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A New Algebraic Inequality and Some Applications in Submanifold Theory

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Abstract: We give a simple proof of the Chen inequality involving the Chen invariant $\delta(k)$ of submanifolds in Riemannian space forms. We derive Chen's first inequality and the Chen–Ricci inequality. Additionally, we establish a corresponding inequality for statistical submanifolds.

Keywords: Riemannian space form; submanifold; Chen invariants; Chen inequalities; statistical manifold; statistical submanifold

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1. Introduction

One of the most important topics of research in the geometry of submanifolds in Riemanian manifolds is to establish sharp relationships between extrinsic and intrinsic invariants of a submanifold.

The most used intrinsic invariants are sectional curvature, scalar curvature and Ricci curvature. The main extrinsic invariant is the squared mean curvature.

There are well-known relationships between the above extrinsic and intrinsic invariants for a submanifold in a Riemannian space form: (generalized) Euler inequality, Chen–Ricci inequality, Wintgen inequality, etc.

In [1,2], B.-Y. Chen introduced a sequence of Riemannian invariants, which are known as Chen invariants. They are different in nature from the classical Riemannian invariants. B.-Y. Chen established optimal relationships between the squared mean curvature and Chen invariants for submanifolds in Riemannian space forms, known as Chen inequalities (see [2]). The proofs of these inequalities use an algebraic inequality, discovered by B.-Y. Chen in [1].

In the present paper, we give simple proofs of some Chen inequalities by using a different algebraic inequality.

Other Chen inequalities were proved in [3] by applying another inequality.

2. Preliminaries

The theory of Chen invariants and Chen inequalities was initiated by B.-Y. Chen [1,2].

Let (M, g) be an n-dimensional ($n \geq 2$) Riemannian manifold, ∇ its Levi–Civita connection and R the Riemannian curvature tensor field on M. The sectional curvature $K(\pi)$ of the plane section $\pi \subset T_pM$, $p \in M$, is defined by

$$K(\pi) = R(e_1, e_2, e_1, e_2) = g(R(e_1, e_2)e_2, e_1),$$

where $\{e_1, e_2\}$ is an orthonormal basis of π.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM. The scalar curvature τ at p is given by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j.

If X is a unit vector tangential to M at p, consider the orthonormal basis $\{e_1 = X, e_2, ..., e_n\}$ of T_pM. The Ricci curvature is defined by

$$\text{Ric}\,(X) = \sum_{j=2}^{n} K(X \wedge e_j).$$

Let L be an r-dimensional subspace of T_pM and $\{e_1, ..., e_r\}$ an orthonormal basis of L, $2 \le r \le n$. The the scalar curvature $\tau(L)$ of L is given by $\tau(L) = \sum_{1 \le \alpha < \beta \le r} K(e_\alpha \wedge e_\beta)$.

In particular, for $r = 2$, $\tau(L)$ is the sectional curvature of L and for $r = n$, $\tau(T_pM) = \tau(p)$ is the scalar curvature of M at p.

B.-Y. Chen introduced a sequence of Riemannian invariants $\delta(n_1, ..., n_l)$, known as Chen invariants (see [2]).

The *Chen first invariant* is $\delta_M = \tau - \inf K$, where

$$(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_pM \text{ plane section}\}.$$

Let $l > 0$ be an integer and $n_1, ..., n_l \ge 2$ integers such that $n_1 < n$ and $n_1 + ... + n_l \le n$. The *Chen invariant* $\delta(n_1, ..., n_l)$ is defined by

$$\delta(n_1, ..., n_l)(p) = \tau(p) - \inf\{\tau(L_1) + ... + \tau(L_l)\},$$

where $L_1, ..., L_l$ are mutually orthogonal subspaces of T_pM with $\dim L_j = n_j$, $j = 1, ..., l$.

For $l = 1$ in particular, one has $\delta(2) = \delta_M$ and $\delta(n-1) = \max \text{Ric}$, with

$$\max \text{Ric}(p) = \max\{\text{Ric}\,(X) | X \in T_pM, g(X, X) = 1\}.$$

We shall consider the Chen invariant $\delta(k)$, which is given by

$$\delta(k)(p) = \tau(p) - \inf \tau(L_k),$$

where L_k is any k-dimensional subspace of T_pM.

