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Set-Valued Symmetric Generalized Strong Vector Quasi-Equilibrium Problems with Variable Ordering Structures

Jing-Nan Li [1], San-Hua Wang [1,2,*] and Yu-Ping Xu [1]

1 Department of Mathematics, Nanchang University, Nanchang 330031, China; lucky.xm06@gmail.com (J.-N.L.); xyp1981192@gmail.com (Y.-P.X.)
2 Post-Doctor Station of Management Science and Engineering, Nanchang University, Nanchang 330031, China
* Correspondence: wangsanhua@ncu.edu.cn or wsh_315@163.com

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Abstract: In this paper, two types of set-valued symmetric generalized strong vector quasi-equilibrium problems with variable ordering structures are discussed. By using the concept of cosmically upper continuity rather than the one of upper semicontinuity for cone-valued mapping, some existence theorems of solutions are established under suitable assumptions of cone-continuity and cone-convexity for the equilibrium mappings. Moreover, the results of compactness for solution sets are proven. As applications, some existence results of strong saddle points are obtained. The main results obtained in this paper unify and improve some recent works in the literature.

Keywords: symmetric generalized strong vector quasi-equilibrium problems; variable ordering structures; existence of solutions; cosmically upper continuity; fixed point theorem

MSC: 65K15; 47J25; 90C33

1. Introduction

Throughout this paper, without special statements, we always assume that D and K are two nonempty closed convex subsets of locally convex Hausdorff topological vector spaces X and Y, respectively. Let Z_1 and Z_2 be two real normed vector spaces. Let $C : D \to 2^{Z_1}$ and $P : K \to 2^{Z_2}$ be two set-valued mappings such that, for any $x \in D$, $y \in K$, $C(x)$ and $P(y)$ are two closed, convex, and pointed cones of Z_1 and Z_2, respectively. Given set-valued mappings $S : D \times K \to 2^D$, $T : D \times K \to 2^K$, $F : D \times K \times D \to 2^{Z_1}$ and $G : D \times K \times K \to 2^{Z_2}$. As we all know, the inclusion relationship cannot be changed from one side to the other, so we consider in this paper the following two types of set-valued symmetric generalized strong vector quasi-equilibrium problems with variable ordering structures (VMSVEP). The first type is to find $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and:

$$(VMSVEP\,1) \qquad \begin{cases} F(\bar{x},\bar{y},\bar{x}) \subseteq F(\bar{x},\bar{y},x) + C(\bar{x}), & \forall x \in S(\bar{x},\bar{y}), \\ G(\bar{x},\bar{y},\bar{y}) \subseteq G(\bar{x},\bar{y},y) + P(\bar{y}), & \forall y \in T(\bar{x},\bar{y}). \end{cases}$$

The second one is to find $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and:

$$(VMSVEP\,2) \qquad \begin{cases} F(\bar{x},\bar{y},x) \subseteq F(\bar{x},\bar{y},\bar{x}) + C(\bar{x}), & \forall x \in S(\bar{x},\bar{y}), \\ G(\bar{x},\bar{y},y) \subseteq G(\bar{x},\bar{y},\bar{y}) + P(\bar{y}), & \forall y \in T(\bar{x},\bar{y}). \end{cases}$$

Special cases:

(1) If F and G are single-valued mappings, then (VMSVEP 1)reduces to the following symmetric generalized strong vector quasi-equilibrium problem with variable ordering structures (VSVEP): find $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and:

$$(VSVEP) \qquad \begin{cases} F(\bar{x},\bar{y},\bar{x}) \in F(\bar{x},\bar{y},x) + C(\bar{x}), & \forall x \in S(\bar{x},\bar{y}), \\ G(\bar{x},\bar{y},\bar{y}) \in G(\bar{x},\bar{y},y) + P(\bar{y}), & \forall y \in T(\bar{x},\bar{y}), \end{cases}$$

which has been studied by Huang, Wang, and Zhang [1]. Further, if $Z_1 = Z_2 = Z$ and $C(x) \equiv -\mathcal{C} \equiv P(y)$, $f(u,y) = F(x,y,u)$, $g(x,v) = G(x,y,v)$ for all $x,u \in D, y,v \in K$, where Z is a real normed linear space, $\mathcal{C} \subseteq Z$ is a closed, convex, and pointed cone, then (VSVEP) reduces to the symmetric strong vector quasi-equilibrium problem (SVEP): find $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and

$$(SVEP) \qquad \begin{cases} f(x,\bar{y}) \in f(\bar{x},\bar{y}) + \mathcal{C}, & \forall x \in S(\bar{x},\bar{y}), \\ g(\bar{x},y) \in g(\bar{x},\bar{y}) + \mathcal{C}, & \forall y \in T(\bar{x},\bar{y}), \end{cases}$$

which has been studied by Gong [2] and Han and Gong [3].

(2) If, for any $x \in D$ and $y \in K$, $C(x) \equiv \mathcal{C}$ and $P(y) \equiv \mathcal{P}$, where $\mathcal{C}$ and $\mathcal{P}$ are closed, convex, and pointed cones of Z_1 and Z_2, respectively, then (VMSVEP 2) reduces to the set-valued symmetric generalized strong vector quasi-equilibrium problem (MSVEP): find $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and:

$$(MSVEP) \qquad \begin{cases} F(\bar{x},\bar{y},x) \subseteq F(\bar{x},\bar{y},\bar{x}) + \mathcal{C}, & \forall x \in S(\bar{x},\bar{y}), \\ G(\bar{x},\bar{y},y) \subseteq G(\bar{x},\bar{y},\bar{y}) + \mathcal{P}, & \forall y \in T(\bar{x},\bar{y}). \end{cases}$$

If, in addition, $Z_1 = Z_2 = Z$, $\mathcal{P} = \mathcal{C}$ and for any $x \in D, y \in K$, $F(x,y,x) \subseteq \mathcal{C}$ and $G(x,y,y) \subseteq \mathcal{P}$, where Z is a real normed linear space, $\mathcal{C}$ is a closed, convex, and pointed cone of Z, then (MSVEP) further collapses to the set-valued symmetric strong vector quasi-equilibrium problem (MVEP): find $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and:

$$(MVEP) \qquad \begin{cases} F(\bar{x},\bar{y},x) \subseteq \mathcal{C}, & \forall x \in S(\bar{x},\bar{y}), \\ G(\bar{x},\bar{y},y) \subseteq \mathcal{C}, & \forall y \in T(\bar{x},\bar{y}), \end{cases}$$

which has been studied by Chen, Huang, and Wen [4].

(3) If, for all $x \in D$, $y,v \in K$, $G(x,y,v) = \{0\}$, $C(x) \equiv \mathcal{C}$, $P(y) \equiv \mathcal{P}$, where $\mathcal{C}$ and $\mathcal{P}$ are two closed, convex, and pointed cone of Z_1 and Z_2, respectively, then (VMSVEP 2) reduces to the following quasi-variational inclusion problem (QVIP): find $(\bar{x},\bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and

$$(QVIP) \qquad F(\bar{x},\bar{y},x) \subseteq F(\bar{x},\bar{y},\bar{x}) + \mathcal{C}, \quad \forall x \in S(\bar{x},\bar{y}),$$

which has been studied by Lin and Tan [5].

Vector equilibrium problem, also called generalized Ky Fan inequality, was proposed and studied by Ansari [6], Bianchi, Hadjisavvas, and Schaible [7], Oettli [8], and other authors. It provides a unified framework for many important problems, such as the optimization problem, the Nash equilibrium problem, the fixed point problem, the complementary problem, the variational inequality, and so on. Today, research on the vector equilibrium problem has achieved fruitful results in many aspects, including optimal conditions, the properties of the solution set, duality theory, well-posedness, stability, and algorithms. Among them, studies on optimal conditions are the most common. Research on duality theory, well-posedness, and stability has also made important progress, while studies on algorithms are relatively less. For details, we refer the readers to [9–18] and the references therein.

