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Article

The Crossing Numbers of Join Products of Paths and Cycles with Four Graphs of Order Five

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Abstract: The main aim of the paper is to establish the crossing numbers of the join products of the paths and the cycles on n vertices with a connected graph on five vertices isomorphic to the graph $K_{1,1,3} \setminus e$ obtained by removing one edge e incident with some vertex of order two from the complete tripartite graph $K_{1,1,3}$. The proofs are done with the help of well-known exact values for the crossing numbers of the join products of subgraphs of the considered graph with paths and cycles. Finally, by adding some edges to the graph under consideration, we obtain the crossing numbers of the join products of other graphs with the paths and the cycles on n vertices.

Keywords: graph; join product; crossing number; cyclic permutation; path; cycle

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1. Introduction

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane. (For the definition of a *drawing*, see also Klešč [1].) One can easily verify that a drawing with the minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no two edges cross more than once, no edge crosses itself, and also no two edges incident with the same vertex cross. Let D ($D(G)$) be a good drawing of the graph G. We denote by $\mathrm{cr}_D(G)$ the number of crossings among edges of G in the drawing D.

Let G_i and G_j be two edge-disjoint subgraphs of G. We denote, by $\mathrm{cr}_D(G_i, G_j)$, the number of crossings between the edges of G_i and edges of G_j, and, by $\mathrm{cr}_D(G_i)$ and $\mathrm{cr}_D(G_j)$, the number of crossings among edges of G_i and of G_j in D, respectively. For any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G by Klešč [1], the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j)\,,$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k)\,.$$

The problem of reducing the number of crossings is interesting in many areas. One of the most popular areas is the implementation of the VLSI layout, which has revolutionized circuit design and had a strong impact on parallel computing. Crossing numbers were also studied to improve the readability of hierarchical structures and automated graphs. The visualized graph should be easy to read and understand. For the sake of clarity of the graphical drawings, the reduction of crossings is likely the most important. Therefore, the investigation on the crossing number of simple graphs is a classical, but very difficult problem. Garey and Johnson [2] proved that determining $\mathrm{cr}(G)$ is an NP-complete problem. Throughout the proofs of paper, we will also use the Kleitman's result [3] on the crossing numbers of the complete bipartite graphs $K_{m,n}$ in the form

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{with}\min\{m,n\} \leq 6.$$

The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices.

Again, using Kleitman's result [3], the crossing numbers for the join product of two paths, the join product of two cycles, and also for the join product of a path and a cycle were determined by Klešč [4]. Notice that a lot of the exact values for crossing numbers of $G + D_n$, $G + P_n$, and of $G + C_n$ for arbitrary graph G at most on four vertices were estimated in [5,6]. The crossing numbers of the join product of many graphs G on five and six vertices with P_n and C_n were also investigated in [1,7–15].

The crossings numbers of the join products of the paths and the cycles with all graphs of order at most four have been well-known for a long time, and therefore it is understandable that our immediate goal is to establish the exact values for the crossing numbers of $G + P_n$ and of $G + C_n$ also for all graphs G of order five. Especially the results of $G_6 + P_n$, $G_9 + P_n$, $G_{11} + P_n$, $G_{14} + P_n$ and of $G_6 + C_n$, $G_9 + C_n$, $G_{11} + C_n$, $G_{14} + C_n$ can be used to determine the crossing number of the join product of the most complicated graph K_5 with the path and the cycle on n vertices. For this purpose, we present a new technique regarding the use of knowledge from the subgraphs whose values of crossing numbers are already known. Due to several possible isomorphisms, the results on the smaller graphs are important to confirm the validity of many conjectures, e.g., Corollary 7.

Let G_{11} be the connected graph on five vertices isomorphic to the graph $K_{1,1,3} \setminus e$ obtained by removing one edge e incident with some vertex of order two from the complete tripartite graph $K_{1,1,3}$, and let $V(G_{11}) = \{v_1, v_2, \ldots, v_5\}$. The crossing number of $G_{11} + D_n$ was determined for any $n \geq 1$ by Staš [12] using the properties of cyclic permutations, where D_n denotes the discrete graph on n vertices. The main aim of the paper is to establish the crossing numbers of the join products $G_{11} + P_n$ and $G_{11} + C_n$, where P_n and C_n are the path and the cycle on n vertices, respectively.