3. An Algebraic Inequality

In this section, we give an algebraic inequality and study its equality case. As an application, we get a simple proof of the Chen inequality for the invariant $\delta(k)$.

Lemma 1. *Let k, n be nonzero natural numbers, $2 \le k \le n-1$, and $a_1, a_2, ..., a_n \in \mathbb{R}$. Then*

$$\sum_{1 \le i < j \le n} a_i a_j - \sum_{1 \le \alpha < \beta \le k} a_\alpha a_\beta \le \frac{n-k}{2(n-k+1)} \left(\sum_{i=1}^{n} a_i \right)^2.$$

Moreover, the equality holds if and only if $\sum_{\alpha=1}^{k} a_\alpha = a_j$, for all $j \in \{k+1, ..., n\}$.

Proof. We prove this Lemma by using the Cauchy–Schwarz inequality. We have

$$\left(\sum_{i=1}^{n} a_i \right)^2 = \left(\sum_{\alpha=1}^{k} a_\alpha + a_{k+1} + ... + a_n \right)^2 \le$$

$$\leq (n-k+1)\left[\left(\sum_{\alpha=1}^{k} a_\alpha\right)^2 + a_{k+1}^2 + ... + a_n^2\right] =$$

$$= (n-k+1)\left(\sum_{i=1}^{n} a_i^2 + 2\sum_{1\leq\alpha<\beta\leq k} a_\alpha a_\beta\right) =$$

$$= (n-k+1)\left[\left(\sum_{i=1}^{n} a_i\right)^2 - 2\sum_{1\leq i<j\leq n} a_i a_j + 2\sum_{1\leq\alpha<\beta\leq k} a_\alpha a_\beta\right],$$

which implies the desired inequality.

The equality holds if and only if we have equality in the Cauchy–Schwarz inequality, i.e., $\sum_{\alpha=1}^{k} a_\alpha = a_j$, for all $j \in \{k+1, ..., n\}$. □

4. Proof of the Chen Inequality for $\delta(K)$

We apply Lemma 1 for obtaining a simple proof of the Chen inequality corresponding to the Chen invariant $\delta(k)$ for submanifolds in Riemannian space forms.

Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c. The Euclidean space $\mathbb{E}^m$, the sphere S^m and the hyperbolic space H^m are the standard examples.

Consider M an n-dimensional submanifold of $\tilde{M}(c)$ and denote by h the second fundamental form of M in $\tilde{M}(c)$. The mean curvature vector $H(p)$ at $p \in M$ is defined by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of T_pM.

The submanifold M is called minimal if the mean curvature vector $H(p)$ vanishes at any $p \in M$.

We recall the Gauss equation (see [4]):

$$R(X,Y,Z,W) = c + g(h(X,Z), h(Y,W)) - g(h(X,W), h(Y,Z)),$$

for all vector fields X, Y, Z, W tangential to M.

Theorem 1. *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. Then, for any $2 \leq k \leq n-1$, one has the following Chen inequality:*

$$\delta(k) \leq \frac{n^2(n-k)}{2(n-k+1)}||H||^2 + \frac{1}{2}[n(n-1) - k(k-1)]c.$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist suitable orthonormal bases $\{e_1, ..., e_n\} \subset T_pM$ and $\{e_{n+1}, ..., e_m\} \subset T_p^\perp M$ such that the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & ... & 0 & 0 & ... & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & ... & a_k & 0 & ... & 0 \\ 0 & ... & 0 & \mu & ... & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 0 & 0 & ... & \mu \end{pmatrix}, \quad \sum_{\alpha=1}^{k} a_\alpha = \mu,$$

$$A_{e_r} = \begin{pmatrix} A_r & 0 \\ 0 & \mathbf{O}_{n-k} \end{pmatrix}, \quad r = n+2, ..., m,$$

where A_r is a symmetric $k \times k$ matrix with trace $A_r = 0$ and $\mathbf{O}_{n-k}$ is the $(n-k) \times (n-k)$ null matrix.

Proof. Let $p \in M$, $L \subset T_pM$ be a k-dimensional subspace and $\{e_1, ..., e_k\}$ be an orthonormal basis of L. We take $\{e_1, ..., e_k, e_{k+1}, ..., e_n\} \subset T_pM$ and $\{e_{n+1}, ..., e_m\} \subset T_p^{\perp}M$ as orthonormal bases, respectively.