On the other hand, time is considered in many situations for multi-objective decision problems. This kind of multi-objective decision optimization problem involving the time factor is also called the vector optimization problem with variable domination structures. It was firstly proposed and studied by Yu [19] in 1974. From then on, it attracted the attention of many people. Today, it has been generalized to and developed into various more general optimization problems with variable domination structures, and a great good deal of practical and theoretical results have been obtained in the literature. Usually, the assumption of upper semi-continuity, which is used commonly for a general set-valued mapping, is imposed on variable ordering structures. It is well known that variable ordering structures can be characterized as a cone-valued mapping. However, just as pointed out by Borde and Crouzeix [20], this assumption is actually not suitable for cone-valued mapping as its special features. In order to deal with cone-valued mapping, Luc [21] introduced the concept of cosmically upper continuity and discussed its properties. Later, Eichfelder and other authors further investigated the properties of cone-valued mapping and applied them to deal with the vector optimization problem (see, for example, [22–24]). Very recently, some authors attempted to employ the concept of cosmically upper continuity to discuss vector variational inequalities and vector equilibrium problems with variable ordering structures (see, for example, [1,17,25]).

Motivated by the works mentioned above, in this paper, we shall use the concept of cosmically upper continuity to discuss (VMSVEP). By applying the famous Fan-Knaster-Kuratowski-Mazurkiewicz (Fan-KKM) theorem and the Kakutani–Fan–Glicksberg fixed point theorem, some existence results of solutions to (VMSVEP) are proposed under suitable assumptions of cone-continuity and cone-convexity for the equilibrium mappings. Further, the conclusions about the compactness of solution sets are also obtained in this paper. As applications, some existence theorems of strong saddle points are proven. Our main results unify and improve some recent works in the literature.

The rest of this paper is organized as follows. In Section 2, we shall give some preliminary concepts and lemmas used in the sequel. In Section 3, the existence of solutions and the compactness of solution sets for (VMSVEP) are discussed. As applications, some existence results of strong saddle points are proven in Section 4.

2. Preliminaries

In this section, we shall give some concepts and known results, which will be used in the sequel.

Let A be a subset, and denote by $conv(A)$, $cl(A)$, and $cone(A)$ the convex, closure, and cone hull of A, respectively. For any real number $r > 0$, we denote by $B_r = \{x \in Z : \|x\| < r\}$ the open ball in a real normed space Z.

Definition 1 ([26])**.** *Let X and Y be two topological spaces. A set-valued mapping $H : X \to 2^Y$ is said to be:*

(i) *upper semi-continuous (u.s.c.) at $x_0 \in X$ if, for each open set $V \subseteq Y$ with $H(x_0) \subseteq V$, there exists a neighborhood U of x_0 such that $H(x) \subseteq V$ for all $x \in U$;*
(ii) *lower semi-continuous (l.s.c.) at $x_0 \in X$ if, for each open set $V \subseteq Y$ with $H(x_0) \cap V \neq \varnothing$, there exists a neighborhood U of x_0 such that $H(x) \cap V \neq \varnothing$ for all $x \in U$;*
(iii) *u.s.c. (l.s.c.) on X if it is u.s.c. (l.s.c.) at every point $x \in X$;*
(iv) *continuous on X if it is both u.s.c. and l.s.c. on X;*
(v) *closed if its graph $Gr(H) := \{(x, y) : x \in X, y \in H(x)\}$ is a closed subset of $X \times Y$.*

Lemma 1 ([26])**.** *Let X and Y be two topological spaces and $H : X \to 2^Y$ be a set-valued mapping.*

(i) *If Y is Hausdorff and H is u.s.c. and compact-valued, then H is closed;*
(ii) *If H is compact-valued, then H is u.s.c. at $x_0 \in X$ if and only if, for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x_0$ and for any net $\{y_\alpha\}$ with $y_\alpha \in H(x_\alpha)$, there exists $y_0 \in H(x_0)$ and a subset $\{y_\beta\} \subseteq \{y_\alpha\}$, such that $y_\beta \to y_0$;*
(iii) *H is l.s.c. at $x_0 \in X$ if and only if, for each $y_0 \in H(x_0)$ and for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x_0$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in H(x_\alpha)$ and $y_\alpha \to y_0$;*
(iv) *If X and Y both are Hausdorff, Y is compact, and the mapping H has nonempty closed values, then H is u.s.c. on X if and only if it is closed.*

Definition 2. *Let X, Y, Z be Hausdorff topological spaces and E be a real Hausdorff topological vector space. Let $C : X \to 2^E$ be a cone-valued mapping. A set-valued mapping $H : X \times Y \times Z \to 2^E$ is said to be:*

(i) *upper (lower) C-continuous at $(x, y, z) \in X \times Y \times Z$ if, for any neighborhood V of the origin in E, there exist neighborhoods U_x, U_y, and U_z of x, y, and z, respectively, such that:*

$$H(\tilde{x}, \tilde{y}, \tilde{z}) \subseteq H(x, y, z) + V + C(x), \;\; \forall (\tilde{x}, \tilde{y}, \tilde{z}) \in U_x \times U_y \times U_z$$
$$(H(x, y, z) \subseteq H(\tilde{x}, \tilde{y}, \tilde{z}) + V - C(x), \;\; \forall (\tilde{x}, \tilde{y}, \tilde{z}) \in U_x \times U_y \times U_z);$$

(ii) *upper (lower) C-continuous on $X \times Y \times Z$ if it is upper (lower) C-continuous at every point $(x, y, z) \in X \times Y \times Z$.*

Remark 1. *When H is single-valued, the above two concepts of upper C-continuity and lower C-continuity coincide with each other (see, for example, Wang, Huang and O'Regan [27]). For simplification, we would call them both C-continuity rather than upper or lower C-continuity in this situation.*

Remark 2. *(i) The above concepts of upper and lower C-continuity for the function of three variables are generalization of the corresponding ones of Mao, Wang, and Huang [25] for the function of one variable; (ii) If, for any $(x, y, z) \in X \times Y \times Z$, $H(x, y, z) = H(x)$ and $C(x) \equiv \mathcal{C}$ ($\mathcal{C}$ is a fixed cone of E), then the above concepts of upper and lower C-continuity collapse to the ones of the upper and lower $\mathcal{C}$-continuity of Luc [28], respectively. In this situation, we can conclude easily that upper semi-continuity implies upper $\mathcal{C}$-continuity.*

As pointed out in the Introduction, the concept of upper semi-continuity used extensively for general set-valued mappings is actually not suitable for cone-valued mappings. To deal with semicontinuity notions also for cone-valued mappings, we introduce the following upper/lower semicontinuity, which can overcome its imperfection originated essentially from its infinite extension.

Definition 3 ([21])**.** *Let X be a topological space, Y be a real normed vector space, and $cl(B_1)$ be the closed unit ball in Y. A cone-valued mapping $C : X \to 2^Y$ is called:*

(i) *cosmically upper continuous (c.u.c.) at $x_0 \in X$ if the set-valued mapping $x \to C(x) \cap cl(B_1)$ is u.s.c. at x_0;*
(ii) *cosmically lower continuous (c.l.c.) at $x_0 \in X$ if the set-valued mapping $x \to C(x) \cap cl(B_1)$ is l.s.c. at x_0;*
(iii) *c.u.c. (c.l.c.) on X if it is c.u.c. (c.l.c.) at every point $x \in X$.*

Remark 3. *It is easy to see that if a cone-valued mapping C is c.u.c. (c.l.c.) at $x_0 \in X$, then the cone-valued mapping $-C$ is also c.u.c. (c.l.c.) at $x_0 \in X$; and vice versa.*

Lemma 2 ([21])**.** *Let X be a topological space, Y be a real normed vector space, and $cl(B_1)$ be the closed unit ball in Y. Let $C : X \to 2^Y$ be a cone-valued mapping.*

(i) *C is c.u.c. at $x_0 \in X$ if and only if, for every bounded closed set $B \subseteq Y$, the set-valued mapping $x \to C(x) \cap B$ is u.s.c. at x_0;*
(ii) *C is c.l.c. at $x_0 \in X$ if and only if it is l.s.c. at x_0.*

Next, we give a nontrivial example, which is *c.u.c.*, but not *u.s.c.*.

