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Sub-Additive Aggregation Functions and Their Applications in Construction of Coherent Upper Previsions

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Abstract: In this paper, we explore the use of aggregation functions in the construction of coherent upper previsions. Sub-additivity is one of the defining properties of a coherent upper prevision defined on a linear space of random variables and thus we introduce a new sub-additive transformation of aggregation functions, called a revenue transformation, whose output is a sub-additive aggregation function bounded below by the transformed aggregation function, if it exists. Method of constructing coherent upper previsions by means of shift-invariant, positively homogeneous and sub-additive aggregation functions is given and a full characterization of shift-invariant, positively homogeneous and idempotent aggregation functions on $[0, \infty]^n$ is presented. Lastly, some concluding remarks are added.

Keywords: sub-additivity; revenue transformation; aggregation function; coherent upper prevision



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1. Introduction

In the subjective probability approach [1], prevision of a random variable is defined by the notion of *coherence* that assures that the prevision of a random variable X is the amount the subject is willing to bet on X such that neither the bettor nor the banker can win or lose for sure. From a mathematical point of view, linear previsions on a linear space of random variables are coherent if and only if they are linear functionals with values bounded by the infimum and the supremum value of the random variable. From these defining properties, we can prove that a linear coherent prevision is a homogeneous functional. Coherent probabilities are obtained when only indicator functions are considered in the domain. Because of the incomplete and inaccurate information, or to represent some preference orderings (e.g., Example 1 of [2]), in some cases it is appropriate to consider non-linear functionals. Coherent upper and lower previsions can be defined as generalizations of linear previsions. In particular, if a linear prevision is defined with respect to a countably additive probability, coherent upper and lower conditional previsions can be defined by the Choquet integral with respect to the outer and inner measures generated by the countably additive probability, which are the natural extensions of an additive probability defined on a σ -field [3]. A new model of coherent upper previsions defined by the Choquet integral with respect to Hausdorff outer measures is proposed in [4]. Also, coherent upper and lower previsions have a behavioural interpretation: the lower prevision of X can be regarded as the supremal buying price for the random variable X and the upper prevision is an infimal selling price. Nevertheless, we observe that there exist coherent upper previsions defined on the linear space of random variables without linear restrictions; an example is the vacuous upper prevision of a random variable, defined as the supremum of the values assumed by that random variable on Ω . So an open problem is to propose new mathematical tools to define coherent upper previsions that cannot be obtained as extensions of linear previsions.

The main motivation behind this paper is to propose a construction method for coherent upper previsions using aggregation functions [5,6] that are prolific and latter-day part of mathematics that found its place both in the theory and applications.

A functional on a linear space of random variables is a coherent upper prevision if and only if it is bounded by the supremum value of the random variable, sub-additive and positively homogeneous. In this contribution, coherent upper previsions are constructed by aggregation functions, which are interpreted as the gains we obtain with a given resource so that the increase in gains, after increase in resources, can be expressed in terms of aggregation functions. We introduce a new type of transformation of aggregation functions, called a revenue transformation, which represents the best upper bound for possible gains; in the paper we consider only revenue transformations that always assume bounded values. Some properties of revenue transformations are proven; in particular, it is proven that the revenue transformation transforms any aggregation function to some other aggregation function that is sub-additive and is bounded below by the transformed function. There already exists a sub-additive transformation of aggregation functions introduced in [7] and is heavily studied by researchers [8–10]. This sub-additive transformation always exists and is bounded above by the aggregation function that is being transformed, whereas examples are given in this paper of aggregation functions that do not have a bounded revenue transformation.

The revenue transformation of the Choquet integral, which is an example of aggregation function, is calculated and an open problem is to determine if it is the Choquet integral with respect to a monotone measure. An example of revenue transformation of Choquet integral with respect to a monotone measure μ , which is not sub-modular, is given such that the transformation is the Choquet integral with respect to a sub-modular monotone measure and it can be used to define a coherent upper prevision.

The paper is organized as follows. In Section 2, some basic preliminaries needed later are given. In Section 3, we define a new sub-additive transformation of aggregation functions, called a revenue transformation, and study its properties. Section 4 consists of a construction method of coherent upper previsions based on aggregation functions. This construction method is heavily exemplified in this section. The last section contains some concluding remarks on the topic.

2. Preliminaries

Let Ω be a non-empty set and let \mathcal{A} be a σ -algebra on Ω such that \mathcal{A} is finite. With this assumption we may, without loss of generality, assume that $\Omega = \{1, 2, \dots, n\}$ and $\mathcal{A} = 2^\Omega$ for some $n \in \mathbb{N}$. This natural number n is fixed throughout. Note that the motivation of the concepts discussed in this paper given in the literature, see, e.g., [3], including illustrative examples, deal mostly with finite Ω only, therefore we focus on this framework only. Though most of concepts we recall or introduce in this paper remain valid also when a general measurable space (Ω, \mathcal{A}) is considered, this is not the case, e.g., with the constructions based on the aggregation functions.

