with $\operatorname{diam}(D) < 2\varepsilon_0$ is contained in a contractible set $U \subset X$. We assume that the numbers ε, which appear below, satisfy $\varepsilon_0 > \varepsilon > 0$.

Let $z_1, z_1' \in \Gamma$ be the given pair of points satisfying $d_1(z_1, z_1') < \varepsilon$. We define $x_1 := p(z_1)$, $x_1' := p(z_1')$ and $x_2 := q(z_1)$, $x_2' := q(z_1')$. It is easy to see that there exist lifts of these points making the diagram commutative.

$$\begin{tikzcd} \widetilde{x}_1, \widetilde{x}_1' \arrow[d, "p_X"'] & \widetilde{z}_1, \widetilde{z}_1' \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"'] & \widetilde{x}_2, \widetilde{x}_2' \arrow[d, "p_Y"'] \\ x_1, x_1' & z_1, z_1' \arrow[l, Leftarrow, "p"'] \arrow[r, "q"] & x_2, x_2' \end{tikzcd}$$

In fact, we choose the lifts $\widetilde{z}_1, \widetilde{z}_1' \in \widetilde{\Gamma}$ of the given $z_1, z_1' \in \Gamma$, and these determine the rest.

Moreover, the assumption $d_1(z_1, z_1') < \varepsilon$ implies $d(x_1, x_1') < \varepsilon$ and $d(x_2, x_2') < \varepsilon$, which in turn implies $x_1' \in K(x_1, \varepsilon)$, $x_2' \in K(x_2, \varepsilon)$, $z_1' \in K(z_1, \varepsilon)$ (here $K(., \varepsilon)$ denotes the open ball). We will show that if $\varepsilon > 0$ is small enough, then the inverse images $p_X^{-1}(K(x_1, \varepsilon))$, $p_X^{-1}(K(x_2, \varepsilon))$, $p_\Gamma^{-1}(K(z_1, \varepsilon))$ split into mutually disjoint parts, each of them sent isometrically onto its image. We will overuse the notation and call these parts by components. Notice that for a sufficiently small $\varepsilon > 0$, $K(x, \varepsilon)$ is connected and the introduced components coincide with the connected components of $p_X^{-1}(K(x_1, \varepsilon)) = \bigcup_\alpha \widetilde{U}_\alpha$.

However, $K(z, \varepsilon)$ need not be connected, because Γ need not be locally connected. So we define the splitting

$$p_\Gamma^{-1}(K(z, \varepsilon)) := \bigcup_{\alpha \in \mathcal{O}_X} p^{-1}(\widetilde{U}_\alpha),$$

where $U_\alpha = \alpha \cdot U_0$ and $\mathcal{O}_X := \{\alpha \colon \widetilde{X} \to \widetilde{X} \mid p_X \alpha = p_X\}$. In other words, we define the components $W_\alpha := p^{-1}(\widetilde{U}_\alpha)$ and we can denote $p_\Gamma^{-1}(K(z, \varepsilon)) = \bigcup_\alpha W_\alpha$.

Our lemma will be proved once we show that: *if $\widetilde{z}_1, \widetilde{z}_1'$ belong to the same component in $p_\Gamma^{-1}(K(z_1, \varepsilon))$, then we can choose the pairs of lifts $\widetilde{x}_1$, $\widetilde{x}_1'$ and $\widetilde{x}_2$, $\widetilde{x}_2'$, which also belong to the same component of the corresponding inverse images $p_X^{-1}(K(x_1, \varepsilon))$, $p_X^{-1}(K(x_2, \varepsilon))$.*

Considering the left hand side of the diagram, the claim is obvious, because $\widetilde{p}$ sends the component U_α into W_α.

Now, let us consider its right hand side. Since Γ need not be locally connected and q is an arbitrary continuous map, we cannot use a similar argument.