Example 1. *Let $X = [0, +\infty) \subseteq \mathbb{R}$. Define a cone-valued mapping $C : X \to 2^{\mathbb{R}^2}$ as follows:*

$$C(x) = cone\ conv(\{(t, t^2) : 0 \le t \le x\} \cup \{(1, 0)\}), \;\; \forall x \in X.$$

By calculation, we can obtain:

$$C(x) = \{(u_1, u_2) \in \mathbb{R}^2 : 0 \le u_1,\ 0 \le u_2 \le xu_1\},\ \ \forall x \in X.$$

From this, we can conclude that C is c.u.c. on X.

Let $x_0 = 1$, then:

$$C(x_0) = \{(u_1, u_2) \in \mathbb{R}^2 : 0 \le u_1,\ 0 \le u_2 \le u_1\}.$$

If we take:

$$V = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 > -1,\ -1 < u_2 < u_1 + e^{-u_1}\},$$

then V is an open set and contains $C(x_0)$ as its true subset, i.e., V is an open neighborhood of $C(x_0)$.

For any $x > x_0 = 1$ and letting it be fixed, we take $u = (x, x^2)$. By the definition, we have $u \in C(x)$. Next, we shall show $u \notin V$. In fact, we can define a function $h : (\frac{1}{2}, +\infty) \to \mathbb{R}$ as follows:

$$h(v) = v^2 - (v + e^{-v}),\ \ \forall v \in (\frac{1}{2}, +\infty).$$

Then, for each $v \in (\frac{1}{2}, +\infty)$,

$$h'(v) = 2v - 1 + e^{-v} > 2v - 1 > 0.$$

This implies that the function h is strictly monotone increasing on the interval $(\frac{1}{2}, +\infty)$. Then, by noting that $x > 1$, we have:

$$h(x) > h(1) = e^{-1} > 0.$$

It follows that $x^2 - (x + e^{-x}) > 0$, i.e., $x^2 > x + e^{-x}$. Hence, $u = (x, x^2) \notin V$. From this, we can know that C is not u.s.c. at $x_0 = 1$, and so, it is not u.s.c. on X.

More examples for a cone-valued mapping that is *c.u.c.*, we refer the reader to [29], Section 3.4.1.

The following concepts of generalized convexity for set-valued mappings are needed in this paper.

Definition 4 ([30])**.** *Let X be a nonempty convex subset of a vector space E and $\mathcal{C}$ be a convex cone of a Hausdorff topological vector space Z. A set-valued mapping $H : X \to 2^Z$ is said to be:*

(i) *upper properly $\mathcal{C}$-quasiconvex if, for every $u_1, u_2 \in X$ and $t \in [0, 1]$, one has:*

either $\quad H(u_1) \subseteq H(tu_1 + (1-t)u_2) + \mathcal{C};$

or $\quad H(u_2) \subseteq H(tu_1 + (1-t)u_2) + \mathcal{C}.$

(ii) *lower properly $\mathcal{C}$-quasiconvex if, for every $u_1, u_2 \in X$ and $t \in [0, 1]$, one has:*

either $\quad H(tu_1 + (1-t)u_2) \subseteq H(u_1) - \mathcal{C};$

or $\quad H(tu_1 + (1-t)u_2) \subseteq H(u_2) - \mathcal{C}.$

Remark 4. *When F reduces to the single-valued mapping, the above concepts of upper and lower properly $\mathcal{C}$-quasiconvexity both coincide with the one of $\mathcal{C}$-quasiconvexity in [2,28]. For simplification, we would call them both properly $\mathcal{C}$-quasiconvexity rather than upper or lower properly $\mathcal{C}$-quasiconvexity in this situation.*

Definition 5 ([31])**.** *Let Z be a real normed vector space and $P \subseteq Z$ be a closed, convex, and pointed cone. A nonempty convex subset $B \subseteq P$ is said to be a base of P if $0 \notin cl(B)$ and $P = cone(B)$.*

Lemma 3 ([31])**.** *Let Z be a real normed vector space and $P \subseteq Z$ be a closed, convex, and pointed cone. Let $P^\wedge = \{f \in Z^* : \exists a > 0, f(x) \ge a\|x\|, \forall x \in P\}$. If P has a bounded closed base, then $P^\wedge \neq \varnothing$.*

The following lemma gives a local property for cosmically upper continuity, which is very useful for dealing with the cone-valued mapping.

Lemma 4 ([25]). *Let E be a Hausdorff topological vector space and X be a nonempty closed convex subset of E. Let Z be a real normed vector space. Let $C : X \to 2^Z$ be a cone-valued mapping such that, for each $x \in X$, $C(x) \subseteq Z$ is a closed, convex, and pointed cone. For any given $x_0 \in X$, if $C(x_0)$ has a bounded closed base and the mapping C is c.u.c. at x_0, then, for any $0 < \varepsilon < l$, there exists a neighborhood U of x_0 such that:*

$$[C(x) + C(x_0)] \cap B_l \subseteq C(x_0) + B_\varepsilon, \ \ \forall x \in U \cap X.$$

Definition 6 ([32]). *Let X be a nonempty subset of a vector space E. A set-valued mapping $G : X \to 2^E$ is called a Knaster-Kuratowski-Mazurkiewicz (KKM)-mapping if, for any finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$, one has:*

$$conv\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i).$$

With the help of the KKM-mapping, we can give the famous nonempty intersection theorem, the Fan-KKM theorem, which will be used in the proof of our main results.

Theorem 1 ([32], Fan-KKM theorem). *Let X be a nonempty subset of a Hausdorff topological vector space E. Assume that $G : X \to 2^E$ is a KKM-mapping such that, for each $x \in X$, $G(x)$ is closed, and at least one is compact. Then, $\bigcap_{x \in X} G(x) \neq \emptyset$.*

The following theorem is a very important tool for establishing our main results.

Theorem 2 ([33], Kakutani–Fan–Glicksberg fixed point theorem). *Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. If $T : X \to 2^X$ is u.s.c. with nonempty closed convex values, then T has at least one fixed point in X.*

3. Main Results

In this section, we will use the cosmically upper continuity condition for the cone-valued mapping to establish some existence results of the solutions and compactness conclusions of the solution sets for (VMSVEP).

Theorem 3. *Let D and K be two nonempty compact convex subsets of X and Y, respectively. Assume that:*

(i) C, P are c.u.c., and for each $x \in D$, $y \in K$, $C(x)$ and $P(y)$ both have bounded closed bases;
(ii) S, T are continuous with nonempty closed and convex values;
(iii) F is lower $-C$-continuous and upper C-continuous on $D \times K \times D$; for any $(x, y, u) \in D \times K \times D$, $F(x, y, u)$ is bounded and $F(x, y, u) + C(x)$ is closed; for each $(x, y) \in D \times K$, $F(x, y, u)$ is lower properly $-C(x)$-quasiconvex in $u \in D$;
(iv) G is lower $-P$-continuous and upper P-continuous on $D \times K \times K$; for any $(x, y, z) \in D \times K \times K$, $G(x, y, z)$ is bounded and $G(x, y, z) + P(y)$ is closed; for each $(x, y) \in D \times K$, $G(x, y, z)$ is lower properly $-P(y)$-quasiconvex in $z \in K$.