In this setting, every random variable $X: \Omega \rightarrow \mathbb{R}$ on (Ω, \mathcal{A}) can be represented as n -tuple $\mathbf{x} \in \mathbb{R}^n$ with i -th coordinate (denoted by x_i) being equal to $X(i)$ for $i = 1, 2, \dots, n$. All such vectors (i.e., random variables) form a vector space \mathbb{R}^n with coordinate-wise addition $+$ and coordinate-wise scalar multiplication (over field \mathbb{R}). Note that coordinate-wise difference $-$, coordinate-wise multiplication \cdot , coordinate-wise division $/$, coordinate-wise supremum \vee , and coordinate-wise infimum \wedge can be also introduced.

Also, a partial order \leq can be introduced by $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$ for all $i \in \{1, 2, \dots, n\}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. By $\mathbf{0}$ we denote the all-zero vector and by $\mathbf{1}$ we denote the all-one vector. An indicator vector $\mathbf{1}_A$ of a set $A \subseteq \Omega$ is given by its i -th coordinate being equal to $\mathbf{1}_A(i)$, where $\mathbf{1}_A$ is the indicator function of the set A . A *positive cone* of a vector space \mathbb{R}^n is the set

$$\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\},$$

i.e., $\mathbb{R}_+^n = [0, \infty]^n$.

Let $\mathbf{x} \in \mathbb{R}^n$ be any vector with coordinates x_i . There exists a permutation $\sigma: \Omega \rightarrow \Omega$ such that

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(n)}.$$

For simplicity, we denote $x_{(1)} = x_{\sigma(1)}, x_{(2)} = x_{\sigma(2)}, \dots, x_{(n)} = x_{\sigma(n)}$. Thus we have $\wedge_i x_i = x_{(1)}$ and $\vee_i x_i = x_{(n)}$. Note that $x_{(i)}$ is just the i -th order statistics of the sample (x_1, \dots, x_n) .

An aggregation function [5,6] is any mapping $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ such that A is non-decreasing, i.e., $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $A(\mathbf{x}) \leq A(\mathbf{y})$; and grounded, i.e., $A(\mathbf{0}) = 0$. In practice, it is convenient to assume that there exists at least one $\mathbf{x} \in \mathbb{R}_+^n$ such that $A(\mathbf{x}) \neq 0$. This restriction is not necessary, but $A(\mathbf{x}) \equiv 0$ is not an interesting aggregation function for us and thus such case will be omitted in proofs (with explicit declaration). We say that an aggregation function A is *sub-additive*, respectively *super-additive*, if and only if

$$A(\mathbf{x} + \mathbf{y}) \leq A(\mathbf{x}) + A(\mathbf{y}), \quad \text{respectively } A(\mathbf{x} + \mathbf{y}) \geq A(\mathbf{x}) + A(\mathbf{y}),$$

holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n$. We say that an aggregation function A is *positively homogeneous* if and only if

$$A(\alpha \mathbf{x}) = \alpha A(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}_+^n$ and all $\alpha \geq 0$. Lastly, we say that an aggregation function A is *shift-invariant* if and only if

$$A(\mathbf{x} + \alpha \mathbf{1}) = A(\mathbf{x}) + \alpha A(\mathbf{1})$$

for all $\mathbf{x} \in \mathbb{R}_+^n$ and all $\alpha \geq -\inf \mathbf{x}$. A $[0, 1]$ -aggregation function is any mapping $A: [0, 1]^n \rightarrow [0, 1]$ such that A is non-decreasing, i.e., $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y} \leq \mathbf{1}$ implies $A(\mathbf{x}) \leq A(\mathbf{y})$; and obeys conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$. Sub-additivity, super-additivity, positive homogeneity, and shift-invariance can be introduced analogously for $[0, 1]$ -aggregation functions.

Note that a full characterization of positively homogeneous and shift-invariant $[0, 1]$ -aggregation functions was given in [6] (Proposition 7.37): A $[0, 1]$ -aggregation function A is shift-invariant and positively homogeneous (interval scale invariant in [6]) if and only if A satisfies the following conditions:

- (i) $A(\mathbf{x}) - A(\mathbf{y}) \leq \vee_i (x_i - y_i)$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ such that $\mathbf{x} \geq \mathbf{y}$ and $x_i = y_i = 0$ for some $i \in \{1, 2, \dots, n\}$;
- (ii) $A(\mathbf{x}) / A(\mathbf{y}) \geq \wedge_i (x_i / y_i)$ for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ such that $\mathbf{x} \leq \mathbf{y}$ and $x_j = y_j = 1$ for some $j \in \{1, 2, \dots, n\}$, with convention $0/0 = 1$.

Observe that then

$$A(\mathbf{x}) = (\vee_i x_i - \wedge_i x_i) A\left(\frac{\mathbf{x} - (\wedge_i x_i) \mathbf{1}}{\vee_i x_i - \wedge_i x_i}\right) + \wedge_i x_i = (x_{(n)} - x_{(1)}) A\left(\frac{\mathbf{x} - x_{(1)} \mathbf{1}}{x_{(n)} - x_{(1)}}\right) + x_{(1)}$$

for all \mathbf{x} such that $\vee_i x_i \neq \wedge_i x_i$ and

$$A(x \mathbf{1}) = x$$

for all $x \in [0, 1]$. See also, e.g., [11,12].

