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Tsallis Log-Scale-Location Models. Moments, Gini Index and Some Stochastic Orders

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Abstract: In this article we give theoretical results for different stochastic orders of a log-scale-location family which uses Tsallis statistics functions. These results describe the inequalities of moments or Gini index according to parameters. We also compute the mean in the case of q -Weibull and q -Gaussian distributions. The paper is aimed at analyzing the order between survival functions, Lorenz curves and (as consequences) the moments together with the Gini index (respectively a generalized Gini index). A real data application is presented in the last section. This application uses only the survival function because the stochastic order implies the order of moments. Given some supplementary conditions, we prove that the stochastic order implies the Lorenz order in the log-scale-location model and this implies the order between Gini coefficients. The application uses the estimated parameters of a Pareto distribution computed from a real data set in a log-scale-location model, by specifying the Kolmogorov–Smirnov p -value. The examples presented in this application highlight the stochastic order between four models in several cases using survival functions. As direct consequences, we highlight the inequalities between the moments and the generalized Gini coefficients by using the stochastic order and the Lorenz order.

Keywords: Gini index; Tsallis statistics; stochastic orders



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1. Introduction

The main purpose of this article is to introduce a family of lifetime distributions, called the Tsallis log-scale-location family. For this family of distributions, we analyze different stochastic orders and also the order between some characteristics of distributions, for example, between the Gini index or the moments. Given the specificity of our research, we will briefly present in the sequel some important topics and we also give some classical and recent references for the main ingredients of our work: (i) survival analysis; (ii) the Gini index and corresponding applications in economy, demography and social sciences in general; (iii) Tsallis statistics and related concepts, like Tsallis exponential and logarithm functions, q -exponential family, maximum entropy principle, etc. (iv) stochastic orders and related properties. Since our work is at the crossroads of these different research directions, we think that it is useful to provide a short overview of some important problems in these fields that are related to our work.

In risk theory, life expectancy is a significant measure. Due to the high average standard of living of some countries lately, there is an interest in studying the extent to which this standard of living is equally accessible to all people. That is why lately many people are studying the measures of variability in terms of lifespan (Anand et al. [1]). The

changes that characterize the changing stage of mortality are measured by variables such as age-specific death rates, life expectancy at birth, probabilities of death and survival function. Survival analysis is used in medicine, biology, social sciences such as economics, engineering (reliability and failure time analysis) and many other sciences. Survival analysis methods depend on the distribution of survival and on the hazard function. Parametric models are in practice easy to adapt and process because they are defined by a small and fixed number of unknown parameters. This allows one to use standard statistical methods in order to carry out statistical inference. These techniques depend on the adequacy of the specific parametric model used. For example, in biomedical applications, nonparametric (e.g., estimator of the survival curve) and semi-parametric (e.g., Cox proportional risk model) models are most important because they have the flexibility to adapt to a wide range of forms of hazard functions. Even so, parametric models are used in biomedical research and may be appropriate when the set of survival data indicates approximately a parametric form.

Another important subject for our work is the Gini coefficient or index. The Gini coefficient is the most common statistical index of diversity or inequality in social sciences (see, e.g., Gini [2,3], Nygard and Sandröm [4], Kakwani [5], Kendall et al. [6], Allison [7]). It is used in econometrics as a standard measure of inter-individual or inter-household inequality in income and wealth (Atkinson [8,9], Sen [10], Anand [11]). Illsey and Le Grand [12], who justified the use of Gini coefficient for the analysis of inequality in health in the 1980s, stressed that the individual-based measurement of inequality in health is a way to a universal comparability of degrees of inequality over time and across countries. They also computed Gini coefficient from distributions of deaths by age in real populations. Other researchers linked the Gini coefficient and other measures of inter-individual inequality in age at death with the life table (Hanada [13], Silber [14], Wilmoth and Horiuchi [15]). Hicks proposed to use the Gini coefficient to adjust average life expectancy for variability in order to construct the inequality-adjusted human development index (Hicks [16]). Recent interesting articles were proposed by Kim and Kim [17], Bonetti et al. [18], Ostasiewicz and Mazurek [19]. Among the recent works related to the Gini index and statistical extensions and applications, we can mention the Gini regressions and the principal component analysis based on the Gini correlation matrix developed in Charpentier et al. [20,21] and linear discriminant analysis based on generalized Gini correlation indexes proposed in Condevaux et al. [22]. It is worth noting that the Gini index can be expressed in terms of the Lorenz curve $L(p)$ as the area between the first diagonal (equality) and the Lorenz curve, divided by the whole area below the diagonal. If we compute the Gini index on a finite sample of observations, it is equal to zero if all individuals die at the same age and it is equal to one if all the individuals except one of them die at birth and this remaining one dies at a positive age.

An important concept in demography is the concept of longevity that denotes the long duration of life and is used as a synonym for high life expectancy. It is well known that a significant increase in longevity has been observed during the past several centuries: in particular, bestpractice life expectancy at birth has risen by 2.5 years per decade since the 1840 (Oeppen and Vaupel [23]). Such increase in life expectancy is one of the consequences of changes in the survival distribution of the population over different cohorts. Evidence suggests that higher life expectancy at birth is associated with a lower concentration of survival times, both cross-country and over time. An empirical analysis of approximately 45 countries for the years 1960–1990 reveals a tight negative association between life expectancy at birth and the Gini coefficient (Shkolnikov et al. [24]). Specifically, during the first three quarters of the 20th century the inter-individual inequality in length of life has been declining.

Several distributions have been proposed to model real lifetime data. The Weibull distribution is one of the most commonly used distributions for this purpose. In practice, it has been shown to be very flexible in modeling various types of lifetime data with monotone failure rates but it is not useful for modeling the bathtub shaped and the

unimodal failure rates, which are common in reliability and biological studies. It is of utmost interest because of its great number of special features and its ability to fit data from various fields, ranging from life data to observations made in economics and business administration, meteorology, hydrology, quality control, acceptance sampling, statistical process control, inventory control, physics, chemistry, geology, geography, astronomy, medicine, psychology, material science, engineering, biology. Iriarte et al. [25] introduced a new probability distribution class (Lambert-F Distributions Class) and they analyzed the hazard rate function. Al-Mofleh et al. [26] proposed a new two-parameter generalized Ramos–Louzada distribution and analyzed the hazard rate function. Gigliarano et al. [27] worked with the log-scale-location model. They analyzed the Gini index and the first moment of this model. Important analyzes on the population were made by Haberman and Renshaw [28], Finkelstein [29], Debon et al. [30], Canudas-Romo [31], Brown et al. [32], Booth and Tickle [33]. Hazra et al. [34] considered the location-scale family of distributions and derived conditions under which the largest order statistic of a set of random variables with different/the same location as well as different/the same scale parameters dominates that of another set of random variables with respect to various stochastic orders.

Recently, the notion of the exponential family has been generalized by Naudts [35–40]. The same definition of generalized exponential family has been introduced in the mathematical literature by Naudts [36,40], Grunwald and Dawid [41], Eguchi [42], Briggs and Beck [43]. This class of models was also derived using the maximum entropy principle in Abe [44], Hanel and Thurner [45] and in the context of game theory by Topsoe [46,47]. The notion of q -exponential family is connected with Amari's family (see Amari [48]), studied in the context of information geometry. The geometric approach is very appealing also in the context of statistical physics (see, for instance, [49,50]). The q -deformed exponential and logarithmic functions were first introduced in Tsallis' statistics in 1994 [51].