Then, (VMSVEP 1) is solvable. Moreover, the solution set is compact in $D \times K$.

Proof. Define two set-valued mappings $A : D \times K \to 2^D$ and $B : D \times K \to 2^K$ as follows: for each $(x, y) \in D \times K$, let:

$$A(x, y) = \{v \in S(x, y) : F(x, y, v) \subseteq F(x, y, u) + C(x), \ \forall u \in S(x, y)\};$$

and:

$$B(x, y) = \{q \in T(x, y) : G(x, y, q) \subseteq G(x, y, p) + P(y), \ \forall p \in T(x, y)\}.$$

The proof is divided into four steps:

(I) $\forall (x, y) \in D \times K$, $A(x, y)$ and $B(x, y)$ are nonempty.

Indeed, take any $(x, y) \in D \times K$, and let it be fixed. For each $u \in S(x, y)$, let:

$$H(u) = \{v \in S(x, y) : F(x, y, v) \subseteq F(x, y, u) + C(x)\}.$$

We can prove that $\bigcap_{u \in S(x,y)} H(u) \neq \varnothing$. It follows $A(x, y) = \bigcap_{u \in S(x,y)} H(u) \neq \varnothing$.

We shall apply the Fan-KKM theorem to show $\bigcap_{u \in S(x,y)} H(u) \neq \varnothing$.

Firstly, for each $u \in S(x, y)$, $H(u)$ is nonempty as $u \in H(u)$. Further, $H(u)$ is a closed subset of X. In fact, let a net $\{v_\alpha\} \subseteq H(u)$ be such that $v_\alpha \to v_0$. Then, by the facts $\{v_\alpha\} \subseteq S(x, y)$ and $S(x, y)$ is closed, we can get $v_0 \in S(x, y) \subseteq D$. As $\{v_\alpha\} \subseteq H(u)$, we also have:

$$F(x, y, v_\alpha) \subseteq F(x, y, u) + C(x), \ \ \forall \alpha.$$

We claim that:

$$F(x, y, v_0) \subseteq F(x, y, u) + C(x). \tag{1}$$

Indeed, if it is false, then there exists some $\omega \in F(x, y, v_0)$ such that:

$$\omega \notin F(x, y, u) + C(x).$$

Since $F(x, y, u) + C(x)$ is closed, there is an open balanced neighborhood U of the origin in Z_1 such that:

$$(\omega + U) \cap (F(x, y, u) + C(x)) = \varnothing. \tag{2}$$

For the above open neighborhood U, since F is lower $-C$-continuous at (x, y, v_0), there must exist a neighborhood V of v_0 such that:

$$F(x, y, v_0) \subseteq F(x, y, v) + U + C(x), \ \ \forall v \in V \cap D.$$

As $v_\alpha \to v_0$, there exists some α_0 such that $v_\alpha \in V \cap D$ for all $\alpha \geq \alpha_0$. It follows that:

$$F(x, y, v_0) \subseteq F(x, y, v_\alpha) + U + C(x), \ \ \forall \alpha \geq \alpha_0.$$

Since $C(x)$ is a convex cone, we have, for any $\alpha \geq \alpha_0$,

$$\omega \in F(x, y, v_0) \subseteq F(x, y, v_\alpha) + U + C(x) \subseteq F(x, y, u) + U + C(x).$$

Noting that U is balanced, we obtain:

$$(\omega + U) \cap (F(x, y, u) + C(x)) \neq \varnothing,$$

which contradicts (2). Thus, (1) holds, and so, $v_0 \in H(u)$. This indicates that $H(u)$ is closed. Further, $H(u)$ is compact as it contains in the compact subset D.

Next, we show that $H : S(x, y) \to 2^{S(x,y)}$ is a KKM mapping.

Indeed, suppose that it is not the case. Then, there exists a finite subset $\{u_1, u_2, \ldots, u_n\}$ of $S(x, y)$ and $\lambda_i \geq 0 (i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n \lambda_i = 1$ such that $u_0 = \sum_{i=1}^n \lambda_i u_i \notin \bigcup_{i=1}^n H(u_i)$. Since $S(x, y)$ is convex, we have $u_0 \in S(x, y)$. It follows that:

$$F(x, y, u_0) \nsubseteq F(x, y, u_i) + C(x), \ \ i = 1, 2, \ldots, n,$$

which contradicts the lower properly $-C(x)$-quasiconvexity of $F(x, y, \cdot)$. Therefore, H is a KKM mapping.

By applying the Fan-KKM theorem, we have $\bigcap_{u \in S(x,y)} H(u) \neq \varnothing$.

Similarly, we can show that $B(x, y)$ is nonempty.

(II) $\forall(x,y) \in D \times K$, $A(x,y)$ and $B(x,y)$ are both closed and convex.

Indeed, for any given $(x,y) \in D \times K$, by (I), we know that the set $A(x,y) = \bigcap_{u \in S(x,y)} H(u)$ is closed as it is the intersection of a family of closed sets.

Next, we prove that the set $A(x,y)$ is convex.

In fact, $\forall v_1, v_2 \in A(x,y), \forall t \in [0,1]$, let $v_t = tv_1 + (1-t)v_2$, then $v_1, v_2 \in S(x,y)$, and for each $u \in S(x,y)$,

$$F(x,y,v_i) \subseteq F(x,y,u) + C(x), \quad i = 1,2.$$

As $S(x,y)$ is convex, we have $v_t \in S(x,y)$. In addition, by the condition of the lower properly $-C(x)$-quasiconvexity of $F(x,y,\cdot)$, there exists some $i_0 = \{1,2\}$, such that:

$$F(x,y,v_t) \subseteq F(x,y,v_{i_0}) + C(x).$$

It follows that:

$$F(x,y,v_t) \subseteq F(x,y,u) + C(x) + C(x) = F(x,y,u) + C(x).$$

This indicates that $v_t \in A(x,y)$. Therefore, $A(x,y)$ is convex.

Similarly, we can show that $B(x,y)$ is closed and convex.

(III) The mappings A and B are both *u.s.c.*.

Notice that D is compact. Then, by (I), (II), and Lemma 1 (iv), to prove A is *u.s.c.*, we need only to show that A is closed, i.e., the graph $Gr(A)$ of A is closed in $D \times K \times D$.

Indeed, let $\{(x_\alpha, y_\alpha, v_\alpha)\} \subseteq Gr(A)$ be an arbitrary net such that $(x_\alpha, y_\alpha, v_\alpha) \to (x_0, y_0, v_0) \in D \times K \times D$, then for each α, we have $v_\alpha \in S(x_\alpha, y_\alpha)$ and:

$$F(x_\alpha, y_\alpha, v_\alpha) \subseteq F(x_\alpha, y_\alpha, u) + C(x_\alpha), \quad \forall u \in S(x_\alpha, y_\alpha).$$

Notice that $S : D \times K \to 2^D$ is *u.s.c.* with nonempty compact values. Then, by Lemma 1 (i), we know that S is closed, and so, $v_0 \in S(x_0, y_0)$. For any given $u_0 \in S(x_0, y_0)$, by the lower semi-continuity of S and Lemma 1 (iii), there exists a net $\{u_\alpha\}$ such that $u_\alpha \in S(x_\alpha, y_\alpha)$ and $u_\alpha \to u_0$. It follows that:

$$F(x_\alpha, y_\alpha, v_\alpha) \subseteq F(x_\alpha, y_\alpha, u_\alpha) + C(x_\alpha), \quad \forall \alpha.$$

We claim that:

$$F(x_0, y_0, v_0) \subseteq F(x_0, y_0, u_0) + C(x_0),$$

which implies that $Gr(A)$ is closed in $D \times K \times D$.