A *monotone measure* [13] is any set function $\mu: \mathcal{A} \rightarrow [0, \infty[$ that is non-decreasing, i.e., $A \subseteq B \subseteq \Omega$ implies $\mu(A) \leq \mu(B)$; and grounded, i.e., $\mu(\emptyset) = 0$. Note that a monotone measure can be viewed as an aggregation function defined on 2^Ω with the partial order \leq being set inclusion. We say that a monotone measure μ is *sub-modular* if and only if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$$

for all $A, B \in 2^\Omega$. Sub-modular monotone measures are also sometimes referred to as 2-alternating monotone measures, see, e.g., [14,15]. If $A = \{x, y, \dots, z\} \in 2^\Omega$ we will write,

for simplicity, $\mu_{xy\dots z}$ instead of $\mu(\{x, y, \dots, z\})$ or μ_A instead of $\mu(A)$. A *conjugate monotone measure* for a monotone measure μ is a monotone measure $\bar{\mu}: 2^\Omega \rightarrow [0, \infty[$ such that

$$\bar{\mu}(A) = \mu(\Omega) - \mu(\Omega \setminus A)$$

for all $A \in 2^\Omega$.

The *Choquet integral* [16] with respect to a monotone measure μ is the operator $\text{Ch}^\mu: \mathbb{R}_+^n \rightarrow [0, \infty[$ such that

$$\text{Ch}^\mu(\mathbf{x}) = x_{(1)}\mu_{A_1} + \sum_{i=2}^k (x_{(i)} - x_{(i-1)})\mu_{A_i},$$

where $A_i = \{j \in \Omega: x_j \geq x_{(i)}\}$ for $i = 1, 2, \dots, n$. Choquet integral is an aggregation function and if μ is sub-modular then Ch^μ is a sub-additive aggregation function. In the theory of imprecise probabilities, an extension of Choquet integral, called an asymmetric Choquet integral, that is an operator $\text{aCh}^\mu: \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\text{aCh}^\mu(\mathbf{x}) = \text{Ch}^\mu(\mathbf{x} \vee \mathbf{0}) - \text{Ch}^{\bar{\mu}}((-\mathbf{x}) \vee \mathbf{0}),$$

is used to construct coherent upper previsions, see, e.g., [17].

A *coherent upper prevision* is a mapping $\bar{P}: \mathbb{R}^n \rightarrow \mathbb{R}$ such that conditions

$$\text{(CUP1)} \quad \bar{P}(\mathbf{x}) \leq \sup \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n;$$

$$\text{(CUP2)} \quad \bar{P}(\alpha \mathbf{x}) = \alpha \bar{P}(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^n \text{ and all } \alpha \in [0, \infty[; \text{ and}$$

$$\text{(CUP3)} \quad \bar{P}(\mathbf{x} + \mathbf{y}) \leq \bar{P}(\mathbf{x}) + \bar{P}(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

hold. Note that condition (CUP2) is positive homogeneity of \bar{P} and condition (CUP3) represents sub-additivity of \bar{P} . A *coherent lower prevision* is any mapping $\underline{P}: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\underline{P}(\mathbf{x}) = -\bar{P}(-\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, where \bar{P} is some coherent upper prevision.

3. Revenue Transformation and Its Properties

Let us start with a notion of a revenue transformation. Imagine that the value of an aggregation function $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ at $\mathbf{y} \in \mathbb{R}_+^n$ represents gains with resources represented by \mathbf{y} . The increase in gains after increasing resources by \mathbf{x} is equal to

$$A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y}),$$

i.e., the difference between gains with increased resources and original resources. Imagine, that the original resources are not known and we ask ourselves a question: What is the maximal gain (or the best upper bound for possible gains) if we increase our resources by \mathbf{x} ? The answer is

$$\sup_{\mathbf{y} \geq \mathbf{0}} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})).$$

This defines the revenue transformation of A that will be denoted by \bar{A} . We would like to ensure that \bar{A} is again an aggregation function and thus we must prevent \bar{A} to being equal to ∞ . Such value is obtainable, see following example.

Example 1. Consider a one-dimensional aggregation function $A: [0, \infty[\rightarrow [0, \infty[$ given by

$$A(x) = x^2.$$

Then one obtains that

$$A(x + y) - A(y) = (x + y)^2 - y^2 = x^2 + 2xy$$

and thus

$$\sup_{y \geq 0} (2xy + y^2) = \infty,$$

for any $x > 0$. As an example for n -dimensional aggregation function that has a similar behaviour is an aggregation function $A: \mathbb{R}_+^n \rightarrow [0, \infty[: \mathbf{x} \mapsto x_1^2 + x_2^2 + \dots + x_n^2$.

One of the existence conditions (and we will prove later that it is the only condition) for \bar{A} to be an aggregation function is

$$\sup_{\mathbf{y} \geq \mathbf{0}} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) < \infty$$

that is required to hold for all $\mathbf{x} \in \mathbb{R}_+^n$. This defines a sub-class of aggregation functions that will be denoted by \mathbb{A}_{rev} . Whenever we refer to any revenue transformation we will always assume that this condition is satisfied and thus we will only work with $A \in \mathbb{A}_{\text{rev}}$ in such a case.