The proofs are done with the help of a lot of well-known exact values for the crossing numbers of the join products of five subgraphs of G_{11} with paths and cycles. These subgraphs are indexed in the order originally designated by Klešč [16] (except in the case of the graph G_0, because it is disconnected), and their planar drawings are presented in Figure 1.

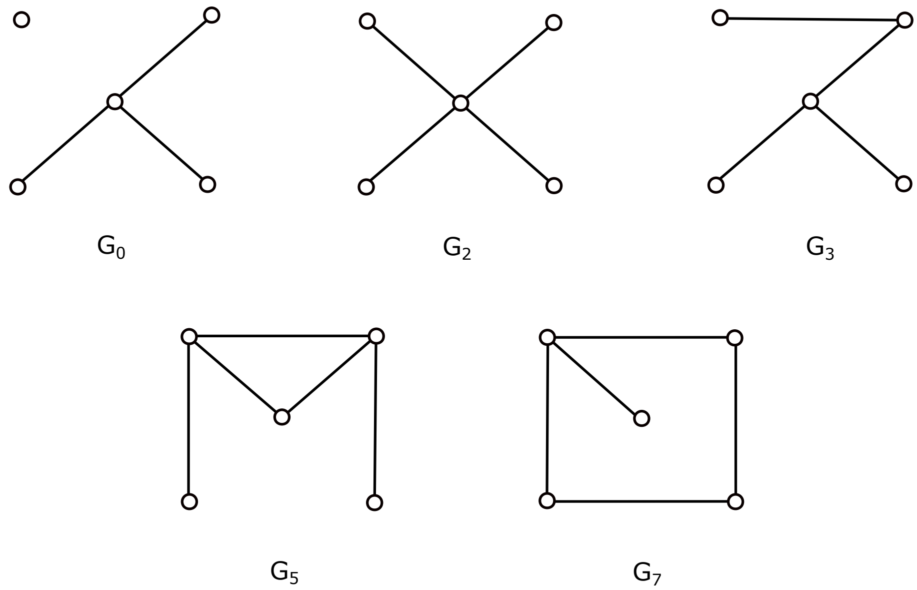

Figure 1. Planar drawings of five graphs G_0, G_2, G_3, G_5, and G_7, which are subgraphs of the graph G_{11}.

The results in Theorems 2 and 3, and in Corollaries 5 and 6 have already been claimed by Li [17] and by Yue et al. [18], respectively. Since these papers do not appear to be available in English, we were unable to verify these results. Clancy et al. [19] also placed an asterisk on a number of the results in their survey to essentially indicate that the mentioned results appeared in journals, which do not have a sufficiently rigorous peer-review process. In certain parts of the presented proofs, it is also possible to simplify the procedure with the help of software generating all cyclic permutations of five elements and its description can be found in Berežný and Buša [20].

2. Cyclic Permutations and Possible Drawings of G_{11}

We consider the join product of the graph G_{11} with the discrete graph D_n, which yields that the graph $G_{11} + D_n$ consists of just one copy of G_{11} and of n vertices $t_1, t_2, \dots, t_n$. Here, each vertex t_i, $i = 1, 2, \dots, n$, is adjacent to every vertex of the graph G_{11}. Let T^i, $1 \le i \le n$, denote the subgraph that is uniquely induced by the five edges incident with the fixed vertex t_i. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{5,n}$ and

$$G_{11} + D_n = G_{11} \cup K_{5,n} = G_{11} \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

Throughout the paper, we also use the same definitions and notation for the good drawings D of the graphs $G_{11} + P_n$ and $G_{11} + C_n$ as in [13,14]. The graph $G_{11} + P_n$ contains $G_{11} + D_n$ as a subgraph, and therefore let P_n^* denote the path induced on n vertices of $G_{11} + P_n$ not belonging to the subgraph G_{11}. The path P_n^* consists of the vertices $t_1, t_2, \dots, t_n$ and of the edges $\{t_i, t_{i+1}\}$ for $i = 1, 2, \dots, n-1$, and thus

$$G_{11} + P_n = G_{11} \cup K_{5,n} \cup P_n^* = G_{11} \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*. \tag{2}$$