We will prove that, for a sufficiently small ε, if the lifts $\widetilde{z}_1$ and $\widetilde{z}_1'$ are chosen in the same component of $p_\Gamma^{-1}(K(z_1, \varepsilon))$, then $\widetilde{x}_2 = \widetilde{q}(\widetilde{z}_1)$ and $\widetilde{x}_2' = \widetilde{q}(\widetilde{z}_1')$ belong to the same component of $p_\Gamma^{-1}(K(x_2, \varepsilon))$. We will proceed by a contradiction. Let us assume that, for a sequence $\varepsilon_0 > \varepsilon_k > 0$ converging to 0, there exist the pairs of points $z_k, z_k' \in \Gamma$ with $d(z_k, z_k') < \varepsilon_k$, and each pair of lifts $\widetilde{z}_k, \widetilde{z}_k' \in \widetilde{X}$ lying in the same component of $p_\Gamma^{-1}(K(z_1, \varepsilon_k))$ implies that the points $q(\widetilde{z}_k), q(\widetilde{z}_k') \in \widetilde{X}$ belong to different components of $p_X^{-1}(K(x_2, \varepsilon_k))$.

By the compactness of Γ, we can assume that z_k converges to a point $z_0 \in \Gamma$. Thus, $K(z_k, \varepsilon_k) \subset K(z_0, \varepsilon_0)$, for almost all k. We choose a component $\widetilde{U} \subset p_\Gamma^{-1}(K(z_0, \varepsilon))$ and define $\widetilde{z}_k := p_\Gamma^{-1}(z_k) \cap \widetilde{U}$, $\widetilde{z}_k' := p_\Gamma^{-1}(z_k') \cap \widetilde{U}$.

Now, z_k' also converges to $z_0 \in \Gamma$ but $\widetilde{q}(\widetilde{z}_k')$, which belong by the assumption to other components, cannot converge to $\widetilde{q}(\widetilde{z}_0)$, because the fibre $p_X^{-1}(q(z_0))$ is discrete. The last fact contradicts to the continuity of $\widetilde{q} \colon \widetilde{\Gamma} \to \widetilde{X}$ in the point $\widetilde{z}_0$, which completes the proof. □

The desired multivalued generalization of Lemma 5 reads as follows.

Lemma 7. *Let $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X$ be an admissible self-map of a compact polyhedron X endowed with the metric d. Then there exists a number $\varepsilon > 0$ such that, for each pair of periodic n-orbits $(z_1, \ldots, z_n)$, $(z_1', \ldots, z_n') \in$* $\mathrm{Coin}((p, q)^n)$ *(see Definition 3): $d_1(z_k, z_k') < \varepsilon$, for $k = 1, \ldots, n$, implies that $z_1, z_1' \in$* $\mathrm{Coin}((p, q)^n)$ *are in the same coincidence Nielsen class, where the semi-metric d_1 on Γ was defined in Definition 4.*

Proof. We have given a multivalued map $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X$ and two ε-close orbits of the length n, namely $(z_1, \cdots, z_n)$ and $(z_1', \cdots, z_n')$. We have to show that z_1 and z_1' belong to the same Nielsen class as the coincidence points of $\mathrm{Coin}((p, q)^n)$.

Let us recall that $\mathrm{Coin}((p,q)^n) = \bigcup_\alpha p_\Gamma(\mathrm{Coin}((\mathrm{id}\times\alpha)(p,q)^n))$, where the summation runs through the set of one representative α from each Reidemester class, is a mutually disjoint sum, and each summand $p_\Gamma(\mathrm{Coin}(\mathrm{id}\times\alpha)(p,q)^n))$ is either the Nielsen class or it is empty.

We will show that there exist orbits $(\widetilde{z}_1,\cdots,\widetilde{z}_n)$ and $(\widetilde{z}'_1,\cdots,\widetilde{z}'_n)$ of $(\widetilde{p},\widetilde{q})$, satisfying $p_\Gamma(\widetilde{z}'_i) = z'_i$, $p_\Gamma(\widetilde{z}_i) = z_i$ and $\widetilde{z}_1, \widetilde{z}'_1 \in \mathrm{Coin}(\mathrm{id}\times\beta)(\widetilde{p},\widetilde{q})^n))$, for some $\beta\in\mathcal{O}_X := \{\alpha\colon \widetilde{X}\to\widetilde{X} \mid p_X\alpha = p_X\}$.