Definition 1. Let $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ be an aggregation function such that $A \in \mathbb{A}_{\text{rev}}$. Then its revenue transformation is a mapping $\bar{A}: \mathbb{R}_+^n \rightarrow [0, \infty[$ given by

$$\bar{A}(\mathbf{x}) = \sup_{\mathbf{y} \geq \mathbf{0}} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y}))$$

for all $\mathbf{x} \in \mathbb{R}_+^n$.

As we hinted before, any revenue transformation of any aggregation function belonging to \mathbb{A}_{rev} yield again an aggregation function. This is stated in the following proposition.

Proposition 1. Let $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ be an aggregation function from \mathbb{A}_{rev} . Then \bar{A} is also an aggregation function.

Proof. Let us start by proving that $\bar{A}(\mathbf{0}) = 0$ holds. It is easy to see that

$$\bar{A}(\mathbf{0}) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{0} + \mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} 0 = 0.$$

Now we will show that \bar{A} is also non-decreasing. Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}_+^n$ be such that $\mathbf{x} \leq \mathbf{z}$. From the monotonicity of A we have $A(\mathbf{x} + \mathbf{y}) \leq A(\mathbf{z} + \mathbf{y})$, or, equivalently, $A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y}) \leq A(\mathbf{z} + \mathbf{y}) - A(\mathbf{y})$ for any $\mathbf{y} \in \mathbb{R}_+^n$ and thus

$$\bar{A}(\mathbf{x}) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) \leq \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{z} + \mathbf{y}) - A(\mathbf{y})) = \bar{A}(\mathbf{z}).$$

Thus, \bar{A} is an aggregation function. \square

Now we give some properties of revenue transformations. These are summarized in the following proposition.

Proposition 2. Let A be an aggregation function. Then

- a \bar{A} is a sub-additive aggregation function;
- b $\bar{A} \geq A$;
- c if A is sub-additive then $\bar{A} = A$;
- d if A is shift-invariant then the diagonal of \bar{A} coincides with the diagonal of A ;
- e if A is shift-invariant then so is \bar{A} ;

f if A is positively homogeneous then so is \bar{A} .

Proof. (a) To see that \bar{A} is a sub-additive aggregation function, let $\mathbf{x}, \mathbf{z} \in \mathbb{R}_+^n$. Then

$$\begin{aligned} \bar{A}(\mathbf{x} + \mathbf{z}) &= \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{z} + \mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{z} + \mathbf{y}) - A(\mathbf{z} + \mathbf{y}) + A(\mathbf{z} + \mathbf{y}) - A(\mathbf{y})) \\ &\leq \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{z} + \mathbf{y}) - A(\mathbf{z} + \mathbf{y})) + \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{z} + \mathbf{y}) - A(\mathbf{y})) \\ &= \sup_{\mathbf{y} \geq \mathbf{z}} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) + \bar{A}(\mathbf{z}) \leq \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) + \bar{A}(\mathbf{z}) = \bar{A}(\mathbf{x}) + \bar{A}(\mathbf{z}) \end{aligned}$$

and thus \bar{A} is a sub-additive aggregation function.

To prove (b) it is sufficient to notice that

$$\bar{A}(\mathbf{x}) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) \geq (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) \Big|_{\mathbf{y}=\mathbf{0}} = A(\mathbf{x}) - A(\mathbf{0}) = A(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}_+^n$ which implies that $\bar{A} \geq A$.

(c) If A is sub-additive then $A(\mathbf{x} + \mathbf{y}) \leq A(\mathbf{x}) + A(\mathbf{y})$ which implies that

$$\bar{A}(\mathbf{x}) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) \leq \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x}) + A(\mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} A(\mathbf{x}) = A(\mathbf{x}),$$

i.e., $\bar{A} \leq A$. Combining this with part (b) of this proposition we obtain that $\bar{A} = A$ as needed.

To see (d) notice that

$$\begin{aligned} \bar{A}(\alpha \mathbf{1}) &= \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\alpha \mathbf{1} + \mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{y} + \alpha \mathbf{1}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{y}) + \alpha A(\mathbf{1}) - A(\mathbf{y})) \\ &= \sup_{\mathbf{y} \in \mathbb{R}_+^n} \alpha A(\mathbf{1}) = \alpha A(\mathbf{1}) = A(\alpha \mathbf{1}), \end{aligned}$$

i.e., the diagonal of \bar{A} coincides with the diagonal of A if A is shift-invariant.