Similarly, the graph $G_{11} + C_n$ contains both $G_{11} + D_n$ and $G_{11} + P_n$ as subgraphs. Let C_n^* denote the subgraph of $G_{11} + C_n$ induced on the vertices $t_1, t_2, \dots, t_n$. Therefore,

$$G_{11} + C_n = G_{11} \cup K_{5,n} \cup C_n^* = G_{11} \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup C_n^*. \tag{3}$$

Let D be a good drawing of the graph $G_{11} + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [21]. We use the notation (12345) if the counter-clockwise order the edges is incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We emphasize that a rotation is a cyclic permutation; that is, (12345), (23451), (34512), (45123), and (51234) denote the same rotation. Thus, $5!/5 = 24$ different $\mathrm{rot}_D(t_i)$ can appear in a drawing of the graph $G_{11} + D_n$.

By $\overline{\mathrm{rot}}_D(t_i)$, we understand the inverse permutation of $\mathrm{rot}_D(t_i)$. In the given drawing D, all subgraphs T^i, $i = 1, \dots, n$ of the graph $G_{11} + D_n$ are divided into three mutually disjoint subsets depending on how many times the edges of the subgraph T^i cross the edges of G_{11} in the considered drawing D. For $i = 1, \dots, n$, $T^i \in R_D$ if $\mathrm{cr}_D(G_{11}, T^i) = 0$, and $T^i \in S_D$ if $\mathrm{cr}_D(G_{11}, T^i) = 1$. Every other subgraph T^i crosses the edges of G_{11} at least twice in D. Clearly, this idea of dividing all subgraphs T^i into three mentioned subsets will be also retained in all drawings of the graphs $G_{11} + P_n$ and $G_{11} + C_n$.

Due to arguments in the proofs of Theorems 2 and 3, at least one of the sets R_D and S_D must be nonempty in any optimal drawing D of $G_{11} + P_n$ and of $G_{11} + C_n$. For $T^i \in R_D \cup S_D$, let F^i denote the subgraph $G_{11} \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $G_{11} + D_n$, and let $D(F^i)$ be its subdrawing induced by D.

According to the Lemmas 1 and 3, we suppose only two possible non isomorphic planar good subdrawings of G_{11} as shown in Figure 2, and where the vertex notation of the graph G_{11} will be explained later.

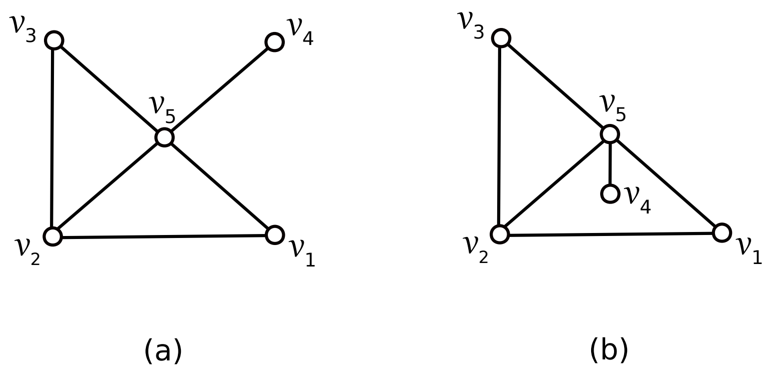

Figure 2. Two possible non isomorphic planar drawings of the graph G_{11}. (**a**) The planar drawing of G_{11} with five vertices in one region; (**b**) the planar drawing of G_{11} with at most four vertices in one region.