Denote as usual by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $i, j = 1, ..., n$, $r \in \{n+1, ..., m\}$, the components of the second fundamental form.

The Gauss equation implies

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j) = \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j) =$$

$$= \frac{n(n-1)}{2} c + \sum_{r=n+1}^{m} \sum_{1 \le i < j \le n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2].$$

Additionally, by the Gauss equation one has

$$\tau(L) = \frac{k(k-1)}{2} c + \sum_{r=n+1}^{m} \sum_{1 \le \alpha < \beta \le k} [h_{\alpha\alpha}^r h_{\beta\beta}^r - (h_{\alpha\beta}^r)^2].$$

Then we get

$$\tau - \tau(L) = \frac{1}{2}[n(n-1) - k(k-1)]c +$$

$$+ \sum_{r=n+1}^{m} \left(\sum_{1 \le i < j \le n} h_{ii}^r h_{jj}^r - \sum_{1 \le \alpha < \beta \le k} h_{\alpha\alpha}^r h_{\beta\beta}^r \right) - \sum_{r=n+1}^{m} \sum_{1 \le i < j \le n; (i,j) \notin \{1,...,k\}^2} (h_{ij}^r)^2.$$

By using the algebraic inequality from the previous section, we obtain

$$\tau - \tau(L) \le \frac{n-k}{2(n-k+1)} \sum_{r=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^r \right)^2 + \frac{1}{2}[n(n-1) - k(k-1)]c =$$

$$= \frac{n^2(n-k)}{2(n-k+1)} ||H||^2 + \frac{1}{2}[n(n-1) - k(k-1)]c,$$

which implies the inequality to prove.

If the equality case holds at a point $p \in M$, then we have equalities in all the inequalities in the proof, i.e.,

$$\begin{cases} \sum_{\alpha=1}^{k} h_{\alpha\alpha}^r = h_{jj}^r, \ \forall j \in \{k+1, ..., n\}, \\ h_{ij}^r = 0, \ \forall 1 \le i < j \le n, (i,j) \notin \{1, ..., k\}^2, \end{cases}$$

for any $r \in \{n+1, ..., m\}$.

If we choose e_{n+1} parallel to $H(p)$, then the shape operators take the above forms. □

Corollary 1. *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. If there exists a point $p \in M$ such that $\delta(k)(p) > \frac{1}{2}[n(n-1) - k(k-1)]c$, then M is not minimal.*

If $k = 1$, we derive Chen's first inequality:

Corollary 2. [1] *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. Then one has*

$$\inf K \geq \tau - \frac{n-2}{2}\left[\frac{n^2}{n-1}||H||^2 + (n+1)c\right].$$

Equality holds at a point $p \in M$ if and only if, with respect to suitable orthonormal bases $\{e_1, ..., e_n\} \subset T_pM$ and $\{e_{n+1}, ..., e_m\} \subset T_p^{\perp}M$, the shape operators take the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & ... & 0 \\ 0 & \mu - a & 0 & ... & 0 \\ 0 & 0 & \mu & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & \mu \end{pmatrix},$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & ... & 0 \\ h_{12}^r & -h_{11}^r & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}, \quad r = n+2, ..., m.$$

Recall that $\delta(n-1) = \max \text{Ric}$. Then, from Theorem 1 we deduce the Chen–Ricci inequality:

Corollary 3. [5] *Let $\tilde{M}(c)$ be an m-dimensional Riemannian space form of constant sectional curvature c and M an n-dimensional submanifold of $\tilde{M}(c)$. Then, for any $p \in M$ and any unit vector X tangential to M, one has*

$$\text{Ric}(X) \leq \frac{n^2}{4}||H||^2 + (n-1)c.$$

We present the following examples:

Example 1. *Let k, n be integers such that $k \geq 2$ and $n \geq 2k-1$. Consider the hypercylinder $M = S^k \times \mathbb{E}^{n-k} \subset \mathbb{E}^{n+1}$.*

Clearly $\delta(k) = \tau = \frac{1}{2}k(k-1)$. Then the equality case of Theorem 1 holds identically if and only $n = 2k-1$, i.e., $M = S^k \times \mathbb{E}^{k-1}$.