The last important concept for our work is the one of stochastic order. It is clear that stochastic orders provide methods of comparing random variables and vectors which are now used in many areas such as statistics, operations research, biomathematics, actuarial sciences, economic theory, queuing theory, risk management and other related fields. For a comprehensive review of the properties and characterizations of stochastic orderings, including a variety of applications, the reader is referred to the monographs of Shaked and Shantikumar [52], Levy [53], Denuit et al. [54], Balakrishnan et al. [55]. Many of these orders have characterizations as so-called integral stochastic orders which is obtained by comparing expectations of functions in a certain class. Lando and Bertoli-Barsotti [56] obtained a method for deriving second-order stochastic dominance between multiparametric families which can be decomposed into a functional composition of two cumulative distributions and a quantile function. The method is applied to stochastic comparisons of order statistics. Recently, Sarabia et al. [57] introduced a general class of multivariate GB2 distributions based on a generalization of the order statistics distribution, its construction resulting in a multivariate GB2 distribution with support above the diagonal. Aijaz et al. [58] introduced a new Hamza two parameter distribution and studied its properties, including the moments, stochastic orderings, Bonferroni and Lorenz curves, Rényi entropy, order statistics, hazard rate function and mean residual function. Analytic representations of the multivariate Lorenz surface for a relevant type of models based on the class of distributions with given marginals described by Sarmanov and Lee have been obtained recently by Sarabia and Jorda [59]. Das and Kayal [60] obtained ordering results for the largest and the smallest order statistics arising from dependent heterogeneous exponentiated location-scale random observations, for the case that the sets of observations follow a common or different Archimedean copulas. Moreover, sufficient conditions for which the usual stochastic order and the reversed hazard rate order between the extreme order statistics hold have been derived. Aijaz et al. [61] proposed the inverse analogue of Ailamujia distribution. The relevant statistical properties of the new distribution investigated include moments, moment generating function, order statistics, survival measures, Shannon entropy, mode and median. Recently, Castaño-Martínez et al. [62] extended the results related to

the increasing convex order of relative spacings for two distributions from consecutive spacings to the case of general spacings. Panja et al. [63] considered stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components with lifetimes following the proportional odds model and component lifetimes joint distribution modeled by Archimedean survival copula. By comparisons of heterogeneous series systems with location-scale family distributed components, Kundu and Chowdhury [64] proved that the systems with dependent series components modeled by Archimedean copula with more dispersion in the location or scale parameters perform better in the sense of the usual stochastic order.

Taking into account all these bibliographic references and associated discussions that we have presented up to this moment, we can state now that the main objective of our work is to introduce a generalized log-scale-location family of distributions that extends existing classes from the literature. Thus we obtain a more flexible model, interesting for lifetime applications in various fields. Then, for this family of distributions, we show that some types of stochastic orders are preserved, under certain conditions. To illustrate our findings, we consider the data from a simpler model existing in the literature and, applying our generalized log-scale-location model, we illustrate graphically that certain stochastic orders are preserved, which is coherent with some of our theoretical results obtained in the article.

The paper is organized as follows. Section 2 introduces some preliminary notions and results. In Section 3 we define the generalized (X_0, q, b, u, x) -log-scale-location model and we analyze the moments of these models in Section 4. Necessary or sufficient conditions for usual stochastic order are derived in Section 5, while necessary conditions for the Lorenz order, order for Gini index and generalized Gini index are obtained in Section 6. The hazard rate order is analyzed in Section 7, while in Section 8 the excess wealth order and convex order are studied. A real data application is presented in Section 9 and some general conclusions of the article are given in the last section.

2. Preliminaries

In this section we introduce some notation and basic definitions that will be used along the article. Except some classical notions and notations, we will give here the definitions of Lorenz curve and Gini index of a random variable, the notions of q -deformed Tsallis exponential function and q -deformed Tsallis logarithm function, the associated q -Weibull and q -Normal distributions, some notions of stochastic ordering and relationships between them.

Let (Ω, \mathcal{F}, P) be a probability space and $X : \Omega \rightarrow \mathbb{R}$ a random variable. We denote by $F_X(x)$ the corresponding distribution function, $F_X(x) = P(X \leq x)$, and by $\bar{F}_X(x)$ the corresponding survival function, $\bar{F}_X(x) = 1 - F_X(x)$, $x \in \mathbb{R}$. We set $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

For a function $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x)_+ = \max(g(x), 0)$ and $g(x)_- = \min(g(x), 0)$. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is:

- (i) non-decreasing (or increasing) if $f(x) \leq f(y)$ for all $x, y \in \mathbb{R}$ with $x \leq y$;
- (ii) non-increasing (or decreasing) if $f(x) \geq f(y)$ for all $x, y \in \mathbb{R}$ with $x \leq y$.

If X is absolutely continuous with respect to the Lebesgue measure, then we denote $f_X(x) = (F_X(x))'$ its density function. We also denote $Q_X(p) = \inf\{x \in \mathbb{R} : p \leq F_X(x)\}$ the inferior quantile function of X . We will also use the notation $F_X^{-1}(p) = Q_X(p)$.

If \bar{F}_X is differentiable, we define the hazard rate function $r_X : \text{Supp}(\bar{F}_X) \rightarrow \mathbb{R}$, $r_X = (-\ln \bar{F}_X)' = \frac{f_X}{\bar{F}_X}$, where for a function $g : \mathbb{R} \rightarrow \mathbb{R}$, $\text{Supp}(g) = \{x \in \mathbb{R} : g(x) \neq 0\}$.

For a positive random variable X with $EX \neq 0$, we introduce the Lorenz curve of X defined by

$$L_X(p) = \frac{\int_0^p Q_X(u) du}{EX}$$

and also the Gini index of X (see Arnold [65,66]) defined

$$G_X = 1 - 2 \int_0^1 L_X(p) dp$$

Other formulas for computing the Gini index of X can also be derived (see, e.g., Gigliarano et al. [27]) are the following:

$$G_X = 1 - \frac{\int_0^\infty \overline{F}_X^2(x) dx}{\int_0^\infty \overline{F}_X(x) dx}$$

and

$$G_X = \frac{\int_0^\infty \int_0^\infty |t_1 - t_2| dF_X(t_1) dF_X(t_2)}{2EX}.$$

For $a \geq 1$, one can define the so-called generalized a -Gini index by

$$G_{X;a} = 1 - \frac{\int_0^\infty (\overline{F}_X)^a(x) dx}{\int_0^\infty \overline{F}_X(x) dx}.$$

We notice that $G_{X;2} = G_X$ and a simple calculation shows that

$$G_{X;a} = 1 - a(a - 1) \int_0^1 (1 - p)^{a-2} L_X(p) dp.$$

Let us now recall the definitions of the q -deformed Tsallis logarithm function and of the q -deformed Tsallis exponential function introduced by Tsallis [51] and also give some properties.

Definition 1. For a real number $q \leq 1$, the q -deformed Tsallis logarithm function is $lt_q : (0, \infty) \rightarrow \mathbb{R}$ with

$$lt_q(x) = \frac{x^{1-q} - 1}{1 - q}, \text{ if } q < 1$$

and

$$lt_q(x) = \ln x, \text{ if } q = 1.$$

All along this article we will use the notations $lt_q(x)$ or $lt_q x$ for the q -deformed Tsallis logarithm function computed in a point x , $x > 0$.

Remark 1. The function lt_q has the following properties:

- (i) $lt_q(1) = 0$.
- (ii) $x \mapsto lt_q(x)$ is strictly non-decreasing function on $(0, \infty)$ because $\frac{\partial}{\partial x} lt_q(x) = x^{-q} > 0$ for all $x > 0$.

Definition 2. For a real number $q \leq 1$, the q -deformed Tsallis exponential function is $ets_q : \mathbb{R} \rightarrow [0, \infty)$ with

$$ets_q(x) = [1 + (1 - q)x]_+^{\frac{1}{1-q}}, \text{ if } q < 1$$

and

$$ets_q(x) = e^x, \text{ if } q = 1.$$

All along this article we will use the notations $ets_q(x)$ or $ets_q x$ for the q -deformed Tsallis exponential function computed in a point $x \in \mathbb{R}$.

Remark 2. The function ets_q has the following properties:

- (i) $ets_q(0) = 1$.
- (ii) $\frac{\partial}{\partial x}ets_q(x) = (1 - q) \cdot (ets_q(x))^q > 0$ for all $x \in \left(-\frac{1}{1-q}, \infty\right)$ and $\frac{\partial}{\partial x}ets_q(x) = 0$ for all $x \in \left(-\infty, -\frac{1}{1-q}\right)$.
- (iii) $x \mapsto ets_q(x)$ is non-decreasing function on \mathbb{R} .
- (iv) For $x > 0$, we have $ets_q(lt_q(x)) = x$.
- (v) For $x \in \left(-\frac{1}{1-q}, \infty\right)$, we have $lt_q(ets_q(x)) = x$.

These functions are equally studied in Naudts [67], that provides also the notion of q -exponential family.

The q -deformed Tsallis exponential function and q -deformed Tsallis logarithm function allow one to introduce new random variables, by analogy with the ones defined using classical exponential and logarithm functions. We will introduce now the notions of q -Weibull distribution and q -Normal distribution.