To prove (e), notice that

$$\begin{aligned} \bar{A}(\mathbf{x} + \alpha \mathbf{1}) &= \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \alpha \mathbf{1} + \mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) + \alpha A(\mathbf{1}) - A(\mathbf{y})) \\ &= \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) + \alpha A(\mathbf{1}) = \bar{A}(\mathbf{x}) + A(\alpha \mathbf{1}). \end{aligned}$$

Now using (d) we have that $A(\mathbf{1}) = \bar{A}(\mathbf{1})$ and thus

$$\bar{A}(\mathbf{x} + \alpha \mathbf{1}) = \bar{A}(\mathbf{x}) + \alpha \bar{A}(\mathbf{1}),$$

i.e., \bar{A} is indeed a shift-invariant aggregation function if A is. Now, to prove (f) notice that

$$\begin{aligned} \bar{A}(\alpha \mathbf{x}) &= \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\alpha \mathbf{x} + \mathbf{y}) - A(\mathbf{y})) = \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\alpha \mathbf{x} + \alpha \mathbf{y}) - A(\alpha \mathbf{y})) \\ &= \alpha \sup_{\mathbf{y} \in \mathbb{R}_+^n} (A(\mathbf{x} + \mathbf{y}) - A(\mathbf{y})) = \alpha \bar{A}(\mathbf{x}) \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}_+^n$ and all $\alpha \geq 0$ if A is positively homogeneous. \square

Example 2. If we take, for the sake of exemplification, a one-dimensional aggregation function $A: [0, \infty[\rightarrow [0, \infty[$ given by

$$A(x) = \begin{cases} x^p, & \text{if } x \in [0, 1], \\ 1, & \text{otherwise,} \end{cases}$$

for any $p > 0$ then we obtain by item c) of Proposition 2 that $\bar{A} = A$ if $p \in]0, 1]$ since A is sub-additive and

$$\bar{A}(x) = \begin{cases} 1 - (1 - x)^p, & \text{if } x \in [0, 1], \\ 1, & \text{otherwise,} \end{cases}$$

if $p > 1$.

Example 3. If we consider a one-dimensional aggregation function $A: [0, \infty[\rightarrow [0, \infty[$ defined by

$$A(x) = \text{sgn}(\alpha) \left(\exp(\alpha x) - 1 \right),$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ then we obtain by c) of Proposition 2 that $\bar{A} = A$ if $\alpha < 0$ since A is sub-additive. On the other hand, if $\alpha > 0$, the revenue transformation of A does not exist.

Example 4. Let us consider $\Omega = \{1, 2\}$ and Choquet integral $\text{Ch}^\mu: \mathbb{R}_+^2 \rightarrow [0, \infty[$. Then one can easily compute its revenue transformation as

$$\overline{\text{Ch}}^\mu(x_1, x_2) = \begin{cases} x_1 \max\{\mu_1, \mu_{12} - \mu_2\} + x_2 \min\{\mu_2, \mu_{12} - \mu_1\}, & \text{if } x_1 \geq x_2, \\ x_1 \min\{\mu_1, \mu_{12} - \mu_2\} + x_2 \max\{\mu_2, \mu_{12} - \mu_1\}, & \text{if } x_1 < x_2, \end{cases}$$

for all $\mathbf{x} \in \mathbb{R}_+^2$, which coincides with Choquet integral $\text{Ch}^{\hat{\mu}}$ where $\hat{\mu}$ is given by

$$\hat{\mu}_{12} = \mu_{12}, \quad \hat{\mu}_i = \mu_i + \max\{0, \mu_{12} - \mu_1 - \mu_2\}$$

for $i \in \{1, 2\}$.

In general, for $n > 2$, it is an open problem whether for any monotone measure μ , $\overline{\text{Ch}}^\mu = \text{Ch}^\nu$ for some monotone measure ν . Clearly, if this is the case, $\nu(A) = \overline{\text{Ch}}(\mathbf{1}_A)$ for any $A \subseteq \Omega$. Note that for any μ , $\nu(A) = \overline{\text{Ch}}^\mu(\mathbf{1}_A)$ defines a sub-modular monotone measure.

Example 5. Consider a Choquet integral Ch^μ on $\Omega = \{1, 2, 3\}$ with respect to a monotone measure μ given by

$$\mu(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ 2/3, & \text{if } |A| \in \{1, 2\}, \\ 1, & \text{if } A = \Omega. \end{cases}$$

This monotone measure is not sub-modular and one can find that $\text{Ch}^\mu(\mathbf{x}) = (x_{(1)} + 2x_{(3)})/3$, i.e., Ch^μ is an OWA (Ordered Weighted Averaging) operator [18]. Its revenue transformation is again Choquet integral $\overline{\text{Ch}}^\mu = \text{Ch}^{\hat{\mu}}$ with respect to a monotone measure $\hat{\mu}$ given by

$$\hat{\mu}(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ 2/3, & \text{if } |A| = 1, \\ 1, & \text{if } |A| \geq 2, \end{cases}$$

and $\text{Ch}^{\hat{\mu}} = (x_{(2)} + 2x_{(3)})/3$; and $\hat{\mu}$ is a sub-modular monotone measure.

Example 6. Let $\Omega = \{1, 2, 3, 4\}$ and let μ be a monotone measure on Ω . Let us continue with Example 2 that we presented in [19]. Note that based on the monotone measure μ introduced there, we can construct a new monotone measure $\nu: 2^\Omega \rightarrow [0, \infty[$ by $\nu(A) = \overline{\text{Ch}}^\mu(\mathbf{1}_A)$ for all $A \in 2^\Omega$.