Moreover, $\max \text{Ric} = \frac{n^2}{4}||H||^2$ if and only if $k = 2$ and $n = 3$, i.e., $M = S^2 \times \mathbb{E}$.

Example 2. *The generalized Clifford torus.*

Let $T = S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}) \subset S^{n+1} \subset \mathbb{E}^{n+2}$, $n > k \geq 1$.

It is known that T is a minimal hypersurface of S^{n+1}, but a non-minimal submanifold of $\mathbb{E}^{n+2}$.

Obviously $\max \text{Ric} = \max\{(k-1)\frac{n}{k}, (n-k-1)\frac{n}{n-k}\}$.

Then $T \subset S^{n+1}$ does not satisfy the equality case of Theorem 1 for $\delta(n-1)$, $\forall n \geq 2$.

If we consider $T \subset \mathbb{E}^{n+2}$, then it does not satisfy the equality case of Theorem 1 for $\delta(n-1)$, $\forall n \geq 2$.

5. A Chen Inequality for Statistical Submanifolds

A statistical manifold is an m-dimensional Riemannian manifold $(\tilde{M}, g)$ endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, which satisfy

$$Zg(X,Y) = g(\tilde{\nabla}_Z X, Y) + g(X, \tilde{\nabla}^*_Z Y), \quad (1)$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$. The connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ are called dual connections (see [6,7]), and it is easily seen that $\left(\tilde{\nabla}^*\right)^* = \tilde{\nabla}$. The pairing $(\tilde{\nabla}, g)$ is said to be a statistical structure. If $\left(\tilde{\nabla}, g\right)$ is a statistical structure on $\tilde{M}^m$, then $\left(\tilde{\nabla}^*, g\right)$ is a statistical structure too [6,8].

Any torsion-free affine connection $\tilde{\nabla}$ on $\tilde{M}$ always has a dual connection given by

$$\tilde{\nabla} + \tilde{\nabla}^* = 2\tilde{\nabla}^0,$$

where $\tilde{\nabla}^0$ is the Levi–Civita connection on $\tilde{M}$.

The dual connections are called conjugate connections in affine differential geometry (see [9]).

Denote by $\tilde{R}$ and $\tilde{R}^*$ the curvature tensor fields of $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. They satisfy

$$g\left(\tilde{R}^*(X,Y)Z, W\right) = -g\left(Z, \tilde{R}(X,Y)W\right). \quad (2)$$

A statistical structure $\left(\tilde{\nabla}, g\right)$ is said to be of constant curvature $\varepsilon \in \mathbb{R}$ if

$$\tilde{R}(X,Y)Z = \varepsilon\{g(Y,Z)X - g(X,Z)Y\}. \quad (3)$$

A statistical structure $\left(\tilde{\nabla}, g\right)$ of constant curvature 0 is called a Hessian structure.

The Equation (2) implies that if $\left(\tilde{\nabla}, g\right)$ is a statistical structure of constant curvature ε, then $\left(\tilde{\nabla}^*, g\right)$ is also a statistical structure of constant curvature ε (obviously, if $(\tilde{\nabla}, g)$ is Hessian, $(\tilde{\nabla}^*, g)$ is also Hessian).

The dual connections are not metric, then we cannot define a sectional curvature in the standard way. A sectional curvature on a statistical manifold was defined by B. Opozda [10].

More precisely, if one considers $p \in \tilde{M}$, π a plane section in $T_p\tilde{M}$ and an orthonormal basis $\{X, Y\}$ of π, then a sectional curvature is defined by

$$\tilde{K}(\pi) = \frac{1}{2}[g(\tilde{R}(X,Y)Y + \tilde{R}^*(X,Y)Y, X)],$$

which is independent of the choice of the orthonormal basis.

Next, we consider a statistical manifold $(\tilde{M}, g)$ and a submanifold M of dimension n of $\tilde{M}$. Then $(M, g|_M)$ is also a statistical manifold with the connection induced by $\tilde{\nabla}$ and induced metric g.

In Riemannian geometry, the fundamental equations are the Gauss and Weingarten formulae and the equations of Gauss, Codazzi and Ricci.