Definition 3. We say that X is q -Weibull distributed with $k, \lambda > 0, q \leq 1$, and denote it by $X \sim q$ -Weibull(k, λ), if

$$f_X(x) = (2 - q) \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} ets_q\left(-\left(\frac{x}{\lambda}\right)^k\right) \cdot 1_{[0, \infty)}(x).$$

In this case:

$$\begin{aligned} \bar{F}_X(x) &= ets_{q'}\left(-\left(\frac{x}{\lambda'}\right)^k\right), \text{ if } x \geq 0 \text{ and} \\ \bar{F}_X(x) &= 1, \text{ if } x < 0, \text{ where} \\ q' &= \frac{1}{2-q}, \lambda' = \frac{\lambda}{(2-q)^{\frac{1}{k}}}. \end{aligned}$$

For the general form of this distribution and further investigations, one can see Picoli et al. [68].

Definition 4. We say that X is q -Normal distributed with parameters $\beta > 0, q \leq 1$, and denote it by $X \sim q$ -N(β, C_q), if

$$\begin{aligned} f_X(x) &= \frac{\sqrt{\beta}}{C_q} ets_q(-\beta x^2), \text{ where} \\ C_q &= \frac{2\sqrt{\pi}\Gamma\left(\frac{1}{1-q}\right)}{(3-q)\sqrt{1-q}\Gamma\left(\frac{3-q}{2(1-q)}\right)}, \text{ if } q < 1, \\ C_q &= \sqrt{\pi}, \text{ if } q = 1. \end{aligned}$$

For further investigations on this distribution and related topics, like q -Central Limit Theorem, one can see Umarov et al. [69].

Let us now recall some definitions of stochastic orders and also some properties of these orders. All these definitions and results can be founded in Shaked and Shantikumar [52].

Definition 5 (cf. [52]). Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random variables. X is said to be smaller than Y in the

- (i) stochastic order (written as $X \prec_{st} Y$) if $\bar{F}_X(x) \leq \bar{F}_Y(x) \forall x \in \mathbb{R}$;
- (ii) hazard rate order (written as $X \prec_{hr} Y$) if $r_X(x) \geq r_Y(x) \forall x \in \text{Supp}(\bar{F}_X) \cap \text{Supp}(\bar{F}_Y)$;
- (iii) Lorenz order (written as $X \prec_{Lorenz} Y$) if $L_X(p) \geq L_Y(p) \forall p \in [0; 1]$;
- (iv) dispersive order (written as $X \prec_{disp} Y$) if $F_X^{-1}(\beta) - F_X^{-1}(\alpha) \leq F_Y^{-1}(\beta) - F_Y^{-1}(\alpha) \forall 0 < \alpha \leq \beta < 1$;
- (v) excess wealth order (written as $X \prec_{ew} Y$) if $\int_{F_X^{-1}(p)}^{\infty} \bar{F}_X(x) dx \leq \int_{F_Y^{-1}(p)}^{\infty} \bar{F}_Y(x) dx \forall p \in (0, 1)$.

Another equivalent definition for stochastic order is given in the next definition.

Definition 6. [cf. [52]] Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random variables. X is said to be smaller than Y in the stochastic order (written as $X \prec_{st} Y$) if $Eu(X) \leq Eu(Y)$, for all non-decreasing functions $u : \mathbb{R} \rightarrow \mathbb{R}$, provided that the means exists.

Definition 7 (cf. [52]). Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random random variables. X is said to be smaller than Y in the convex order (written as $X \prec_{cx} Y$) if $Eu(X) \leq Eu(Y)$, for all convex functions $u : \mathbb{R} \rightarrow \mathbb{R}$, provided that the means exists.

The following two results concern some properties of the stochastic order and of the dispersive order, respectively.

Theorem 1. [cf. [52]] Let $X : \Omega \rightarrow \mathbb{R}$ random variable and the functions $\varphi_1, \varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$. If $\varphi_1(x) \leq \varphi_2(x) \forall x \in \mathbb{R}$ then $\varphi_1(X) \prec_{st} \varphi_2(X)$.

Theorem 2. [cf. [52], Theorem 3.B.10, p. 152] Let $X, Y : \Omega \rightarrow \mathbb{R}$ be two random variables such that $X \prec_{st} Y$.

(i) If $X \prec_{disp} Y$, then $\varphi(X) \prec_{disp} \varphi(Y)$ for all non-decreasing convex or non-increasing concave functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$.

(ii) If $X \prec_{disp} Y$, then $\varphi(Y) \prec_{disp} \varphi(X)$ for all non-increasing convex or non-decreasing concave functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$.

Let us now state some well known properties between these stochastic orders.

Proposition 1 (cf. [52], Theorem 1.B.1, p. 18). $X \prec_{hr} Y \Rightarrow X \prec_{st} Y$.

Proposition 2. [cf. [52]] $X \prec_{Lorenz} Y \Rightarrow G_X \leq G_Y$.

Proposition 3 (cf. [52]). $X \prec_{disp} Y \Rightarrow X \prec_{Lorenz} Y$.

Proposition 4. [cf. [52], 2007, p. 166] If $EX = EY$ is finite then $X \prec_{ew} Y \Rightarrow X \prec_{cx} Y$. In particular, if $Var(Y)$ is finite, then $X \prec_{ew} Y \Rightarrow Var(X) \leq Var(Y)$.

Proposition 5 (cf. [52], 3.C.9, p. 166). $X \prec_{disp} Y \Rightarrow X \prec_{ew} Y$.

3. Generalized (X_0, q, b, u, x) -Log-Scale-Location Model

In this section we propose a new log-scale-location class of lifetime distributions that extend existing classes presented in [27]. We compute several characteristics of these distributions, like the hazard rate, the Gini index and the generalized a -Gini index.

Definition 8. For a real random variable X_0 , functions $u : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R} \rightarrow (0, \infty)$ and $x \in \mathbb{R}$, $q \in [0, 1]$, we say that the positive random variable T follows the (X_0, q, b, u, x) -log-scale-location model if

$$\bar{F}_T(t) = \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right), \text{ if } t > 0$$

and

$$\bar{F}_T(t) = 1, \text{ if } t \leq 0.$$

In this definition of the (X_0, q, b, u, x) -log-scale-location model, the real number x stands for any continuous covariate on which the distribution of the lifetime T depends on. One possible generalization of this model would be the introduction of several covariates.

For $q = 1$ we obtain the model (5) from Gigliarano et al. [27].

If the random variable X_0 is absolutely continuous with respect to the Lebesgue measure, with density f_{X_0} , then we can immediately obtain several characteristics of the random variable T , namely the density $f_T(t)$, the hazard rate r_T , the Gini index G_T and the generalized a -Gini index $G_{T;a}$:

$$f_T(t) = f_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) \cdot \frac{t^{-q}}{b(x)} \cdot 1_{(0,\infty)}(t), \text{ where } t \in \mathbb{R},$$

$$r_T(t) = \frac{t^{-q}}{b(x)} r_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right), \text{ where } t > 0,$$

$$G_T = 1 - \frac{\int_0^\infty \bar{F}_T^2(t) dt}{\int_0^\infty \bar{F}_T(t) dt} = 1 - \frac{\int_0^\infty \bar{F}_{X_0}^2 \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}{\int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt},$$

and

$$G_{T;a} = 1 - \frac{\int_0^\infty \bar{F}_{X_0}^a \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}{\int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt} = 1 - \frac{\int_0^\infty \bar{F}_{X_0}^a \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}{ET}.$$

We also have the alternative expression of the generalized a -Gini index in terms of the Lorenz curve of T, L_T :

$$G_{T;a} = 1 - a(a - 1) \int_0^1 (1 - p)^{a-2} L_T(p) dp.$$

4. The Moments of T

In this section we analyze the stochastic order and, as a consequence, the inequalities between the moments of two log-scale-location models. Next theorem provides a formula for computing the mean of T .