The values of the chosen monotone measure μ and the related monotone measure ν can be found in Table 1. Note that ν is a sub-modular monotone measure and also, if $\overline{\text{Ch}}^\mu$ is Choquet integral then it would coincide with Ch^ν .

Table 1. Values of capacities μ and ν from Example 6.

A	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$\mu(A)$	0	1/4	0	1/4	0	1/2	1/2	1/2	1/2	1/2	1/2
$\nu(A)$	0	1/2	1/2	1/2	1/2	1	1	1	1	1/2	1
A	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$						
$\mu(A)$	3/4	1/2	3/4	1/2	1						
$\nu(A)$	1	1	1	1	1						

4. Constructions of Coherent Upper Previsions

In this section, we will construct coherent upper previsions using aggregation functions with special properties. Analogous construction process as in [4,19] will be adopted.

Proposition 3. Let $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ be a sub-additive, positively homogeneous and shift-invariant aggregation function such that $A(\mathbf{1}) \neq 0$. Then the mapping $\text{cup}: \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\text{cup}(\mathbf{x}) = \frac{A(\mathbf{x} - (\inf \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$ defines a coherent upper prevision.

Proof. First of all, note that for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} - (\inf \mathbf{x})\mathbf{1} \geq \mathbf{0}$, and thus the definition of cup is valid. Since \mathbb{R}^n is a linear space to prove that cup defines a coherent upper prevision we will prove that cup obeys conditions (CUP1)-(CUP3). To see that (CUP1) holds, notice that

$$\begin{aligned} \text{cup}(\mathbf{x}) &= \frac{A(\mathbf{x} - (\inf \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{x} \leq \frac{A(\sup(\mathbf{x} - (\inf \mathbf{x})\mathbf{1})\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{x} \\ &= (\sup \mathbf{x} - \inf \mathbf{x}) \frac{A(\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{x} = \sup \mathbf{x} - \inf \mathbf{x} + \inf \mathbf{x} = \sup \mathbf{x} \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$. To see (CUP2) it is enough to consider

$$\begin{aligned} \text{cup}(\alpha \mathbf{x}) &= \frac{A(\alpha \mathbf{x} - (\inf \alpha \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \inf \alpha \mathbf{x} = \frac{A(\alpha \mathbf{x} - \alpha(\inf \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \alpha \inf \mathbf{x} \\ &= \alpha \frac{A(\mathbf{x} - (\inf \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \alpha \inf \mathbf{x} = \alpha \left(\frac{A(\mathbf{x} - (\inf \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{x} \right) = \alpha \text{cup}(\mathbf{x}) \end{aligned}$$

for all $\alpha \geq 0$ and all $\mathbf{x} \in \mathbb{R}^n$. Lastly, it remains to show that the condition (CUP3) holds, i.e., to show that cup is sub-additive. Because infimum is a super-additive aggregation function, we know that $\inf(\mathbf{x} + \mathbf{y}) \geq \inf \mathbf{x} + \inf \mathbf{y}$ and thus $r = \inf(\mathbf{x} + \mathbf{y}) - \inf \mathbf{x} - \inf \mathbf{y} \geq 0$. Note that

$$\text{cup}(\mathbf{x} + \mathbf{y}) = \frac{A(\mathbf{x} + \mathbf{y} - (\inf(\mathbf{x} + \mathbf{y}))\mathbf{1})}{A(\mathbf{1})} + \inf(\mathbf{x} + \mathbf{y}).$$

Now, from $-\inf(\mathbf{x} + \mathbf{y}) = -\inf \mathbf{x} - \inf \mathbf{y} - r$, we obtain that

$$\text{cup}(\mathbf{x} + \mathbf{y}) = \frac{A(\mathbf{x} + \mathbf{y} - (r + \inf \mathbf{x} + \inf \mathbf{y})\mathbf{1}) + \overbrace{rA(\mathbf{1}) - rA(\mathbf{1})}^0}{A(\mathbf{1})} + \inf(\mathbf{x} + \mathbf{y}).$$

Using the shift-invariance of A and the fact that $r \geq 0$ we have that

$$\text{cup}(\mathbf{x} + \mathbf{y}) = \frac{A\left(\left(\mathbf{x} - (\inf \mathbf{x})\mathbf{1}\right) + \left(\mathbf{y} - (\inf \mathbf{y})\mathbf{1}\right)\right)}{A(\mathbf{1})} - r + \inf(\mathbf{x} + \mathbf{y}).$$

Noticing that $\inf(\mathbf{x} + \mathbf{y}) - r = \inf \mathbf{x} + \inf \mathbf{y}$ and by the sub-additivity of A we finally obtain that

$$\text{cup}(\mathbf{x} + \mathbf{y}) \leq \frac{A(\mathbf{x} - (\inf \mathbf{x})\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{x} + \frac{A(\mathbf{y} - (\inf \mathbf{y})\mathbf{1})}{A(\mathbf{1})} + \inf \mathbf{y} = \text{cup}(\mathbf{x}) + \text{cup}(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, i.e., cup is sub-additive. \square

Remark 1. Moreover, we may assume that A is also idempotent, i.e., $A(\alpha\mathbf{1}) = \alpha$ for all $\alpha \geq 0$. With the fact that A is shift-invariant it is enough to consider only the fact that $A(\mathbf{1}) = 1$. Assuming this extra condition, the coherent upper prevision can be constructed by

$$\text{cup}(\mathbf{x}) = A(\mathbf{x} - (\inf \mathbf{x})\mathbf{1}) + \inf \mathbf{x},$$

which simplifies the construction.