As usual, we denote by $\Gamma\left(T^\perp M\right)$ the set of the sections of the bundle normal to M.

In our case, for any $X, Y \in \Gamma(TM)$, according to [8], the corresponding Gauss formulae are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),$$
$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y),$$

where $h, h^* : \Gamma(TM) \times \Gamma(TM) \to \Gamma(T^\perp M)$ are symmetric and bilinear, called the imbedding curvature tensor (see [6,8]) of M in $\tilde{M}$ for $\tilde{\nabla}$ and the imbedding curvature tensor of M in $\tilde{M}$ for $\tilde{\nabla}^*$, respectively.

In [8], it was also proven that (∇, g) and (∇^*, g) are dual statistical structures on M.

Since h and h^* are bilinear, there are linear transformations A_ξ and A^*_ξ on TM defined by

$$g\left(A_\xi X, Y\right) = g(h(X,Y), \xi),$$

$$g\left(A^*_\xi X, Y\right) = g(h^*(X,Y), \xi),$$

for any $\xi \in \Gamma\left(T^\perp M\right)$ and $X, Y \in \Gamma(TM)$.

Further (see [8]), the corresponding Weingarten formulae are

$$\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^\perp_X \xi,$$

$$\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi,$$

for any $\xi \in \Gamma\left(T^\perp M\right)$ and $X \in \Gamma(TM)$. The connections $\nabla^\perp$ and $\nabla^{*\perp}$ are Riemannian dual connections with respect to the induced metric on $\Gamma\left(T^\perp M\right)$.

Let $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_m\}$ be orthonormal tangential and normal frames, respectively, on M. Then the mean curvature vector fields are defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h^\alpha_{ii}\right)e_\alpha,\ h^\alpha_{ij} = g(h(e_i, e_j), e_\alpha),$$

and

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h^{*\alpha}_{ii}\right)e_\alpha,\ h^{*\alpha}_{ij} = g(h^*(e_i, e_j), e_\alpha),$$

for $1 \le i, j \le n$ and $n+1 \le \alpha \le m$.

The Gauss equations for the dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively, are given by (see [8])

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + g(h(X,Z), h^*(Y,W)) - \\ &\quad - g(h^*(X,W), h(Y,Z)), \\ g\left(\tilde{R}^*(X,Y)Z,W\right) &= g(R^*(X,Y)Z,W) + g(h^*(X,Z), h(Y,W)) - \\ &\quad - g(h(X,W), h^*(Y,Z)), \end{aligned}$$

Geometric inequalities for statistical submanifolds in statistical manifolds with constant curvature were obtained in [11].

In this section we prove the Chen inequality corresponding to the Chen invariant $\delta(k)$ for statistical submanifolds in statistical manifolds of constant curvature.

We consider an m-dimensional statistical manifold $\tilde{M}(\varepsilon)$ of constant curvature ε and an n-dimensional statistical submanifold M. Let $p \in M$ and L be a k-dimensional subspace of T_pM. Denote by $\{e_1, ..., e_k\}$ an orthonormal basis of L, $\{e_1, ..., e_k, e_{k+1}, ..., e_n\}$ an orthonormal basis of T_pM and $\{e_{n+1}, ..., e_m\}$ an orthonormal basis of $T^\perp_p M$, respectively.