Theorem 3. *Let a random variable X_0 and the positive random variable T follows the (X_0, q, b, u, x) -log-scale-location model, with $q < 1$. If there exists $\lim_{r \rightarrow \infty} \left[r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) \right] \in \mathbb{R}$ then*

$$E(T) = [(1 - q)b(x)]^{\frac{1}{1-q}} \lim_{r \rightarrow \infty} \left[r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) \right] + \int_{-\frac{1}{1-q} + u(x)}^\infty f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

Proof. $E(T) = \int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt.$

We make the change of variable $r = \frac{lt_q(t) - u(x)}{b(x)}$ and we obtain

$$\begin{aligned} & \int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt = \int_{-\frac{1}{1-q} + u(x)}^\infty \bar{F}_{X_0}(r) \cdot \frac{\partial}{\partial r} ets_q(b(x) \cdot r + u(x)) dr \\ &= \bar{F}_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) \Big|_{-\frac{1}{1-q} + u(x)}^\infty - \int_{-\frac{1}{1-q} + u(x)}^\infty -f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr \\ &= \lim_{r \rightarrow \infty} \bar{F}_{X_0}(r) \cdot [1 + (1 - q)(b(x) \cdot r + u(x))]_+^{\frac{1}{1-q}} \\ & \quad - \bar{F}_{X_0} \left(-\frac{1}{1-q} + u(x) \right) ets_q \left(b(x) \cdot \left(-\frac{1}{1-q} + u(x) \right) u(x) \right) \\ & \quad + \int_{-\frac{1}{1-q} + u(x)}^\infty f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr \end{aligned}$$

$$\begin{aligned}
 &= \lim_{r \rightarrow \infty} \bar{F}_{X_0}(r) \cdot [1 + (1 - q)(b(x) \cdot r + u(x))]_+^{\frac{1}{1-q}} \\
 &\quad + \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr \\
 &= \lim_{r \rightarrow \infty} \bar{F}_{X_0}(r) \cdot [1 + (1 - q)(b(x) \cdot r + u(x))]_+^{\frac{1}{1-q}} \\
 &\quad + \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr \\
 &= \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \cdot \bar{F}_{X_0}(r) \cdot \left[\frac{1}{r} + (1 - q) \left(b(x) + \frac{u(x)}{r} \right) \right]_+^{\frac{1}{1-q}} \\
 &\quad + \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr \\
 &= [(1 - q)b(x)]^{\frac{1}{1-q}} \cdot \lim_{r \rightarrow \infty} \left[r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) \right] + \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.
 \end{aligned}$$

□

In particular, for $\bar{F}_{X_0}(r) = e^{-e^r}$, we have

$$\lim_{r \rightarrow \infty} \left[r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) \right] = \lim_{r \rightarrow \infty} \left[r^{\frac{1}{1-q}} \cdot e^{-e^r} \right] = \lim_{r \rightarrow \infty} e^{\frac{1}{1-q} \ln r - e^r} = 0$$

and, for $q \rightarrow 1, q < 1$, it results that

$$\begin{aligned}
 E(T) &= \int_{-\infty}^{\infty} f_{X_0}(r) \cdot e_1^{Ts}(b(x) \cdot r + u(x)) dr = \int_{-\infty}^{\infty} f_{X_0}(r) \cdot e^{b(x) \cdot r + u(x)} dr = \\
 &e^{u(x)} \cdot \int_{-\infty}^{\infty} f_{X_0}(r) \cdot e^{b(x) \cdot r} dr = e^{u(x)} \cdot b(x) \cdot \int_{-\infty}^{\infty} \frac{e^{b(x) \cdot r}}{b(x)} \cdot f_{X_0}(r) dr = e^{u(x)} \cdot E\left(e^{b(x) \cdot R}\right),
 \end{aligned}$$

where R is a random variable with survival function $\bar{F}_{X_0}(r) = e^{-e^r}$. It is worth noticing that we have thus obtained the Proposition 2 from Gagliarano et al. [27].

In the next results we investigate several particular cases, namely the cases where the baseline distribution of X_0 is a q -Weibull distribution, a q -Normal distribution or a bounded distribution.

Corollary 1. Let $X_0 \sim q_1 - Weibull(k, \lambda)$ and T following the (X_0, q, b, u, x) -log-scale-location model, $q < 1, q_1 \leq 1$, with $q'_1 = \frac{1}{2 - q_1}, \lambda' = \frac{\lambda}{(2 - q_1)^{\frac{1}{k}}}$. Then

$$E(T) = \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

Proof. We have

$$\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) = \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \left[1 - (1 - q'_1) \left(\frac{r}{\lambda'} \right)^k \right]_+^{\frac{1}{1-q'_1}}.$$

$q_1 \leq 1 \Rightarrow q'_1 \leq 1$; if $q_1 < 1$ then $q'_1 < 1$. In this case we have $\lim_{r \rightarrow \infty} 1 - (1 - q'_1) \left(\frac{r}{\lambda'} \right)^k = -\infty$. Thus there exists $r_0 > 0$ such that $1 - (1 - q'_1) \left(\frac{r}{\lambda'} \right)^k < 0 \forall r \geq r_0$. It results

$$\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \left[1 - (1 - q'_1) \left(\frac{r}{\lambda'} \right)^k \right]_+^{\frac{1}{1-q'_1}} = \lim_{r \rightarrow \infty} \left(r^{\frac{1}{1-q}} \cdot 0 \right) = \lim_{r \rightarrow \infty} 0 = 0.$$

Thus

$$E(T) = \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

If $q_1 = 1$ then $q'_1 = 1$. In this, case we have

$$\bar{F}_{X_0}(r) = e_1^{Ts} \left(- \left(\frac{x}{\lambda'} \right)^k \right) = e^{-\left(\frac{r}{\lambda'}\right)^k}.$$

It is obvious that $\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) = \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} e^{-\left(\frac{r}{\lambda'}\right)^k} = 0$, thus

$$E(T) = \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

□

Corollary 2. Let $X_0 \sim q_1 - N(\beta, C_{q_1})$ and T following the (X_0, q, b, u, x) -log-scale-location model, $q < 1, q_1 \leq 1$. Then

$$E(T) = \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

Proof. We have

$$\begin{aligned} \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) &= \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \int_r^{\infty} \frac{\sqrt{\beta}}{C_q} ets_q(-\beta x^2) dx \\ &= \frac{\sqrt{\beta}}{C_q} \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \int_r^{\infty} [1 - \beta(1 - q_1)x^2]_{+}^{\frac{1}{1-q_1}} dx. \end{aligned}$$

If $q_1 < 1$, then $\lim_{x \rightarrow \infty} [1 - \beta(1 - q_1)x^2] = -\infty$; thus there exists an $x_0 > 0$ such that $1 - \beta(1 - q_1)x^2 < 0 \forall x \geq x_0$. It results

$$\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \int_r^{\infty} [1 - \beta(1 - q_1)x^2]_{+}^{\frac{1}{1-q_1}} dx = \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \int_r^{\infty} 0 dx = \lim_{r \rightarrow \infty} (r \cdot 0) = \lim_{r \rightarrow \infty} 0 = 0$$

and then $\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) = 0$. Thus

$$E(T) = \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

If $q_1 = 1$, then $f_{X_0}(x) = \sqrt{\frac{\beta}{\pi}} e^{-\beta x^2}$.

$$\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \int_r^{\infty} \frac{\sqrt{\beta}}{C_q} e_1^{Ts}(-\beta x^2) dx = \sqrt{\frac{\beta}{\pi}} \lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \int_r^{\infty} e^{-\beta x^2} dx = \lim_{r \rightarrow \infty} \frac{\int_r^{\infty} e^{-\beta x^2} dx}{r^{-\frac{1}{1-q}}} =$$

$$\lim_{r \rightarrow \infty} \frac{\left(\int_r^{\infty} e^{-\beta x^2} dx \right)'}{\left(r^{-\frac{1}{1-q}} \right)'} = \lim_{r \rightarrow \infty} r^{\frac{-2+q}{1-q}} e^{-\beta r^2} = \lim_{r \rightarrow \infty} \frac{1}{r^{\frac{2-q}{1-q}} e^{\beta r^2}} = 0.$$

It results that $\lim_{r \rightarrow \infty} r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) = 0$. Thus

$$E(T) = \int_{-\frac{1}{1-q} + u(x)}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

□

Corollary 3. Let X_0 be a real random variable with the property that there exists $M \in \mathbb{R}$ such that $X_0 \leq M$ a.s. Then:

(i) For $q < 1$, there exists

$$E(T) = \int_{-\frac{1}{1-q} + \frac{u(x)}{b(x)}}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr.$$

(ii) For $q = 1$, there exists

$$E(T) = e^{u(x)} \cdot E\left(e^{b(x) \cdot R}\right).$$

Proof. Since $X_0 \leq M$ a.s., then $\bar{F}_{X_0}(r) = 0$ for all $r > M$ and we obtain that $\Rightarrow \lim_{r \rightarrow \infty} \left[r^{\frac{1}{1-q}} \bar{F}_{X_0}(r) \right] = 0$.