Indeed, suppose to the contrary that there exists some $\omega_0 \in F(x_0, y_0, v_0)$ such that:

$$\omega_0 \notin F(x_0, y_0, u_0) + C(x_0).$$

Notice that $F(x_0, y_0, u_0) + C(x_0)$ is closed. There must exist some $\varepsilon > 0$, such that:

$$[\omega_0 + B_\varepsilon] \cap [F(x_0, y_0, u_0) + B_\varepsilon + C(x_0)] = \varnothing. \tag{3}$$

Since F is lower $-C$-continuous at (x_0, y_0, v_0) and upper C-continuous at (x_0, y_0, u_0), there exists some $\alpha_1 > 0$ such that, for any $\alpha > \alpha_1$,

$$\begin{aligned} \omega_0 \in F(x_0, y_0, v_0) &\subseteq F(x_\alpha, y_\alpha, v_\alpha) + (1/2)B_\varepsilon + C(x_0) \\ &\subseteq F(x_\alpha, y_\alpha, u_\alpha) + C(x_\alpha) + (1/2)B_\varepsilon + C(x_0) \\ &\subseteq F(x_0, y_0, u_0) + (1/2)B_\varepsilon + C(x_0) + C(x_\alpha) + (1/2)B_\varepsilon + C(x_0) \\ &= F(x_0, y_0, u_0) + B_\varepsilon + C(x_0) + C(x_\alpha). \end{aligned} \tag{4}$$

Observing that B_ε is balanced, we have B_ε=$-B_\varepsilon$. It follows that:

$$[\omega_0 - F(x_0, y_0, u_0) + B_\varepsilon] \cap [C(x_\alpha) + C(x_0)] \neq \emptyset, \ \forall \alpha > \alpha_1. \tag{5}$$

As $F(x_0, y_0, u_0)$ is bounded, we can take a positive number $l > \varepsilon$ such that $\omega_0 - F(x_0, y_0, u_0) + B_\varepsilon \subseteq B_l$. On the other hand, since C is *c.u.c.* at x_0 and $C(x_0)$ has a bounded closed base, we can conclude from Lemma 4 that there exists some α_2 such that:

$$[C(x_\alpha) + C(x_0)] \cap B_l \subseteq C(x_0) + B_\varepsilon, \ \forall \alpha > \alpha_2. \tag{6}$$

Take any $\overline{\alpha}$ such that $\overline{\alpha} > \alpha_1$ and $\overline{\alpha} > \alpha_2$. Then, by (5) and the fact $\omega_0 - F(x_0, y_0, u_0) + B_\varepsilon \subseteq B_l$, we have, for any $\alpha > \overline{\alpha}$,

$$[\omega_0 - F(x_0, y_0, u_0) + B_\varepsilon] \cap [C(x_\alpha) + C(x_0)] \cap B_l \neq \emptyset.$$

This, together with (6), yields:

$$[\omega_0 - F(x_0, y_0, u_0) + B_\varepsilon] \cap [C(x_0) + B_\varepsilon] \neq \emptyset.$$

It follows that:

$$[\omega_0 + B_\varepsilon] \cap [F(x_0, y_0, u_0) + B_\varepsilon + C(x_0)] \neq \emptyset, \tag{7}$$

which contradicts (3). Therefore, $Gr(A)$ is closed.

By similar arguments, we can show that B is *u.s.c.*.

Define $M : D \times K \to 2^{D \times K}$ by:

$$M(x, y) = (A \times B)(x, y) = (A(x, y), B(x, y)), \ \forall (x, y) \in D \times K.$$

Then, M is *u.s.c.* with nonempty, closed, and convex values. According to the Kakutani–Fan–Glicksberg fixed point theorem, there exists some $(\bar{x}, \bar{y}) \in D \times K$ such that $(\bar{x}, \bar{y}) \in M(\bar{x}, \bar{y})$. It follows that $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$ and:

$$\begin{cases} F(\bar{x}, \bar{y}, \bar{x}) \subseteq F(\bar{x}, \bar{y}, x) + C(\bar{x}), & \forall x \in S(\bar{x}, \bar{y}), \\ G(\bar{x}, \bar{y}, \bar{y}) \subseteq G(\bar{x}, \bar{y}, y) + P(\bar{y}), & \forall y \in T(\bar{x}, \bar{y}). \end{cases}$$

This indicates that $(\bar{x}, \bar{y})$ is a desired solution of (VMSVEP 1).

(IV) The solution set $\mathcal{S}$ of (VMSVEP 1) is a compact subset of $D \times K$.

Noting that D and K are both compact subsets, we only show that $\mathcal{S}$ is closed. In fact, let $\{(x_\alpha, y_\alpha)\} \subseteq \mathcal{S}$ be an arbitrary net with $(x_\alpha, y_\alpha) \to (x_0, y_0) \in D \times K$. Then, for each α, we have $x_\alpha \in S(x_\alpha, y_\alpha)$, $y_\alpha \in T(x_\alpha, y_\alpha)$ and:

$$\begin{cases} F(x_\alpha, y_\alpha, x_\alpha) \subseteq F(x_\alpha, y_\alpha, x) + C(x_\alpha), & \forall x \in S(x_\alpha, y_\alpha), \\ G(x_\alpha, y_\alpha, y_\alpha) \subseteq G(x_\alpha, y_\alpha, y) + P(y_\alpha), & \forall y \in T(x_\alpha, y_\alpha). \end{cases}$$

By proceeding with the same arguments as those used in the proof of (III), we can show that $x_0 \in S(x_0, y_0)$, $y_0 \in T(x_0, y_0)$ and:

$$\begin{cases} F(x_0, y_0, x_0) \subseteq F(x_0, y_0, x) + C(x_0), & \forall x \in S(x_0, y_0), \\ G(x_0, y_0, y_0) \subseteq G(x_0, y_0, y) + P(y_0), & \forall y \in T(x_0, y_0). \end{cases}$$

Thus $(x_0, y_0) \in \mathcal{S}$, which implies that $\mathcal{S}$ is closed. $\square$

Remark 5. *(1) In Theorem 3, if, for any $x \in D$ and $y \in K$, $C(x) \equiv \mathcal{C}$ and $P(y) \equiv \mathcal{P}$ ($\mathcal{C}$ and $\mathcal{P}$ are two closed, convex, and pointed cones of Z_1 and Z_2, respectively), then Condition (i) and the bounded assumptions of $F(x,y,u)$ and $G(x,y,z)$ in Conditions (iii) and (iv), respectively, can be dropped. In fact, all these hypotheses were used in the proof of Theorem 3 only to establish (6) and further to obtain the consequence (7). When $C(x) \equiv \mathcal{C}$ and $P(y) \equiv \mathcal{P}$, by modifying (4), we can directly prove (7), but without the help of (6). Indeed, when $C(x) \equiv \mathcal{C}$ and $P(y) \equiv \mathcal{P}$, (4) can be modified as follows:*

$$\begin{aligned}
\omega_0 \in F(x_0, y_0, v_0) &\subseteq F(x_\alpha, y_\alpha, v_\alpha) + (1/2)B_\varepsilon + \mathcal{C} \\
&\subseteq F(x_\alpha, y_\alpha, u_\alpha) + \mathcal{C} + (1/2)B_\varepsilon + \mathcal{C} \\
&\subseteq F(x_0, y_0, u_0) + (1/2)B_\varepsilon + \mathcal{C} + \mathcal{C} + (1/2)B_\varepsilon + \mathcal{C} \\
&= F(x_0, y_0, u_0) + B_\varepsilon + \mathcal{C}.
\end{aligned}$$

It follows that:

$$[\omega_0 + B_\varepsilon] \cap [F(x_0, y_0, u_0) + B_\varepsilon + \mathcal{C}] \neq \varnothing,$$

i.e., (7) holds;

(2) In Theorem 3, if, for any $(x,y,v) \in D \times K \times K$, $G(x,y,v) \equiv \{0\}$, then the requirement of the continuity for T can be weakened as the one of upper semi-continuity. In fact, the key of the proof of Theorem 3 is to prove that the mappings A and B are both u.s.c. with nonempty, closed, and convex values. When $G(x,y,v) \equiv \{0\}$, we can see easily that $B(x,y) = T(x,y)$. Therefore, the fact that B is u.s.c. with nonempty, closed, and convex values can be derived by assuming that T is u.s.c. with nonempty, closed, and convex values.