The Gauss equation implies

$$\begin{aligned}
\tau &= \frac{1}{2}\sum_{1\leq i<j\leq n}[g(R(e_i,e_j)e_j,e_i)+g(R^*(e_i,e_j)e_j,e_i)] = \\
&= \frac{n(n-1)}{2}\varepsilon+\frac{1}{2}\sum_{1\leq i<j\leq n}[g(h^*(e_i,e_i),h(e_j,e_j))+g(h(e_i,e_i),h^*(e_j,e_j))- \\
&\quad -2g(h(e_i,e_j),h^*(e_i,e_j))] = \\
&= \frac{n(n-1)}{2}\varepsilon+\frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\leq i<j<n}(h_{ii}^{*r}h_{jj}^{r}+h_{ii}^{r}h_{jj}^{*r}-2h_{ij}^{r}h_{ij}^{*r}) = \\
&= \frac{n(n-1)}{2}\varepsilon+\frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}[(h_{ii}^{r}+h_{ii}^{*r})(h_{jj}^{r}+h_{jj}^{*r})-h_{ii}^{r}h_{jj}^{r}-h_{ii}^{*r}h_{jj}^{*r}- \\
&\quad -(h_{ij}^{r}+h_{ij}^{*r})^2+(h_{ij}^{r})^2+(h_{ij}^{*r})^2] = \\
&= \frac{n(n-1)}{2}\varepsilon+\sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}\Big\{2[h_{ii}^{0r}h_{jj}^{0r}-(h_{ij}^{0r})^2]-\frac{1}{2}[h_{ii}^{r}h_{jj}^{r}-(h_{ij}^{r})^2]- \\
&\quad -\frac{1}{2}[h_{ii}^{*r}h_{jj}^{*r}-(h_{ij}^{*r})^2]\Big\},
\end{aligned} \tag{4}$$

where h^0 is the second fundamental form of the Riemannian submanifold M.

We denote by τ_0 the scalar curvature with respect to the Levi–Civita connection and by $\tilde{\tau}_0 = \sum_{1\leq i<j\leq n}\tilde{K}_0(e_i\wedge e_j)$.

The Gauss equation with respect to the Levi–Civita connection gives

$$\tau_0 = \tilde{\tau}_0 + \sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}[h_{ii}^{0r}h_{jj}^{0r}-(h_{ij}^{0r})^2]. \tag{5}$$

By substituting Equation (5) into (4), we get

$$\begin{aligned}
\tau = \ & 2(\tau_0-\tilde{\tau}_0)+\tfrac{n(n-1)}{2}\varepsilon-\tfrac{1}{2}\textstyle\sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}[h_{ii}^{r}h_{jj}^{r}-(h_{ij}^{r})^2]- \\
& -\tfrac{1}{2}\textstyle\sum_{r=n+1}^{m}\sum_{1\leq i<j\leq n}[h_{ii}^{*r}h_{jj}^{*r}-(h_{ij}^{*r})^2].
\end{aligned} \tag{6}$$

By using Gauss equation, we have

$$\tau(L) = \frac{1}{2}\sum_{1\le\alpha<\beta\le k}[g(R(e_\alpha,e_\beta)e_\beta,e_\alpha) + g(R^*(e_\alpha,e_\beta)e_\beta,e_\alpha)] =$$

$$= \frac{k(k-1)}{2}\varepsilon + \frac{1}{2}\sum_{1\le\alpha<\beta\le k}[g(h^*(e_\alpha,e_\alpha),h(e_\beta,e_\beta)) + g(h(e_\alpha,e_\alpha),h^*(e_\beta,e_\beta)) -$$
$$- 2g(h(e_\alpha,e_\beta),h^*(e_\alpha,e_\beta))] =$$
$$= \frac{k(k-1)}{2}\varepsilon + \frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le k}(h^{*r}_{\alpha\alpha}h^r_{\beta\beta} + h^r_{\alpha\alpha}h^{*r}_{\beta\beta} - 2h^r_{\alpha\beta}h^{*r}_{\alpha\beta}) =$$
$$= \frac{k(k-1)}{2}\varepsilon + \frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le k}[(h^r_{\alpha\alpha}+h^{*r}_{\alpha\alpha})(h^r_{\beta\beta}+h^{*r}_{\beta\beta}) -$$
$$- h^r_{\alpha\alpha}h^r_{\beta\beta} - h^{*r}_{\alpha\alpha}h^{*r}_{\beta\beta} - (h^r_{\alpha\beta}+h^{*r}_{\alpha\beta})^2 + (h^r_{\alpha\beta})^2 + (h^{*r}_{\alpha\beta})^2] =$$
$$= \frac{k(k-1)}{2}\varepsilon + \sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le k}\Big\{2[h^{0r}_{\alpha\alpha}h^{0r}_{\beta\beta} - (h^{0r}_{\alpha\beta})^2] - \frac{1}{2}[h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2] -$$
$$- \frac{1}{2}[h^{*r}_{\alpha\alpha}h^{*r}_{\beta\beta} - (h^{*r}_{\alpha\beta})^2]\Big\} =$$
$$= 2\tau_0(L) - 2\tilde{\tau}_0(L) + \frac{k(k-1)}{2}\varepsilon -$$
$$- \frac{1}{2}\sum_{r=n+1}^{m}\sum_{1\le\alpha<\beta\le k}\Big\{[h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2] + [h^{*r}_{\alpha\alpha}h^{*r}_{\beta\beta} - (h^{*r}_{\alpha\beta})^2]\Big\}.$$