Then $E(T) = \int_{-\frac{1}{1-q} + \frac{u(x)}{b(x)}}^{\infty} f_{X_0}(r) \cdot ets_q(b(x) \cdot r + u(x)) dr$.

Applying Theorem 3 we obtain the desired result. \square

5. Stochastic Order of These Models

If we want give an order between the moments of a random variable, it is complicated to compute all the moments and then establish an order. For this reason, we give a theorem which characterizes the stochastic order of these models.

Theorem 4. Let X_0 be a random variable, let T_1 be a positive random variable that follows the (X_0, q, b, u, x_1) -log-scale-location model and let T_2 be a positive random variable that follows the (X_0, q, b, u, x_2) -log-scale-location model. Then $T_1 \prec_{st} T_2$ if and only if

$$\frac{u(x_1)}{b(x_1)} \leq \frac{u(x_2)}{b(x_2)},$$

$$b(x_1) \leq b(x_2)$$

and

$$\frac{1}{1-q} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}.$$

Proof. Let us consider $T_1 \prec_{st} T_2$. We have:

$$\bar{F}_{T_1}(t) \leq \bar{F}_{T_2}(t) \quad \forall t > 0 \Leftrightarrow$$

$$\bar{F}_{X_0} \left(\frac{lt_q(t) - u(x_1)}{b(x_1)} \right) \leq \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x_2)}{b(x_2)} \right) \quad \forall t > 0 \Leftrightarrow$$

$$\frac{lt_q(t) - u(x_1)}{b(x_1)} \geq \frac{lt_q(t) - u(x_2)}{b(x_2)} \quad \forall t > 0 \Leftrightarrow$$

$$\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)} \quad \forall t > 0.$$

For $t = 1$ we have $\frac{u(x_1)}{b(x_1)} \leq \frac{u(x_2)}{b(x_2)}$.

We have $\lim_{t \rightarrow \infty} lt_q(t) = \infty$ and $\lim_{t \rightarrow 0} lt_q(t) = -\frac{1}{1-q}$. Then:

$$\lim_{t \rightarrow \infty} \left[\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) \right] \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}$$

and

$$\lim_{\substack{t \rightarrow 0 \\ t > 0}} \left[\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) \right] \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}.$$

Then $b(x_1) \leq b(x_2)$ and $\frac{1}{q-1} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}.$

Now, let us prove the converse. For $t > 0$, we have

$$\begin{aligned} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) &\geq \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot \lim_{\substack{t \rightarrow 0 \\ t > 0}} lt_q(t) = \\ &\frac{1}{1-q} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}. \end{aligned}$$

Thus $\bar{F}_{T_1}(t) \leq \bar{F}_{T_2}(t).$ □

The next proposition is a particular case where we examine the behavior of the survival distribution.

Proposition 6. *If u is constant and b is a non-decreasing (non-increasing) function, then:*

- (i) *for $t > ets_q(u)$, $x \mapsto \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right)$ is a non-decreasing (non-increasing) function;*
 - (ii) *for $t < ets_q(u)$, $x \mapsto \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right)$ is a non-increasing (non-decreasing) function;*
- and
- (iii) *for $t = ets_q(u)$, $x \mapsto \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right)$ is a constant function.*

Proof. We consider only the case where b is a non-decreasing function (the non-increasing case can be proved similarly). Then:

- (i) If $t > ets_q(u)$, i.e., $lt_q(t) > u$, and is we assume that $x_1 < x_2$, then

$$\bar{F}_{X_0} \left(\frac{lt_q(t) - u}{b(x_1)} \right) \leq \bar{F}_{X_0} \left(\frac{lt_q(t) - u}{b(x_2)} \right);$$

- (ii) If $t < ets_q(u)$, i.e., $lt_q(t) < u$, and if we assume that $x_1 < x_2$, then

$$\bar{F}_{X_0} \left(\frac{lt_q(t) - u}{b(x_1)} \right) \geq \bar{F}_{X_0} \left(\frac{lt_q(t) - u}{b(x_2)} \right);$$

- (iii) If $t = ets_q(u)$, i.e., $lt_q(t) = u$, and if we assume that $x_1 < x_2$, then

$$\bar{F}_{X_0} \left(\frac{lt_q(t) - u}{b(x_1)} \right) = \bar{F}_{X_0}(0) = \bar{F}_{X_0} \left(\frac{lt_q(t) - u}{b(x_2)} \right).$$

□

This proposition generalizes Corollary 1 from Gigliarano et al. [27] when $q \rightarrow 1, q < 1.$

The next two results are consequences of Theorem 4.

Corollary 4. *Let is consider a random variable X_0 and a random variable T_1 following the (X_0, q, b, u, x_1) -log-scale-location model, a random variable T_2 following the (X_0, q, b, u, x_2) -log-scale-location model. If*

$$\begin{aligned} \frac{u(x_1)}{b(x_1)} &\leq \frac{u(x_2)}{b(x_2)}, \\ b(x_1) &\leq b(x_2) \end{aligned}$$

and

$$\frac{1}{1-q} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)},$$

then

$$E(T_1^k) \leq E(T_2^k) \text{ for all } k \in \mathbb{N}, k \geq 1.$$

Proof. This is a consequence of Theorem 5 and Definition 6. \square

Proposition 7. Let T_1 be a random variable following the (X_1, q, b, u, x_1) -log-scale-location model and let T_2 be a random variable following the (X_2, q, b, u, x_2) -log-scale-location model, with $X_1 \sim q_0 - \text{Weibull}(k_1, \lambda_1)$, $X_2 \sim q_0 - \text{Weibull}(k_2, \lambda_2)$.

Then $T_1 \prec_{st} T_2$ if and only if

$$\frac{(2-q_0)^{\frac{1}{k_1}} u(x_1)}{\lambda_1 b(x_1)} \leq \frac{(2-q_0)^{\frac{1}{k_2}} u(x_2)}{\lambda_2 b(x_2)},$$

$$\frac{\lambda_1 b(x_1)}{(2-q_0)^{\frac{1}{k_1}}} \leq \frac{\lambda_2 b(x_2)}{(2-q_0)^{\frac{1}{k_2}}},$$

and

$$\frac{1}{1-q} \left(\frac{(2-q_0)^{\frac{1}{k_1}}}{\lambda_1 b(x_1)} - \frac{(2-q_0)^{\frac{1}{k_2}}}{\lambda_2 b(x_2)} \right) \geq \frac{(2-q_0)^{\frac{1}{k_1}} u(x_1)}{\lambda_1 b(x_1)} - \frac{(2-q_0)^{\frac{1}{k_2}} u(x_2)}{\lambda_2 b(x_2)}.$$

Proof. Let us consider $T_1 \prec_{st} T_2$. We have:

$$\bar{F}_{T_1}(t) \leq \bar{F}_{T_2}(t) \quad \forall t > 0 \Leftrightarrow$$

$$\bar{F}_{X_0} \left(\frac{lt_q(t) - u(x_1)}{b(x_1)} \right) \leq \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x_2)}{b(x_2)} \right) \quad \forall t > 0 \Leftrightarrow$$

$$e_{q_0}^{T_s} \left(- \left(\frac{lt_q(t) - u(x_1)}{b(x_1)} \right)^{k_1} \right) \leq e_{q_0}^{T_s} \left(- \left(\frac{lt_q(t) - u(x_2)}{b(x_2)} \right)^{k_2} \right) \quad \forall t > 0 \Leftrightarrow$$

$$\frac{lt_q(t) - u(x_1)}{b(x_1)} \geq \frac{lt_q(t) - u(x_2)}{b(x_2)} \quad \forall t > 0 \Leftrightarrow$$

$$\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)} \quad \forall t > 0.$$

For $t = 1$ we have $\frac{u(x_1)}{b(x_1)} \leq \frac{u(x_2)}{b(x_2)}$.

We have $\lim_{t \rightarrow \infty} lt_q(t) = \infty$ and $\lim_{t \rightarrow 0} lt_q(t) = -\frac{1}{1-q}$. Then:

$$\lim_{t \rightarrow \infty} \left[\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) \right] \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}$$

and

$$\lim_{t \rightarrow 0} \left[\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \cdot lt_q(t) \right] \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}.$$

Then $b(x_1) \leq b(x_2)$ and $-\frac{1}{1-q} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)} \right) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}$.