Applying Lemma 6 to z_1, z'_1 and x_1, x'_1, we get the elements $\widetilde{z}_1$, $\widetilde{z}'_1$ and $\widetilde{x}_2$, $\widetilde{x}'_2$. Then we repeat the same for z_2, z'_2 and x_2, x'_2, and we get $\widetilde{z}_2$, $\widetilde{z}'_2$. We follow it, until we get $\widetilde{z}_{n+1}$, $\widetilde{z}'_{n+1}$, satisfying $p_\Gamma(\widetilde{z}_{n+1}) = z_{n+1} = z_1$, $p_\Gamma(\widetilde{z}'_{n+1}) = z'_{n+1} = z'_1$. This implies $\widetilde{z}_{n+1} = \beta\widetilde{z}_1$, $\widetilde{z}'_{n+1} = \beta'\widetilde{z}'_1$, for some $\beta,\beta'\in\mathcal{O}_X$. However, by the construction, the points $\widetilde{z}_i$, $\widetilde{z}'_i$ belong to a contractible open set which is mapped by p_X homeomorphically. Hence, $\beta=\beta'$ which completes the proof. □

We are ready to give the first main theorem.

Theorem 2. *Let $\varphi\colon X \multimap X$ be an admissible self-map of a compact polyhedron X, determined by the admissible pair (p,q), i.e., $\varphi(\cdot) = q(p^{-1}(\cdot))$, where $X \xLeftarrow{p} \Gamma \xrightarrow{q} X$. Then the inequality*

$$h(p,q) \geq \log N_H^\infty(p,q) := \log\max\left\{1, \limsup_{m\to\infty}\left(N_H((p,q)^m)\right)^{\frac{1}{m}}\right\} \tag{2}$$

holds for the topological entropy $h(p,q)$ of (p,q), defined in Definition 4, and the asymptotic Nielsen number $N_H^\infty(p,q)$ (modulo the subgroup H), where $N_H((p,q)^m)$ denotes the coincidence Nielsen number (modulo the subgroup H) of the pair $(p,q)^m$, defined in Section 3.

Proof. By Lemma 7, for a sufficiently small $\varepsilon>0$, there exist $N_H((p,q)^m)$ (m,ε)-separated m-orbits. This implies $N_H((p,q)^m)\leq s(m,\varepsilon)$, and subsequently

$$\log\max\left\{1, \limsup_{m\to\infty}\left(N_H((p,q)^m)\right)^{\frac{1}{m}}\right\} \leq \log\max\left\{1, \limsup_{m\to\infty}\left(s(m,\varepsilon)\right)^{\frac{1}{m}}\right\} = h_{\mathrm{se}}(p,q;\varepsilon).$$

Since $h(p,q) := \lim_{\varepsilon\to 0} h_{\mathrm{se}}(p,q;\varepsilon)$, we arrive at the inequality (2), which completes the proof. □

On tori, Theorem 2 can be still improved by means of the equalities (1) in the following way.

Theorem 3. *Let $\varphi\colon \mathbb{T}^n \multimap \mathbb{T}^n$ be an admissible map, determined by the admissible pair (p,q) such that $p^{-1}(x)$ is an R_δ-set, for every $x\in\mathbb{T}^n$. Then the relations*

$$h(p,q) \geq \log N^\infty(p,q) = \log|L^\infty(\rho)| := \log\max\left\{1, \limsup_{m\to\infty}\sqrt[m]{|L(\rho^m)|}\right\} \tag{3}$$

hold for the topological entropy $h(p,q)$ of (p,q) and the asymptotic Lefschetz number $L^\infty(\rho)$ of ρ, where $\rho\colon\mathbb{T}^n\to\mathbb{T}^n$ is a single-valued continuous map, which is admissibly homotopic to φ.

Proof. The relations (3) follow directly from those in (1) and (2). The existence of single-valued continuous map ρ, which is admissibly homotopic to φ, is guaranteed by Proposition 3. □

Remark 3. *Let us note that, in difference to the topological invariants like the Nielsen number and the Lefschetz number, the topological entropy is not invariant under homotopy.*

As an illustrative example of the calculation by means of (3), we can give the following one.