Remark 2. As another remark, note that a positively homogeneous and idempotent aggregation function $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ is fully determined by its restriction to $[0, 1]^n$ because

$$A(\mathbf{x}) = (\sup \mathbf{x})A\left(\frac{\mathbf{x}}{\sup \mathbf{x}}\right),$$

for any $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, where $\mathbf{x}/(\sup \mathbf{x}) \in [0, 1]^n$. Also note that the given restriction $A|_{[0,1]^n}$ is a $[0, 1]$ -aggregation function and thus we can characterize all positively homogeneous, shift-invariant and idempotent aggregation functions. To authors' best knowledge, this characterization has not been published anywhere, yet.

Proposition 4. An aggregation function $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ is positively homogeneous, shift-invariant and idempotent if and only if there exists a positively homogeneous and shift-invariant $[0, 1]$ -aggregation function $B: [0, 1]^n \rightarrow [0, 1]$ such that $A|_{[0,1]^n} = B$ and

$$A(\mathbf{x}) = (\sup \mathbf{x})A\left(\frac{\mathbf{x}}{\sup \mathbf{x}}\right)$$

whenever $\mathbf{x} \neq \mathbf{0}$.

Remark 3. Note that there are positively homogeneous and shift-invariant aggregation functions that are not idempotent. As a trivial example of such aggregation function we give the sum function (with $n > 1$).

Coherent lower previsions are obtained by the conjugacy property

$$\text{clp}(\mathbf{x}) = -\text{cup}(-\mathbf{x}) = -\frac{A(-\mathbf{x} - (\inf(-\mathbf{x}))\mathbf{1})}{A(\mathbf{1})} - \inf(-\mathbf{x}) = -\frac{A(-\mathbf{x} - (\sup(\mathbf{x}))\mathbf{1})}{A(\mathbf{1})} - \sup(\mathbf{x})$$

We can observe that if the aggregation function A , that is considered to define a coherent upper prevision, is linear, then $\text{clp}(\mathbf{x}) = -\text{cup}(-\mathbf{x})$ and a linear prevision is obtained.

Coherent upper probabilities can be obtained by Proposition 3 when only indicator vectors are considered.

The coherent upper probability of an event $E \subseteq \Omega$ is defined by

$$\bar{P}(E) = \text{cup}(\mathbf{1}_E) = \frac{A(\mathbf{1}_E)}{A(\mathbf{1})}.$$

and the coherent lower probability is obtained by the conjugacy property,

$$\text{clp}(\mathbf{1}_E) = 1 - \text{cup}(\mathbf{1}_{E^c})$$

Example 7. Let $A: \mathbb{R}_+^n \rightarrow [0, \infty[$ be an aggregation function defined by $A(\mathbf{x}) = \sup \mathbf{x}$. Note that A satisfies all requirements of Proposition 3 and thus A can be used to construct a coherent upper prevision. This construction leads to $\text{cup}(\mathbf{x}) = \sup \mathbf{x}$ which is the vacuous upper prevision.

Example 8. The Choquet integral $\text{Ch}^{\hat{\mu}}$ defined in Example 5 by the revenue transformation of Ch^{μ} (which is not a coherent upper prevision because it is not sub-additive), is a coherent upper prevision since it is sub-additive, positively homogeneous and such that

$$\text{Ch}^{\hat{\mu}}(\mathbf{x}) = \frac{x_{(2)} + 2x_{(3)}}{3} \leq x_{(3)} = \sup \mathbf{x}.$$

Coherent upper probabilities are obtained when only indicator vectors $\mathbf{1}_E$, for all $E \in 2^\Omega$, are considered. These are given by

$$\bar{P}(\mathbf{1}_E) = \text{cup}(\mathbf{1}_E) = \frac{\text{Ch}^{\hat{\mu}}(\mathbf{1}_E)}{\text{Ch}^{\hat{\mu}}(\mathbf{1})} = \text{Ch}^{\hat{\mu}}(\mathbf{1}_E) = \hat{\mu}(E) = \begin{cases} 0, & \text{if } E = \emptyset, \\ 2/3, & \text{if } |E| = 1, \\ 1, & \text{if } |E| \geq 2, \end{cases}$$

In the following example an aggregation function which cannot be used to define a coherent upper prevision is given.

Example 9. Let $\Omega = \{1, 2, \dots, n\}$. The aggregation function $A(\mathbf{x}) = \sum_{i=1}^n x_i^2$ can not be used to construct a coherent upper prevision, because it is not sub-additive; moreover the revenue transformation \bar{A} does not exist as shown in Example 1.