Now, let us prove the converse. For $t > 0$, we have

$$\left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)}\right) \cdot lt_q(t) \geq \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)}\right) \cdot \lim_{\substack{t \rightarrow 0 \\ t > 0}} lt_q(t) = \frac{1}{1-q} \left(\frac{1}{b(x_1)} - \frac{1}{b(x_2)}\right) \geq \frac{u(x_1)}{b(x_1)} - \frac{u(x_2)}{b(x_2)}.$$

Thus $\bar{F}_{T_1}(t) \leq \bar{F}_{T_2}(t)$. \square

6. Lorenz Order and Gini Index

In this section we give some results on stochastic orderings according to Lorenz curve and Gini index. We also give some results related to the generalized Gini index.

The next theorem characterizes the dispersive order.

Theorem 5. *If b is a non-decreasing (non-increasing) function and T_1 is a positive random variable following the (X_0, q, b, u, x_1) -log-scale-location model, T_2 is a positive random variable following the (X_0, q, b, u, x_2) -log-scale-location model, $q \in (0; 1)$, then*

$$lt_q T_1 \prec_{disp} lt_q T_2 \left(lt_q T_2 \prec_{disp} lt_q T_1 \right).$$

Proof. We consider only the case where b is a non-decreasing function (the non-increasing case can be proved similarly). Let us consider $0 < \alpha \leq \beta < 1$ and $b_1 = b(x_1), b_2 = b(x_2), u_1 = u(x_1), u_2 = u(x_2)$, with $x_1 < x_2$. It results that $b_1 = b(x_1) \leq b(x_2) = b_2$. Let $Y_1 = lt_q T_1, Y_2 = lt_q T_2, \beta = F_{Y_1}(y_\beta) = 1 - \bar{F}_{X_0}\left(\frac{y_\beta - u_1}{b_1}\right), \alpha = F_{Y_1}(y_\alpha) = 1 - \bar{F}_{X_0}\left(\frac{y_\alpha - u_1}{b_1}\right)$.

Then $F_{Y_1}^{-1}(\beta) = u_1 + b_1 \cdot \bar{F}_{X_0}^{-1}(1 - \beta)$ and $F_{Y_1}^{-1}(\alpha) = u_1 + b_1 \cdot \bar{F}_{X_0}^{-1}(1 - \alpha)$. It results

$$F_{Y_1}^{-1}(\beta) - F_{Y_1}^{-1}(\alpha) = b_1 \cdot \left(\bar{F}_{X_0}^{-1}(1 - \beta) - \bar{F}_{X_0}^{-1}(1 - \alpha) \right).$$

Similarly, we have

$$F_{Y_2}^{-1}(\beta) - F_{Y_2}^{-1}(\alpha) = b_2 \cdot \left(\bar{F}_{X_0}^{-1}(1 - \beta) - \bar{F}_{X_0}^{-1}(1 - \alpha) \right).$$

Then

$$F_{Y_1}^{-1}(\beta) - F_{Y_1}^{-1}(\alpha) \leq F_{Y_2}^{-1}(\beta) - F_{Y_2}^{-1}(\alpha) \Leftrightarrow b_1 \cdot \left(\bar{F}_{X_0}^{-1}(1 - \beta) - \bar{F}_{X_0}^{-1}(1 - \alpha) \right) \leq b_2 \cdot \left(\bar{F}_{X_0}^{-1}(1 - \beta) - \bar{F}_{X_0}^{-1}(1 - \alpha) \right).$$

But $\bar{F}_{X_0}^{-1}(1 - \beta) \geq \bar{F}_{X_0}^{-1}(1 - \alpha)$. Then

$$b_1 \cdot \left(\bar{F}_{X_0}^{-1}(1 - \beta) - \bar{F}_{X_0}^{-1}(1 - \alpha) \right) \leq b_2 \cdot \left(\bar{F}_{X_0}^{-1}(1 - \beta) - \bar{F}_{X_0}^{-1}(1 - \alpha) \right) \Leftrightarrow b_1 \leq b_2.$$

Therefore $Y_1 \prec_{disp} Y_2 \Leftrightarrow b_1 \leq b_2$. Thus $lt_q T_1 \prec_{disp} lt_q T_2$. \square

The next theorem characterizes the Lorenz order.

Theorem 6. *If b is a non-decreasing (non-increasing) function and T_1 is a positive random variable following the (X_0, q, b, u, x_1) -log-scale-location model, T_2 is a positive random variable following the (X_0, q, b, u, x_2) -log-scale-location model, $q \in (0; 1)$, with $T_1 \prec_{st} T_2$ ($T_2 \prec_{st} T_1$) then*

$$T_1 \prec_{Lorenz} T_2 \left(T_2 \prec_{Lorenz} T_1 \right).$$

Proof. As previously, we consider only the case where b is a non-decreasing function (the non-increasing case can be proved similarly). From Theorem 6 we have $lt_q T_1 \prec_{disp} lt_q T_2$. Notice that $T_1 \prec_{st} T_2$ and the function

$\varphi : \mathbb{R} \rightarrow (0, \infty)$, $\varphi(x) = ets_q(x)$ is increasing and convex. Then $ets_q(lt_q T_1) \prec_{disp} ets_q(lt_q T_2)$. It results $T_1 \prec_{disp} T_2$. Thus $T_1 \prec_{Lorenz} T_2$. \square

Let us now state a consequence of this result.

Corollary 5. *If b is a non-decreasing (non-increasing) function and T is a positive random variable following the (X_0, q, b, u, x) -log-scale-location model, $q \in (0; 1)$, then the function*

$$x \mapsto 1 - \frac{\int_0^\infty \bar{F}_{X_0}^2 \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}{\int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}$$

is non-decreasing (non-increasing).

Proof. This is a direct consequence of Theorem 7. \square

This result shows that, under the class of (X_0, q, b, u, x) -log-scale-location model, the Gini index is non-decreasing (non-increasing) as x increases, if the shape parameter $b(x)$ is non-decreasing (non-increasing).

Theorem 7 and Corollary 5 generalize Theorem 1 from Gigliarano et al. [27] when $q \rightarrow 1, q < 1$.

Let us now focus on the generalized a -Gini index. The following two results characterize the generalized a -Gini index.

Proposition 8. *Let X, Y be two random variables. If $X \prec_{Lorenz} Y$, then $G_{X;a} \leq G_{Y;a}$.*

Proof. $X \prec_{Lorenz} Y \Rightarrow L_X(p) \geq L_Y(p) \forall p \in [0; 1] \Rightarrow G_{X;a} \leq G_{Y;a}$. \square

The next result generalizes Theorem 7 and Corollary 5.

Corollary 6. *If b is a non-decreasing (non-increasing) function and T follows the (X_0, q, b, u, x) -log-scale-location model, $q \in (0; 1)$, then:*

(i) *The function $x \mapsto 1 - \frac{\int_0^\infty \bar{F}_{X_0}^a \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}{\int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}$ is non-decreasing (non-increasing), for $a \in (-\infty; 0] \cup [1; \infty)$;*

(ii) *The function $x \mapsto 1 - \frac{\int_0^\infty \bar{F}_{X_0}^a \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}{\int_0^\infty \bar{F}_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right) dt}$ is non-increasing (non-decreasing), for $a \in (0; 1)$.*

Proof. This is a consequence of Proposition 8. \square

This result shows that, under the class of (X_0, q, b, u, x) -log-scale-location model, the a -Gini index is non-decreasing (non-increasing) as x increases, if the shape parameter $b(x)$ is non-decreasing (non-increasing).

7. The Hazard Rate Order

In this section we give results for the hazard rate order and hazard rate functions.

Theorem 7. *If b, u, r_{X_0} are non-decreasing (non-increasing) functions and T_1 is a positive real random variable following the (X_0, q, b, u, x_1) -log-scale-location model, T_2 is a positive real random variable following the (X_0, q, b, u, x_2) -log-scale-location model, then $T_1 \prec_{hr} T_2$ ($T_2 \prec_{hr} T_1$).*

Proof. The proof will be done for non-decreasing b, u, r_{X_0} (the non-increasing case can be proved similarly). For $x_1 \leq x_2$ and $t > 0$, we have

$$\frac{t^{-q}}{b(x_1)} \geq \frac{t^{-q}}{b(x_2)}, \quad \frac{lt_q(t) - u(x_1)}{b(x_1)} \geq \frac{lt_q(t) - u(x_2)}{b(x_2)}.$$

Then

$$r_{T_1}(t) = \frac{t^{-q}}{b(x_1)} r_{X_0} \left(\frac{lt_q(t) - u(x_1)}{b(x_1)} \right) \geq \frac{t^{-q}}{b(x_2)} r_{X_0} \left(\frac{lt_q(t) - u(x_2)}{b(x_2)} \right) = r_{T_2}(t).$$

□

Proposition 9. If $X_0 \sim q_1 - \text{Weibull}(k, \lambda)$ and T is a positive real random variable that follows the (X_0, q, b, u, x) -log-scale-location model, with b, u non-decreasing, then $x \mapsto \frac{t^{-q}}{b(x)} r_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right)$ is non-increasing.