Similarly, if, for any $(x,y,u) \in D \times K \times D$, $F(x,y,u) \equiv \{0\}$, then the requirement of the continuity for S can be weakened as the one of upper semi-continuity.

Remark 6. *For (VMSVEP 1), by replacing C and P with $-C$ and $-P$, respectively, Fu, Wang, and Li [34] also proved an existence result of Theorem 2.3. However, it is different from Theorem 3 of this paper. The main difference lies in the requirements of the continuity for the cone-valued mappings C and P. In fact, these two cone-valued mappings C and P both impose the assumption of the upper semi-continuity in Theorem 2.3 of Fu, Wang, and Li [34], while they both impose the one of cosmically upper continuity in Theorem 3 of this paper. In general, for a cone-valued mapping, the requirement of cosmically upper continuity is much easier to satisfy than the one of upper semi-continuity, as demonstrated by the example below.*

Example 2. *Let $D = [-2, +2] \subseteq \mathbb{R}$. Define a cone-valued mapping $C : D \to 2^{\mathbb{R}^2}$ as follows:*

$$C(x) = \{(u_1, u_2) : u_1 \geq 0, u_2 \geq xu_1\}, \ \forall x \in D.$$

It is easy to see that C is c.u.c. on D.

Let $x_0 = 1 \in D$, then:

$$C(x_0) = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 \geq 0,\ u_2 \geq u_1\}.$$

If we take:

$$V = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 > -1,\ u_2 > u_1 - e^{-u_1}\},$$

then V is an open set and contains $C(x_0)$ as its true subset, i.e., V is an open neighborhood of $C(x_0)$.

For any $x : 0 < x < x_0 = 1$ and letting it be fixed, we take any positive number u_1 such that $u_1 > \frac{1}{1-x} > 1$. Then, by the definition, we have $u = (u_1, xu_1) \in C(x)$. Next, we shall show $u \notin V$. In fact, since $u_1 > \frac{1}{1-x} > 1$, we have:

$$u_1 + \ln u_1 > \ln u_1 > \ln \frac{1}{1-x} = -\ln(1-x) > 0.$$

It follows that:

$$u_1 > -\ln u_1 - \ln(1-x) = -\ln((1-x)u_1),$$

i.e., $-u_1 < \ln((1-x)u_1)$*. Then, we have* $e^{-u_1} < (1-x)u_1$*. That is* $xu_1 < u_1 - e^{-u_1}$*, which implies that* $u = (u_1, xu_1) \notin V$*. From this, we know that* C *is not u.s.c. at* $x_0 = 1$*.*

Further, by similar arguments, we can prove that C *is not u.s.c. at every point of* $(-2, 2]$*.*

If F and G are both reduced to single-valued mappings, then we have the following conclusion.

Corollary 1. *Let* D *and* K *be as in Theorem 3. Assume that Conditions (i) and (ii) of Theorem 3 and the following conditions hold:*

(iii)′ $f : D \times K \times D \to Z_1$ *is* $-C$*-continuous and* C*-continuous; for any* $(x, y) \in D \times K$*,* $f(x, y, u)$ *is properly* $-C(x)$*-quasiconvex in* $u \in D$*;*

(iv)′ $g : D \times K \times K \to Z_2$ *is* $-P$*-continuous and* P*-continuous; for any* $(x, y) \in D \times K$*,* $g(x, y, z)$ *is properly* $-P(y)$*-quasiconvex in* $z \in K$*.*

Then (VSVEP) is solvable. Moreover, the solution set is compact in $D \times K$*.*

Proof. Replace F and G by f and g, respectively. Then, it is easy to check that all the conditions of Theorem 3 are satisfied. Thus, the conclusions follow immediately from Theorem 3. □

Remark 7. *In Corollary* 3.1*, if, for any* $x \in D$ *and* $y \in K$*,* $C(x) \equiv \mathcal{C}$ *and* $P(y) \equiv \mathcal{P}$ *(*$\mathcal{C}$ *and* $\mathcal{P}$ *are two closed, convex, and pointed cones of* Z_1 *and* Z_2*, respectively), then Condition (i) can be canceled.*

Remark 8. *Huang, Wang, and Zhang [1] also studied the existence of solutions for (VSVEP) and proved their main result in Theorem 3.1. However, Theorem 3.1 of [1] is different from Corollary 1 of this paper. The main difference lies in the requirement of continuity for the equilibrium mappings* f *and* g*. Indeed,* f *and* g *are assumed to be continuous in Theorem 3.1 of [1], while they are assumed to be* C*-continuous and* $-C$*-continuous in Corollary 1 of this paper.*

Corollary 2. *Let* D *and* K *be as in Theorem 3. Let* $\mathcal{C} \subseteq Z_1$ *and* $\mathcal{P} \subseteq Z_2$ *be two closed, convex, and point cones. Suppose that Condition (ii) of Theorem 3 and the following conditions hold:*

(iii)″ $f : D \times K \to Z_1$ *is* $-\mathcal{C}$*-continuous and* $\mathcal{C}$*-continuous; for each* $y \in K$*,* $f(x, y)$ *is properly* $\mathcal{C}$*-quasiconvex in* $x \in D$*;*

(iv)″ $g : D \times K \to Z_2$ *is* $-\mathcal{P}$*-continuous and* $\mathcal{P}$*-continuous; for each* $x \in D$*,* $g(x, y)$ *is properly* $\mathcal{P}$*-quasiconvex in* $y \in K$*.*

Then, the following symmetric generalized strong vector quasi-equilibrium problems (SGVEP) is solvable, i.e., there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that* $\bar{x} \in S(\bar{x}, \bar{y})$*,* $\bar{y} \in T(\bar{x}, \bar{y})$ *and:*

$$(SGVEP) \qquad \begin{cases} f(x, \bar{y}) \in f(\bar{x}, \bar{y}) + \mathcal{C}, & \forall x \in S(\bar{x}, \bar{y}), \\ g(\bar{x}, y) \in g(\bar{x}, \bar{y}) + \mathcal{P}, & \forall y \in T(\bar{x}, \bar{y}). \end{cases}$$

Moreover, the solution set of (SGVEP) is compact in $D \times K$*.*

Proof. For any $x, u \in D$ and $y, z \in K$, let:

$$\tilde{f}(x, y, u) = f(u, y), \;\; \tilde{g}(x, y, z) = g(x, z);$$

and:

$$C(x) = -\mathcal{C}, \;\; P(y) = -\mathcal{P}.$$

Then, it is easy to check that all the conditions of Corollary 1 are satisfied, and so the conclusions follow immediately from Corollary 1. □

In Corollary 2, if $Z_1 = Z_2$ and $\mathcal{C} = \mathcal{P}$, then we have the following result for (SVEP).

Corollary 3. *Let D and K be as in Theorem 3. Let $\mathcal{C}$ be a closed, convex, and point cone of a real normed vector space Z. Assume that:*

(i) *S, T are continuous and both have nonempty closed and convex values;*
(ii) *$f : D \times K \to Z$ is $-\mathcal{C}$-continuous and $\mathcal{C}$-continuous; for any $y \in K$, $f(x, y)$ is properly $\mathcal{C}$-quasiconvex in $x \in D$;*
(iii) *$g : D \times K \to Z$ is $-\mathcal{C}$-continuous and $\mathcal{C}$-continuous; for any $x \in D$, $g(x, y)$ is properly $\mathcal{C}$-quasiconvex in $y \in K$.*

Then (SVEP) is solvable. Moreover, the solution set is compact in $D \times K$.