By subtracting the last equation from (4), we obtain

$$(\tau - \tau(L)) - 2(\tau_0 - \tau_0(L)) \ge 2(\tilde{\tau}_0(L) - \tilde{\tau}_0) + \frac{1}{2}[n(n-1) - k(k-1)]\varepsilon -$$
$$- \frac{1}{2}\sum_{r=n+1}^{m}\left(\sum_{1\le i<j\le n}h^r_{ii}h^r_{jj} - \sum_{1\le\alpha<\beta\le k}h^r_{\alpha\alpha}h^r_{\beta\beta}\right) -$$
$$- \frac{1}{2}\sum_{r=n+1}^{m}\left(\sum_{1\le i<j\le n}h^{*r}_{ii}h^{*r}_{jj} - \sum_{1\le\alpha<\beta\le k}h^{*r}_{\alpha\alpha}h^{*r}_{\beta\beta}\right).$$

We denote by $\max\tilde{K}_0(p)$ the maximum of the Riemannian sectional curvature function of $\tilde{M}(\varepsilon)$ restricted to 2-plane sections of the tangent space T_pM, $p \in M$. Obviously

$$\tilde{\tau}_0 - \tilde{\tau}_0(L) \le \frac{1}{2}[n(n-1) - k(k-1)]\max\tilde{K}_0(p).$$

On the other hand, by using Lemma 1, one has

$$\sum_{1\le i<j\le n}h^r_{ii}h^r_{jj} - \sum_{1\le\alpha<\beta\le k}h^r_{\alpha\alpha}h^r_{\beta\beta} \le \frac{n-k}{2(n-k+1)}\left(\sum_{i=1}^{n}h^r_{ii}\right)^2,$$

$$\sum_{1\le i<j\le n}h^{*r}_{ii}h^{*r}_{jj} - \sum_{1\le\alpha<\beta\le k}h^{*r}_{\alpha\alpha}h^{*r}_{\beta\beta} \le \frac{n-k}{2(n-k+1)}\left(\sum_{i=1}^{n}h^{*r}_{ii}\right)^2.$$

It follows that

$$\tau - \tau(L) \ge 2(\tau_0 - \tau_0(L)) + \frac{1}{2}[n(n-1) - k(k-1)](\varepsilon - 2\max\tilde{K}_0(p)) -$$

$$-\frac{n^2(n-k)}{2(n-k+1)}[||H||^2+||H^*||^2].$$

We state the following result.

Theorem 2. *Let M be an n-dimensional statistical submanifold of an m-dimensional statistical manifold $\tilde{M}(\varepsilon)$ of constant curvature. Then, for any $p \in M$ and any k-plane section L of T_pM, we have:*

$$\tau_0-\tau_0(L)\leq\frac{1}{2}(\tau-\tau(L))+\frac{n^2(n-k)}{4(n-k+1)}[||H||^2+||H^*||^2]+$$

$$+\frac{1}{2}[n(n-1)-k(k-1)](\max\tilde{K}_0(p)-\frac{\varepsilon}{2}).$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist orthonormal bases $\{e_1,...,e_n\}$ of T_pM and $\{e_{n+1},...,e_m\}$ of $T_p^{\perp}M$ such that

$$\begin{cases}\sum_{\alpha=1}^{k}h^r_{\alpha\alpha}=h^r_{jj},\ \forall j\in\{k+1,...,n\},\\ \sum_{\alpha=1}^{k}h^{*r}_{\alpha\alpha}=h^{*r}_{jj},\ \forall j\in\{k+1,...,n\},\\ h^r_{ij}=h^{*r}_{ij}=0,\ \forall 1\leq i<j\leq n,(i,j)\notin\{1,...,k\}^2,\end{cases}$$

for any $r \in \{n+1,...,m\}$.

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