Example 1. *Let $\varphi\colon \mathbb{T}^n \multimap \mathbb{T}^n$ be an admissible map, determined by the admissible pair (p,q) such that $p^{-1}(x)$ is an R_δ-set, for every $x\in\mathbb{T}^n$. Let φ be admissibly homotopic to a single-valued endomorphism $\rho\colon\mathbb{T}^n\to\mathbb{T}^n$,* *defined by an integer* $(n \times n)$*-matrix* A*. Let* $(\lambda_1, \cdots, \lambda_n)$ *be the eigenvalues of* A*. Then, according to* (3) *in Theorem* 3*,*

$$h(p,q) \geq \log |L^\infty(\rho)| = \sum_{|\lambda_k|>1} \log |\lambda_k|, \textit{ provided } \lambda_k \neq 1, \textit{ for all } k = 1, \ldots, n, \tag{4}$$

where

$$|L^\infty(\rho)| = \begin{cases} 1, & \textit{if } \prod_{k=1}^n (1-\lambda_k) = 0, \\ \prod_{|\lambda_k|>1} |\lambda_k| & \textit{otherwise}. \end{cases}$$

In particular, for $n = 1$*,* $h(p,q) \geq \log |d| = \log |A|$*, where* d *stands for the topological degree of* ρ*.*

5. Application to Impulsive Differential Inclusions on Tori

Consider the impulsive vector differential inclusion

$$\begin{cases} x' \in F(t,x), & t \neq t_j := t_0 + j\omega, \text{ for some given } \omega > 0, \\ x(t_j^+) = I(x(t_j^-)), & j \in \mathbb{Z}, \end{cases} \tag{5}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that

$$F(t,x) \equiv F(t+\omega, x), F(t, \ldots, x_j, \ldots) \equiv F(t, \ldots, x_j + 1, \ldots), \; j = 1, \ldots, n, \tag{6}$$

and $I\colon \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued continuous impulsive mapping such that

$$I(\ldots, x_j, \ldots) \equiv I(\ldots, x_j + 1, \ldots) \; (\mathrm{mod}\, 1), \; j = 1, \ldots, n, \tag{7}$$

where $x = (x_1, \ldots, x_n)$. Furthermore, $x(t_j^+) := \lim_{t \to t_j^+} x(t)$, $j \in \mathbb{Z}$, i.e., $x(t_j^+)$ stands for the right limit, $x(t_j^-) := \lim_{t \to t_j^-} x(t)$, $j \in \mathbb{Z}$, i.e., $x(t_j^-)$ stands for the left limit.

Let us recall that a multivalued map $F\colon \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is, under (6), upper-Carathéodory, provided:

(i) $F(\cdot, x)\colon [0,\omega] \multimap \mathbb{R}^n$ is Lebesgue measurable, for every $x \in [0,1]^n$, i.e., $\{t \in [0,\omega]\colon F(\cdot,x) \subset U\}$ is Lebesgue measurable, for each open $U \subset [0,1]^n$,
(ii) $F(t,\cdot)\colon [0,1]^n \multimap \mathbb{R}$ is u.s.c., for almost all (a.a.) $t \in [0,\omega]$, (cf. Definition 1),
(iii) F has convex and compact values $\{F(t,x)\}$, for all $(t,x) \in [0,\omega] \times [0,1]^n$,
(iv) $|y| \leq r(t)(1 + |x|)$, for all $(t,x) \in [0,\omega] \times [0,1]^n$, $y \in F(t,x)$, where $r\colon [0,\omega] \to [0,\infty)$ is a Lebesgue integrable function.

If $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is single-valued, then it is called Carathéodory.

By a solution of (5), we understand a (piece-wise) absolutely continuous vector function $x \in \mathrm{AC}((t_j, t_{j+1}), \mathbb{R}^n)$, $j \in \mathbb{Z}$, satisfying (5), for a.a. $t \in \mathbb{R} \setminus \{t_j\}_{j \in \mathbb{Z}}$.

Let us notice that conditions (6) and (7) allow us to consider the inclusion (5) on the factor space $\mathbb{R}^n/\mathbb{Z}^n$.