Proof. It is clear that $r_{X_0}(x) = \frac{k}{\lambda^k} \cdot \frac{x^{k-1}}{1 - (1 - q_1^k) \left(\frac{x}{\lambda}\right)^k}$ is a non-decreasing function of x . This implies that $x \mapsto \frac{t^{-q}}{b(x)} r_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right)$ is non-increasing. □

Proposition 10. If $X_0 \sim \mu$, where μ can be $(1 - p)\delta_0 + p\delta_1$ ($p \in (0, 1)$), Geometric(p), Binomial(n, p), Poisson(λ), Negbin(n, p), Unif(a, b), Gamma(v, a), $N(\mu, \sigma)$, Beta($m + 1, n + 1$), $\mathcal{X}^2(n)$ and T follows the (X_0, q, b, u, x_1) -log-scale-location model, with b, u non-increasing (non-decreasing), then $x \mapsto \frac{t^{-q}}{b(x)} r_{X_0} \left(\frac{lt_q(t) - u(x)}{b(x)} \right)$ is non-increasing (non-decreasing).

Proof. We have that r_{X_0} is non-decreasing. □

8. The Excess Wealth and Convex Order

In this section we analyze the excess wealth and convex orders. Theorem 9 gives a sufficient condition for excess wealth order of two (X_0, q, b, u, x) -log-scale-location models, while Theorem 10 gives sufficient conditions for convex order of these models.

Theorem 8. Let us consider some positive real random variables T_1 and T_2 . If b is non-decreasing (non-increasing) and T_1 follows the (X_0, q, b, u, x_1) -log-scale-location model, while T_2 follows the (X_0, q, b, u, x_2) -log-scale-location model, $q \in (0; 1)$, with $T_1 \prec_{st} T_2$ ($T_2 \prec_{st} T_1$) then

$$T_1 \prec_{ew} T_2 \quad (T_2 \prec_{ew} T_1).$$

Proof. Without loss of generality we take b to be non-decreasing.

From Theorem 6 we have $lt_q T_1 \prec_{disp} lt_q T_2$. We have $T_1 \prec_{st} T_2$ and the function $\varphi : \mathbb{R} \rightarrow (0, \infty)$, $\varphi(x) = ets_q(x)$ is increasing convex. Then $ets_q(lt_q T_1) \prec_{disp} ets_q(lt_q T_2)$.

It results that $T_1 \prec_{disp} T_2$. Thus $T_1 \prec_{ew} T_2$. □

Theorem 9. If b is a non-decreasing (non-increasing) function, T_1 is a positive random variable that follows the (X_0, q, b, u, x_1) -log-scale-location model, T_2 is a positive random variable that follows the (X_0, q, b, u, x_2) -log-scale-location model, $q \in (0; 1)$, with $ET_1 = ET_2$ and $T_1 \prec_{ew} T_2$ ($T_2 \prec_{ew} T_1$) then

$$T_1 \prec_{cx} T_2 \quad (T_2 \prec_{cx} T_1).$$

Proof. It results from Theorem 9 and Proposition 4. □

The next result is a consequence of Theorem 9 and Proposition 4.

Corollary 7. If b is a non-decreasing (non-increasing) function, T_1 is a positive random variable that follows the (X_0, q, b, u, x_1) -log-scale-location model, T_2 is a positive random variable that follows the (X_0, q, b, u, x_2) -log-scale-location model, $q \in (0; 1)$, with $Var(T_2)$ finite ($Var(T_1)$ finite) and $T_1 \prec_{ew} T_2$ ($T_2 \prec_{ew} T_1$) then

$$Var(T_1) \leq Var(T_2) \quad (Var(T_2) \leq Var(T_1)).$$

Proof. It results from Theorem 9 and Proposition 4. \square

9. Real Data Application

In this section we illustrate the theoretical results obtained in the paper. We use the data and the estimated parameters of a Pareto distribution from Nadarajah et al. [70]. The data represents automobile insurance claims from a large midwestern US property. As we already mentioned, the Pareto distribution $Pa(b, c)$ is used in [70] for this application.

Let $X_0 \sim Pa(b, c)$, $u, b : [0, 20) \rightarrow \mathbb{R}$, $u(x) = -4$, $b(x) = 0.2 - \frac{x}{100}$. Let also T_1, T_2, T_3, T_4 be random variables that follow the $(X_0, q, b, u, 0)$ -log-scale-location, $(X_0, q, b, u, 5)$ -log-scale-location, $(X_0, q, b, u, 10)$ -log-scale-location, $(X_0, q, b, u, 15)$ -log-scale-location models, respectively.

In each plots we will represent $y = \bar{F}_T(t) = \bar{F}_{X_0}\left(\frac{lt_q(t)-u(x)}{b(x)}\right)$ when x takes the values 0, 5, 10 and 15 for two Pareto distribution with parameters estimated $\hat{\alpha}$ and $\hat{\beta}$ and different values of q . The survival functions are very important because their ordering implies the ordering between the moments of Gini coefficients (when the means are equal). The estimated values of the parameters are given in Table 1.

Table 1. The estimated parameters $\hat{\alpha}, \hat{\beta}$ for different values of q and the p -values based on one-sample Kolmogorov–Smirnov test.

$\hat{\alpha}$	$\hat{\beta}$	q	K-S p -Value
0.2897983	25	0.1	0.143
0.2897983	25	0.5	0.143
0.2897983	25	1	0.143
0.2949511	35	0.1	0.290
0.2949511	35	0.5	0.290
0.2949511	35	1	0.290

For each of the following graphs, the black line corresponds to the case $x = 0$, the red line corresponds to the case $x = 5$, the green line corresponds to the case $x = 10$ and the yellow one corresponds to the case $x = 15$.

Figure 1 displays the plot of y for $\hat{\alpha} = 0.2897983, \hat{\beta} = 25, q = 0.1, u(x) = -4, b(x) = 0.2 - \frac{x}{100}$.

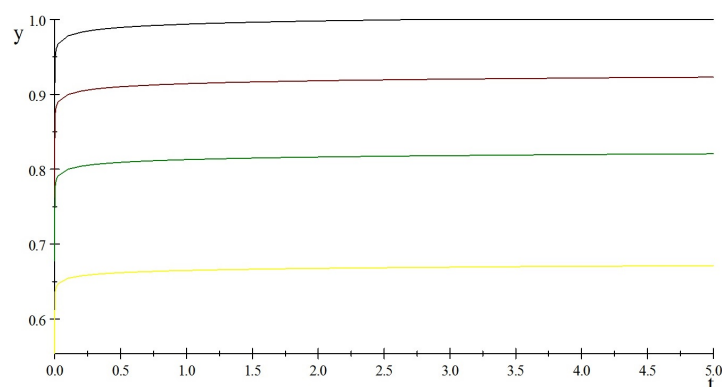


Figure 1. The plot of y for $(\hat{\alpha} = 0.2897983, \hat{\beta} = 25, q = 0.1, u(x) = -4, b(x) = 0.2 - \frac{x}{100})$.

Figure 2 displays the plot of y for $\hat{\alpha} = 0.2897983, \hat{\beta} = 25, q = 0.5, u(x) = -4, b(x) = 0.2 - \frac{x}{100}$.