Remark 9. *Corollary 3 above improves Theorem 3.1 of Gong [2] by weakening the continuity conditions of the equilibrium mappings f and g. In fact, f and g are assumed to be continuous in Theorem 3.1 of Gong [2], while they are assumed to be $-C$-continuous and C-continuous in Corollary 3 of this paper. It is easy to see that a continuous mapping must be C-continuous and $-C$-continuous, but the converse is not necessarily true (see, for example, [28], pp. 22–23).*

Remark 10. *Corollary 3 of this paper improves Theorem 2.1 of Fu [35] in the following aspects: (i) the conclusion is enhanced from a weak problem to a strong one; (ii) the requirements of continuity are weakened on the equilibrium mappings. In fact, the equilibrium mappings f and g are both assumed to be continuous in Theorem 2.1 of Fu [35], while they are assumed to be $-C$-continuous and C-continuous in Corollary 3 of this paper.*

As for the problem of (VMSVEP 2), we have:

Theorem 4. *Let D and K be as in Theorem 3. Suppose that:*

(i) *C, P are c.u.c. and for each $x \in D$, $y \in K$, $C(x)$ and $P(y)$ both have bounded closed bases;*
(ii) *S, T are continuous with nonempty closed and convex values;*
(iii) *F is lower $-C$-continuous and upper C-continuous on $D \times K \times D$; for any $(x, y, u) \in D \times K \times D$, $F(x, y, u)$ is bounded and $F(x, y, u) + C(x)$ is closed; for each $(x, y) \in D \times K$, $F(x, y, u)$ is upper properly $C(x)$-quasiconvex in $u \in D$;*
(iv) *G is lower $-P$-continuous and upper P-continuous on $D \times K \times K$; for any $(x, y, z) \in D \times K \times K$, $G(x, y, z)$ is bounded and $G(x, y, z) + P(y)$ is closed; for each $(x, y) \in D \times K$, $G(x, y, z)$ is upper properly $P(y)$-quasiconvex in $z \in K$.*

Then (VMSVEP 2) is solvable. Moreover, the solution set is compact in $D \times K$.

Proof. Define two set-valued mappings $A : D \times K \to 2^D$ and $B : D \times K \to 2^K$ as follows: for each $(x, y) \in D \times K$, let:

$$A(x, y) = \{v \in S(x, y) : F(x, y, u) \subseteq F(x, y, v) + C(x),\ \forall u \in S(x, y)\};$$

and:

$$B(x, y) = \{q \in T(x, y) : G(x, y, p) \subseteq G(x, y, q) + P(y),\ \forall p \in T(x, y)\}.$$

Then, by proceeding with the remaining arguments as used in the proof of Theorem 3, we can obtain the conclusions. □

Remark 11. *In the above Theorem 4, (1) if, for any $x \in D$ and $y \in K$, $C(x) \equiv \mathcal{C}$ and $P(y) \equiv \mathcal{P}$ ($\mathcal{C}$ and $\mathcal{P}$ are two closed, convex, and pointed cones of Z_1 and Z_2, respectively), then Condition (i) and the bounded assumptions of $F(x, y, u)$ and $G(x, y, z)$ in Conditions (iii) and (iv), respectively, can be dropped; (2) if, for any $(x, y, v) \in D \times K \times K$, $G(x, y, v) \equiv \{0\}$, then the requirement of continuity for T can be weakened as the one* *of upper semi-continuity. Similarly, if, for any $(x, y, u) \in D \times K \times D$, $F(x, y, u) \equiv \{0\}$, then the requirement of continuity for S can be weakened as the one of upper semi-continuity.*

Corollary 4. *Let D and K be as in Theorem 3. Let $\mathcal{C}$ and $\mathcal{P}$ be closed, convex, and pointed cones of Z_1 and Z_2, respectively. Suppose that:*

(i) *S, T are continuous with nonempty closed and convex values;*
(ii) *F is lower $-\mathcal{C}$-continuous and upper $\mathcal{C}$-continuous on $D \times K \times D$; for any $(x, y, u) \in D \times K \times D$, $F(x, y, u) + \mathcal{C}$ is closed; for each $(x, y) \in D \times K$, $F(x, y, u)$ is upper properly $\mathcal{C}$-quasiconvex in $u \in D$;*
(iii) *G is lower $-\mathcal{P}$-continuous and upper $\mathcal{P}$-continuous on $D \times K \times K$; for any $(x, y, z) \in D \times K \times K$, $G(x, y, z) + \mathcal{P}$ is closed; for each $(x, y) \in D \times K$, $G(x, y, z)$ is upper properly $\mathcal{P}$-quasiconvex in $z \in K$.*

Then (MSVEP) is solvable. Moreover, its solution set is compact in $D \times K$.

Proof. For any $x \in D$ and $y \in K$, let:

$$C(x) \equiv \mathcal{C} \text{ and } P(y) \equiv \mathcal{P}.$$

Then, the conclusions can be derived immediately from Theorem 4 and Remark 11. □

From Corollary 4, we can obtain the following results for (MVEP).

Corollary 5. *Let D and K be as in Theorem 3. Let $\mathcal{C}$ be a closed, convex, and point cone of a real normed vector space Z. Suppose that:*

(i) *S, T are continuous with nonempty closed and convex values;*
(ii) *$F : D \times K \times D \to 2^Z$ is lower $-\mathcal{C}$-continuous and upper $\mathcal{C}$-continuous on $D \times K \times D$; for any $(x, y, u) \in D \times K \times D$, $F(x, y, u) + \mathcal{C}$ is closed; for each $(x, y) \in D \times K$, $F(x, y, x) \subseteq \mathcal{C}$ and $F(x, y, u)$ is upper properly $\mathcal{C}$-quasiconvex in $u \in D$;*
(iii) *$G : D \times K \times K \to 2^Z$ is lower $-\mathcal{C}$-continuous and upper $\mathcal{C}$-continuous on $D \times K \times K$; for any $(x, y, z) \in D \times K \times K$, $G(x, y, z) + \mathcal{C}$ is closed; for each $(x, y) \in D \times K$, $G(x, y, y) \subseteq \mathcal{C}$ and $G(x, y, z)$ is upper properly $\mathcal{C}$-quasiconvex in $z \in K$.*

Then (MVEP) is solvable. Moreover, its solution set is compact in $D \times K$.

Proof. Let $Z_1 = Z_2 = Z$, $\mathcal{P} = \mathcal{C}$. Then, the conclusions can be derived directly from Corollary 4. □

Remark 12. *Chen, Huang, and Wen [4] also studied the existence of solutions and the compactness of the solution set for (MVEP) and established their main result in Theorem 3.1. However, Corollary 5 of this paper improves Theorem 3.1 of [4] mainly in the following two aspects: (a) Assumption (i) of Theorem 3.1 of [4] is dropped; (b) the assumptions that F and G both have compact values in Theorem 3.1 of [4] are weakened as $F(x, y, u) + \mathcal{C}$ and $G(x, y, z) + \mathcal{C}$ are closed subsets for all $x, u \in D$ and $y, z \in K$. In fact, if F and G both have compact values, i.e., for any $x, u \in D$ and $y, z \in K$, $F(x, y, u)$ and $G(x, y, z)$ are both compact subsets, then, by the closedness of the subset $\mathcal{C}$, we can conclude easily that the sets $F(x, y, u) + \mathcal{C}$ and $G(x, y, z) + \mathcal{C}$ are closed. However, the converse is not necessarily true.*

Corollary 6. *([5] Theorem 3.6) Let D and K be as in Theorem 3. Let $\mathcal{C}$ be a closed, convex, and point cone of a real normed vector space Z. Suppose that:*

(i) *S is continuous with nonempty closed and convex values; T is u.s.c. with nonempty closed and convex values;*
(ii) *$F : D \times K \times D \to 2^Z$ is lower $-\mathcal{C}$-continuous and upper $\mathcal{C}$-continuous on $D \times K \times D$; for any $(x, y, u) \in D \times K \times D$, $F(x, y, u) + \mathcal{C}$ is closed; for each $(x, y) \in D \times K$, $F(x, y, u)$ is upper properly $\mathcal{C}$-quasiconvex in $u \in D$.*

Then (QVIP) is solvable. Moreover, its solution set is compact in $D \times K$.