Hence, assuming (6) and (7), consider the composition $\widehat{I} \circ \widehat{T}_{\mu\omega}\colon \mathbb{R}^n/\mathbb{Z}^n \times [0,1] \multimap \mathbb{R}^n/\mathbb{Z}^n$, where the ($\mu$-parametrized) Poincaré translation operator $\widehat{T}_{\mu\omega}\colon \mathbb{R}^n/\mathbb{Z}^n \times [0,1] \multimap \mathbb{R}^n/\mathbb{Z}^n$ along the trajectories of (5) considered on $\mathbb{R}/\mathbb{Z}$, is defined by

$$\begin{aligned} \widehat{T}_{\mu\omega}([x_0]) := \{\widehat{x}(t_0 + \mu\omega) \in \mathbb{R}^n/\mathbb{Z}^n \colon \widehat{x}(\cdot) \in \mathrm{AC}(\mathbb{R}, \mathbb{R}^n/\mathbb{Z}^n) \text{ is a solution of} \\ x' \in F(t,x) \text{ considered on } \mathbb{R}^n/\mathbb{Z}^n, \text{ such that } \widehat{x}(t_0) = [x_0]\}, \end{aligned} \tag{8}$$

and $\widehat{I}\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is the factorization of the impulsive mapping $I\colon \mathbb{R}^n \to \mathbb{R}^n$, i.e., $[\tau \circ I(x)] = \widehat{I}([x])$, for every $x \in \mathbb{R}^n$.

Proposition 5. (cf. [24] Proposition 3.2) *The above composition $\widehat{I \circ T_{\mu\omega}} := \widehat{I} \circ \widehat{T}_{\mu\omega}$, $\mu \in [0,1]$, where $\widehat{T}_{\mu\omega}$ is the (factorized) Poincaré translation operator defined in (8) and $\widehat{I}$ is the factorized impulsive mapping, builds an admissible homotopy bridge between $\widehat{I \circ T_\omega}$ and $\widehat{I}$.*

For the desired application of Theorem 3 to (5), the following correspondence is crucial.

Lemma 8. (cf. [24] Lemma 4.1) *There exists a selected pair $(\widehat{p}, \widehat{q})$ which determines the (admissible) operator $\widehat{I \circ T_\omega}$, treated in Proposition 5, such that $\widehat{p}^{-1}([x])$ is an R_δ-set, for every $[x] \in \mathbb{R}^n/\mathbb{Z}^n$. Moreover, there is a one-to-one correspondence, for every $k \in \mathbb{N}$, between the irreducible k-orbits of coincidences (for their definition and more details, see [23,24]) of the mapping*

$$\mathbb{R}^n/\mathbb{Z}^n \overset{\widehat{p}}{\Longleftarrow} \widehat{Z} \overset{\widehat{q}}{\longrightarrow} \mathbb{R}^n/\mathbb{Z}^n \tag{9}$$

and $k\omega$-periodic $(\mathrm{mod}\ 1)$ solutions of the impulsive inclusion (5), provided (6) and (7) hold.

Since the operator $\widehat{I \circ T_\omega}$ is, according to Proposition 5 and Lemma 8, admissible in the sense of Definition 2, and such that $\widehat{p}^{-1}([x])$ is an R_δ-set, for every $[x] \in \mathbb{R}^n/\mathbb{Z}^n$, and it is also admissibly homotopic to the factorization $\widehat{I}$ of the impulsive mapping I, satisfying (7), we can use the formulas (1)–(4) for the calculation of the (asymptotic) Nielsen and Lefschetz numbers.

It is also natural to introduce the following definition.

Definition 5. *The impulsive differential inclusion (5) has, under (6) and (7), a positive topological entropy if $h(\widehat{p}, \widehat{q}) > 0$ holds for the admissible pair $(\widehat{p}, \widehat{q})$, determining the Poincaré translation operator $\widehat{T}_\omega$ defined in (8).*

In view of the above arguments, Theorem 3 can be immediately applied to (5) as follows.