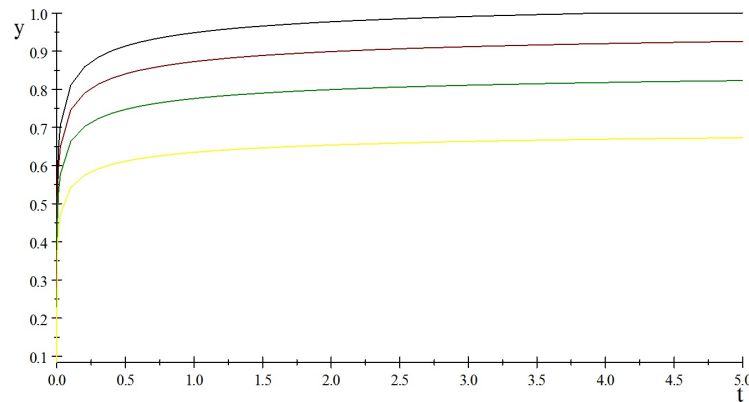


Figure 2. The plot of y for $(\hat{\alpha} = 0.2897983, \hat{\beta} = 25, q = 0.5, u(x) = -4, b(x) = 0.2 - \frac{x}{100})$.

Figure 3 displays the plot of y for $\hat{\alpha} = 0.2897983, \hat{\beta} = 25, q = 1, u(x) = -4, b(x) = 0.2 - \frac{x}{100}$.

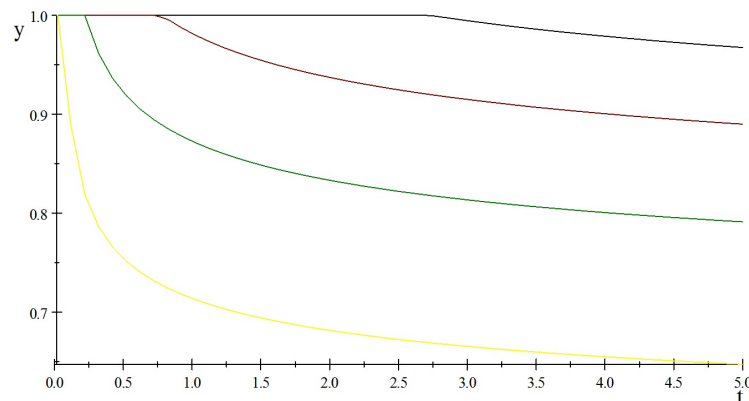


Figure 3. The plot of y for $(\hat{\alpha} = 0.2897983, \hat{\beta} = 25, q = 1, u(x) = -4, b(x) = 0.2 - \frac{x}{100})$.

Figure 4 displays the plot of y for $\hat{\alpha} = 0.2949511, \hat{\beta} = 35, q = 0.1, u(x) = -4, b(x) = 0.2 - \frac{x}{100}$.

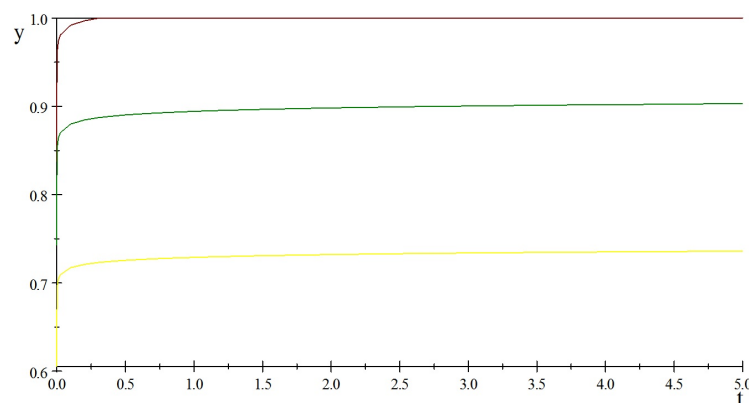


Figure 4. The plot of y for $(\hat{\alpha} = 0.2949511, \hat{\beta} = 35, q = 0.1, u(x) = -4, b(x) = 0.2 - \frac{x}{100})$.

Figure 5 displays the plot of y for $\hat{\alpha} = 0.2949511, \hat{\beta} = 35, q = 0.5, u(x) = -4, b(x) = 0.2 - \frac{x}{100}$.

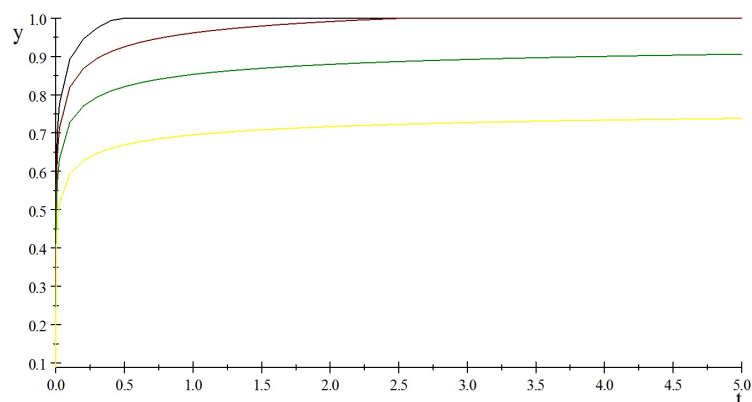


Figure 5. The plot of y for $(\hat{\alpha} = 0.2949511, \hat{\beta} = 35, q = 0.5, u(x) = -4, b(x) = 0.2 - \frac{x}{100})$.

Figure 6 displays the plot of y for $\hat{\alpha} = 0.2949511, \hat{\beta} = 35, q = 1, u(x) = -4, b(x) = 0.2 - \frac{x}{100}$.

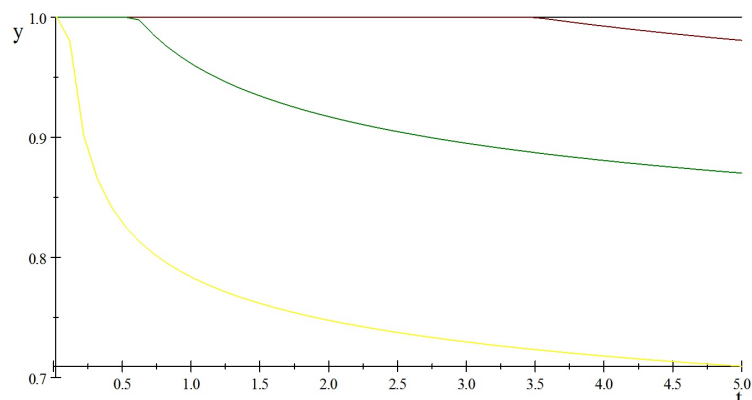


Figure 6. The plot of y for $(\hat{\alpha} = 0.2949511, \hat{\beta} = 35, q = 1, u(x) = -4, b(x) = 0.2 - \frac{x}{100})$.

From these six graphs we observe that, for $q = 1$, the functions are convex and, for $q \in \{0.1, 0.5\}$, the functions are concave. We observe also that, for $q = 0.5$, the graphs of these four functions are closer than in the other cases. Another conclusion is that the function $x \mapsto \bar{F}_{X_0}(\frac{It_q(t)-u(x)}{b(x)})$ is increasing on $\{0, 5, 10, 15\}$. This implies that

$$T_1 \prec_{st} T_2 \prec_{st} T_3 \prec_{st} T_4.$$

It yields that

$$E(T_1^k) \leq E(T_2^k) \leq E(T_3^k) \leq E(T_4^k), k \in \mathbb{N}, k \geq 1.$$

Moreover, from Theorem 7 we have

$$T_1 \prec_{Lorenz} T_2 \prec_{Lorenz} T_3 \prec_{Lorenz} T_4.$$

Then

$$G_{T_1;a} \leq G_{T_2;a} \leq G_{T_3;a} \leq G_{T_4;a}.$$

10. Conclusions

In this article we propose a new generalized log-scale-location family of distributions and we gave results on different stochastic orders for this generalized log-scale-location family that uses the Tsallis statistics. For this family of lifetime distributions, we have studied different stochastic orders, the moments and Gini indexes according to the parameters. On the one hand, the interest in the research work that we proposed in this article comes from the fact that we have developed new classes of lifetimes that extend existing classes from the literature. Thus we obtain a modeling tool that is more flexible, from a certain point of view, than the ones existing in the literature. On the other hand, for the models that we define in this work, we show that some types of stochastic orders are preserved, under certain conditions. Having in mind various potential fields of applications for this family of lifetime distributions (e.g., risk theory, reliability, survival analysis, epidemiology, insurance, demography), these stochastic orderings are extremely important. Last but not least, our research is a contribution to the growing literature of Tsallis statistical applications.

As for the future work, it would be interesting to study this topic for other values of the Tsallis parameter and also to carry out some extended simulations and detailed real data applications of the type of models and techniques developed in the present article.

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