Proof. Let $Z_1 = Z_2 = Z$, $\mathcal{P} = \mathcal{C}$. For any $x \in D$ and $y, v \in K$, let:

$$C(x) \equiv \mathcal{C}, \ \ P(y) \equiv \mathcal{P} \ \text{ and } \ G(x,y,v) \equiv \{0\}.$$

Then, the conclusions can be made immediately from Theorem 4 and Remark 11. □

4. Applications

In this section, we shall apply Theorem 4 to discuss vector saddle point problems.

Definition 7. *Let X and Y be two Hausdorff topological spaces and D and K be two nonempty subsets of X and Y, respectively. Let Z be a real Hausdorff topological vector space and $\mathcal{C}$ be a closed, convex, and pointed cone of Z. Given a set-valued mapping $L : D \times K \to 2^Z$, a point $(\bar{x}, \bar{y}) \in D \times K$ is said to be a strong saddle point of L in $D \times K$ if:*

$$\begin{cases} L(x,\bar{y}) \subseteq L(\bar{x},\bar{y}) + \mathcal{C}, & \forall x \in D, \\ L(\bar{x},y) \subseteq L(\bar{x},\bar{y}) - \mathcal{C}, & \forall y \in K. \end{cases}$$

Definition 8. *Let $X, Y, Z, D, K, \mathcal{C}$ and L be as in Definition 7. Let $S : D \times K \to 2^D$ and $T : D \times K \to 2^K$ be two set-valued mappings. A point $(\bar{x}, \bar{y}) \in X \times Y$ is said to be a strong saddle point of L in $D \times K$ with constraints if $\bar{x} \in S(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x},\bar{y})$ and:*

$$\begin{cases} L(x,\bar{y}) \subseteq L(\bar{x},\bar{y}) + \mathcal{C}, & \forall x \in S(\bar{x},\bar{y}), \\ L(\bar{x},y) \subseteq L(\bar{x},\bar{y}) - \mathcal{C}, & \forall y \in T(\bar{x},\bar{y}). \end{cases}$$

Remark 13. *The above concepts of a strong saddle point and a strong saddle point with constraints are set-valued generalizations of the corresponding ones of Gong [2].*

Theorem 5. *Let X, Y, D, and K be as in Theorem 4. Let $\mathcal{C}$ be a closed, convex, and pointed cone of a real normed vector space Z. Let $S : D \times K \to 2^D$, $T : D \times K \to 2^K$ and $L : D \times K \to 2^Z$ be all set-valued mappings. Suppose that:*

(i) *S, T are continuous with nonempty closed and convex values;*
(ii) *L is lower $-\mathcal{C}$-continuous and upper $\mathcal{C}$-continuous on $D \times K$; for any $(x,y) \in D \times K$, $L(x,y) + \mathcal{C}$ is closed; for each $y \in K$, $L(x,y)$ is upper properly $\mathcal{C}$-quasiconvex in $x \in D$;*
(iii) *L is lower $\mathcal{C}$-continuous and upper $-\mathcal{C}$-continuous on $D \times K$; for any $(x,y) \in D \times K$, $L(x,y) - \mathcal{C}$ is closed; for each $x \in D$, $L(x,y)$ is upper properly $-\mathcal{C}$-quasiconvex in $y \in K$.*

Then, L has at least one strong saddle point with constraints in $D \times K$. Moreover, the set of all strong saddle points with constraints for L is compact in $D \times K$.

Proof. Let $Z_1 = Z_2 = Z$. Define set-valued mappings $C : D \to 2^{Z_1}$, $P : K \to 2^{Z_2}$, $F : D \times K \times D \to 2^{Z_1}$, and $G : D \times K \times K \to 2^{Z_2}$ as follows: for each $x, u \in D$ and $y, v \in K$, let:

$$C(x) \equiv \mathcal{C}, \ \ P(y) \equiv -\mathcal{C}, \ \ F(x,y,u) = L(u,y), \ \ G(x,y,v) = L(x,v).$$

Then, by the assumptions and Remark 11, it is easy to see that all the conditions of Theorem 4 are satisfied. Thus, the conclusions follows immediately from Theorem 4. □

When L is a single-valued mapping, we have the following conclusion.

Corollary 7. *Let $X, Y, Z, D, K, \mathcal{C}, S$, and T be as in Theorem 5. Let $L : D \times K \to Z$ be a single-valued mapping. Suppose that:*

(i) *S, T are continuous with nonempty closed and convex values;*
(ii) *L is $-\mathcal{C}$-continuous and $\mathcal{C}$-continuous on $D \times K$;*
(iii) *for each $y \in K$, $L(x, y)$ is properly $\mathcal{C}$-quasiconvex in $x \in D$;*
(iv) *for each $x \in D$, $L(x, y)$ is properly $-\mathcal{C}$-quasiconvex in $y \in K$.*

Then, L has at least one strong saddle point with constraints in $D \times K$. Moreover, the set of all strong saddle points with constraints for L is compact in $D \times K$.

Remark 14. *Corollary 7 above improves Theorem 4.1 of Gong [2] by weakening the continuity condition of the mapping L. In fact, L is assumed to be continuous in Theorem 4.1 of Gong [2], while it is assumed to be $-C$-continuous and C-continuous in Corollary 7 of this paper.*

In Corollary 7, if $S(x, y) \equiv D$ and $T(x, y) \equiv K$ for all $(x, y) \in D \times K$, then we have the following results about the strong saddle point.

Corollary 8. *Let $X, Y, Z, D, K, \mathcal{C}$, and L be as in Theorem 5. Suppose that Conditions (ii)–(iv) of Corollary 7 hold. Then, L has at least one strong saddle point in $D \times K$. Moreover, the set of all strong saddle points for L is compact in $D \times K$.*

5. Conclusions

The main purpose of this paper is to investigate the existence of solutions for set-valued symmetric generalized strong vector quasi-equilibrium problems with variable ordering structures (VMSVEP). The greatest difficulty is dealing with the variable ordering structures, which are characterized by a family of cones or a cone-valued mapping. In the past few years, many researchers have made great efforts and established a large quantity of results for various vector optimization problems with variable ordering structures. In most of these results, the assumption of upper semi-continuity, which is used commonly for set-valued mapping, is imposed on the variable ordering structures (i.e., the cone-valued mapping). However, just as pointed out by Borde and Crouzeix [20], this assumption is actually not suitable for cone-valued mapping as its particular characters. Therefore, it is necessary to discuss these problems again. In this paper, we use the concept of cosmically upper continuity for cone-valued mapping, which was firstly proposed by Luc [21], to discuss (VMSVEP). By applying the famous Fan-KKM theorem and Kakutani–Fan–Glicksberg fixed point theorem, the existence results of solutions for (VMSVEP) are proposed under suitable assumptions of cone-continuity and cone-convexity for the equilibrium mappings. Further, the conclusions about the compactness of solution sets are obtained in this paper. As applications, some existence results of strong saddle points are also obtained. The main results obtained in this paper unify and improve some recent works in the literature. As future work, we would generalize and extend the key concept of cosmically upper continuity from a normed vector space to a more general topological vector space and use it to establish some new existence results for various vector optimization problems with variable ordering structures. Further, we will explore practical applications of our research results in medical image registration, portfolio optimization, location problems, and so on.

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