Theorem 4. *The impulsive differential inclusion (5) has, under (6) and (7), a positive topological entropy in the sense of Definition 5, provided*

$$\limsup_{m\to\infty} \sqrt[m]{|L(\widehat{I}^m)|} > 1. \tag{10}$$

Proof. Because of an admissible homotopy $(\widehat{p}, \widehat{q}) \sim \widehat{I}$ (cf. Proposition 5), where $\widehat{p}^{-1}([x])$ is an R_δ-set, for every $[x] \in \mathbb{R}^n/\mathbb{Z}^n$ (cf. Lemma 8), we have (in view of (3))

$$h(\widehat{p}, \widehat{q}) \geq \log N^\infty(\widehat{p}, \widehat{q}) = \log |L^\infty(\widehat{I})| := \log \max \left\{ 1, \limsup_{m\to\infty} \sqrt[m]{|L(\widehat{I}^m)|} \right\}. \tag{11}$$

Thus, $h(\widehat{p}, \widehat{q}) > 0$ holds, when $\limsup_{m\to\infty} \sqrt[m]{|L(\widehat{I}^m)|} > 1$, as claimed. □

Example 2. (continued Example 1) *Consider (5), under (6) and (7). Let the impulsive mapping $\widehat{I} := \tau \circ I \colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be an endomorphism defined by the integer $(n \times n)$-matrix A like in Example 1. Since $(\widehat{p}, \widehat{q}) \sim \widehat{I}$, condition (10) can be improved, by means of (4) and (11), into the form $\sum_{|\lambda_k|>1} \log |\lambda_k| > 0$, provided $\lambda_k \neq 1$, for all $k = 1, \ldots, n$. It happens, when $\lambda_k \neq 1$, for all $k = 1, \ldots, n$, and at least one of the eigenvalues λ_k satisfies $|\lambda_k| > 1$. In particular, for $n = 1$, $h(\widehat{p}, \widehat{q}) \geq \log |d| = \log |I(1) - I(0)| > 0$, where d stands for the topological degree of $\widehat{I}$, when $|I(1) - I(0)| > 1$.*

6. Concluding Remarks

If

$$N^{\infty}(p,q) := \limsup_{m\to\infty} \sqrt[m]{N((p,q)^m)} > 1$$

and

$$|L^{\infty}(p,q)| := \limsup_{m\to\infty} \sqrt[m]{|L((p,q)^m)|} > 1$$

hold for the asymptotic Nielsen and Lefschetz numbers $N^{\infty}(p,q)$ and $L^{\infty}(p,q)$, then the sequences $\{N((p,q)^m)\}$ and $\{|L((p,q)^m)|\}$ of the Nielsen and Lefschetz numbers grow exponentially. Thus, there exists $m^* \in \mathbb{N}$ such that, for infinitely many $m \geq m^*$, the sequences $\{N((p,q)^m)\}$ and $\{|L((p,q)^m)|\}$ are strictly (exponentially) increasing.

Therefore, according to ([23] Theorem 5.3), the admissible map $\mathbb{T}^n \xLeftarrow{p} \Gamma \xrightarrow{q} \mathbb{T}^n$ such that $p^{-1}([x])$ is an R_δ-set, for every $[x] \in \mathbb{T}^n$, then admits an irreducible m-orbit of coincidences, for infinitely many $m \geq m^*$.

In view of one-to-one correspondence, described in Lemma 8, we can give the last theorem as follows.

Theorem 5. *There exists a positive integer m^* such that the impulsive differential inclusion (5) admits, under (6) and (7), an $m\omega$-periodic $(\mathrm{mod}\ 1)$ solution, for infinitely many $m \geq m^*$, provided (10) holds for the Lefschetz numbers $L(\widehat{I}^m)$ of the m-th iterates of $\widehat{I} := \tau \circ I$.*

Proof. The proof is quite analogous to the one in ([24] Theorem 4.3). □

Remark 4. *For $n = 1$, the impulsive differential inclusion (5) admits, under (6) and (7), an $m\omega$-periodic $(\mathrm{mod}\ 1)$ solution, for every $m \in \mathbb{N}$, provided $|I(1) - I(0)| > 1$ or $|I(1) - I(0)| < -2$. If $|I(1) - I(0)| = -2$, then an $m\omega$-periodic $(\mathrm{mod}\ 1)$ solution of (5) exists, for every $m \in \mathbb{N} \setminus \{2\}$. For more details, see ([24] Theorem 4.6).*

Remark 5. *For $n = 2, 3$, the more explicit conditions than (10) for the minimal sets of periods can be found in [29–31].*

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