Example 10. Let $\Omega = \{1, 2, \dots, n\}$. The aggregation function $A(\mathbf{x}) = \sum_i^n x_i$ can be used to construct a coherent upper prevision, because it is linear and homogeneous; by Proposition 3 we obtain that $\text{cup}(\mathbf{x})$ is the mean and the coherent probability of an event is defined by the counting measure, which is the Hausdorff measure of order 0. In this setting, $\bar{A} = A$.

In the following theorem a construction of aggregation functions, which are idempotent, positively homogeneous, shift invariant and sub-additive are proposed. They are applied in the construction of coherent upper conditional previsions.

Theorem 1. Denote by K_n the class of all n -ary aggregation functions on $[0, \infty[$, which are idempotent, positively homogeneous, shift invariant and sub-additive, then for any $n, k \in \mathbb{N}$, let $A \in K_n$ and E_1, E_2, \dots, E_n be non-empty subsets of $\{1, 2, \dots, k\}$ with cardinalities k_1, k_2, \dots, k_n , respectively, so that $E_i = \{e_{i,1}, e_{i,2}, \dots, e_{i,k_i}\}$, where $e_{i,1} < \dots < e_{i,k_i}$, $s_i: \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ are permutations and $B_i \in K_{k_i}$, then the aggregation function C defined by

$$C(x_1, \dots, x_k) = A\left(B_1(x_{s_1(e_{1,1})}, \dots, x_{s_1(e_{1,k_1})}), \dots, B_n(x_{s_n(e_{n,1})}, \dots, x_{s_n(e_{n,k_n})})\right)$$

belongs to K_k .

Proof. Observe first that each function $C_i: [0, \infty[^k \rightarrow [0, \infty[$ given by

$$C_i(x_1, \dots, x_k) = B_i(x_{s_i(e_{i,1})}, \dots, x_{s_i(e_{i,k_i})}),$$

for $i = 1, 2, \dots, n$, is an aggregation function from the class K_k . Then, for any $c \in [0, \infty[$,

$$C(\mathbf{c}) = A(C_1(\mathbf{c}), \dots, C_n(\mathbf{c})) = A(c, \dots, c) = c,$$

i.e., C is idempotent. Similarly, one can show the positive homogeneity, shift invariance and sub-additivity of C , which ensures $C \in K_k$. \square

The previous construction method can be used to define coherent upper previsions if $C(\mathbf{1}) \neq 0$.

Corollary 1. Let $A \in K_n$ be a weighted arithmetic mean, i.e., a linear prevision given by

$$A(x_1, \dots, x_n) = \sum c_i x_i,$$

and let B_1, \dots, B_n be given as in Theorem 1. Then the convex combination

$$C(x_1, \dots, x_k) = \sum_{i=1}^n c_i B_i(x_{e_{i,1}}, \dots, x_{e_{i,k_i}})$$

belongs to K_k .

Also multi-step Choquet integrals [20–22], which are multi-step aggregation functions and which are not Choquet integrals, in general, can be used to construct coherent upper previsions.

Example 11. Let $n = 2, k = 3, A = \max, B_1(x_1, x_2) = (\min\{x_1, x_2\} + 2 \max\{x_1, x_2\})/3$, and $B_2(x_3) = x_3$ then

$$C(x_1, x_2, x_3) = \max\left\{\frac{\min\{x_1, x_2\} + 2 \max\{x_1, x_2\}}{3}, x_3\right\},$$

and C belongs to K_3 and $C(\mathbf{1}) = 1 \neq 0$ hence it can be used, by Proposition 1 and Remark 1, to build the following coherent upper prevision

$$\text{cup}(x_1, x_2, x_3) = \frac{C(x_1, x_2, x_3)}{C(\mathbf{1})} = \max\left\{\frac{\min\{x_1, x_2\} + 2 \max\{x_1, x_2\}}{3}, x_3\right\},$$

Note that A, B_1 and B_2 are OWA operators and thus Choquet integrals, but C cannot be expressed as a Choquet integral.

5. Conclusions

In this paper a construction method, based on sub-additive positively homogeneous and shift-invariant aggregation functions is proposed to define coherent upper previsions. Moreover a sub-additive transformation of an aggregation function, named revenue transformation is introduced to obtain a sub-additive aggregation function. Under some mild additional constraints, this transformed aggregation function is then used to define a coherent upper prevision. As a distinguished example one can recall the Choquet integral. If the considered monotone measure μ is sub-modular then the related asymmetric Choquet integral Ch^μ is a coherent upper prevision. If μ is not sub-modular, clearly, Ch^μ is not a coherent upper prevision, but based on our proposal one can define a coherent upper prevision $\overline{\text{Ch}}^\mu$. It is an interesting open problem whether the operator $\overline{\text{Ch}}^\mu$ is comonotone additive and thus it can be represented as a Choquet integral Ch^ν , where ν is a sub-modular monotone measure given by $\nu(E) = \overline{\text{Ch}}^\mu(\mathbf{1}_E)$.

We have considered finite σ -algebras \mathcal{A} only, being inspired by the motivation and basic examples of coherent upper (lower) previsions known from the literature. In our future work, we aim to focus on constructions of coherent upper (lower) previsions on general measurable spaces (Ω, \mathcal{A}) .

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