3. The Crossing Number of $G_{11} + P_n$

Lemma 1. *For $n \geq 2$, if D is any good drawing of the join product $G_{11} + P_n$ with $\mathrm{cr}_D(G_{11}) \geq 1$, then there are at least $n(n-1)+1$ crossings in D.*

Proof. Let us consider any good drawing D of $G_{11} + P_n$ with $\mathrm{cr}_D(G_{11}) \geq 1$. In the rest of the paper, suppose that let v_1, v_3, and v_2 be the vertex notation of two vertices of degree two and of one vertex of degree three in the considered good subdrawing of the graph G_{11}, respectively. Since no two edges incident with the same vertex cross, there is at least one crossing on the edge v_1v_2 or v_2v_3 in the subdrawing of G_{11} induced by D. By removing both these edges from the graph G_{11}, we obtain a subgraph isomorphic to the graph G_2. The exact value for the crossing number of the graph $G_2 + P_n$ is given in [12], i.e., $\mathrm{cr}(G_2 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor = n(n-1)$, which yields that there are at least $n(n-1)+1$ crossings in D. □

As the same argument with the removing of the edges v_1v_2 and v_2v_3 from the graph G_{11} can be also applied for two possible planar subdrawings of G_{11} in D, the proof of Corollary 1 can be omitted.

Corollary 1. *Let D be any good drawing of the join product $G_{11} + P_n$, $n \geq 2$, with $\mathrm{cr}_D(G_{11}) = 0$ and also with the vertex notation of G_{11} given in Figure 2a or Figure 2b. If any of the edges v_1v_2 or v_2v_3 is crossed in D, then there are at least $n(n-1)+1$ crossings in the drawing D.*

In the proof of Theorem 2, several parts are based on the previous Lemma 1, Corollary 1, and on the following theorem presented in [12].

Theorem 1 (See [12] Corollary 1)**.** $\mathrm{cr}(G_{11} + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ *for $n \geq 1$.*

The exact values of the crossing numbers for many small graphs can be calculated using an algorithm located on a website http://crossings.uos.de/ (accessed on 10 October 2020). This system also generates verifiable formal proofs, like those described by Chimani and Wiedera [22]. However, the capacity of this system is unfortunately limited.

Lemma 2. $\mathrm{cr}(G_{11} + P_2) = 3$ *and* $\mathrm{cr}(G_{11} + P_3) = 7$.

Theorem 2. $\mathrm{cr}(G_{11} + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 1 = n(n-1) + 1$ *for $n \geq 2$.*

Proof. In Figure 3, the edges of $K_{5,n}$ cross each other $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i, $i = 1, \ldots, \lceil \frac{n}{2} \rceil$ on the right side does not cross the edges of G_{11}, and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n$ on the left side crosses the edges of G_{11} exactly twice. The path P_n^* crosses G_{11} once, and thus $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 1$ crossings appear among the edges of the graph $G_{11} + P_n$ in this drawing. Thus, $\mathrm{cr}(G_{11} + P_n) \leq n(n-1) + 1$. Lemma 2 confirms this result for $n = 2$ and $n = 3$. We prove the reverse inequality by induction on n. Now, let us suppose that for some $n \geq 4$, there is a drawing D for which

$$\mathrm{cr}_D(G_{11} + P_n) < n(n-1) + 1, \tag{4}$$

and that

$$\mathrm{cr}(G_{11} + P_m) = m(m-1) + 1 \text{ for any } 2 \leq m < n. \tag{5}$$

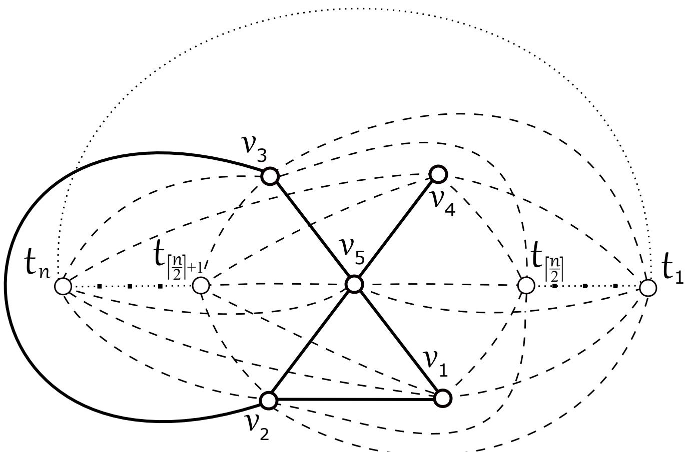

Figure 3. The good drawing of $G_{11} + P_n$ with $n(n-1) + 1$ crossings.

Since the graph $G_{11} + D_n$ is a subgraph of $G_{11} + P_n$, by Theorem 1, the edges of $G_{11} + P_n$ are crossed exactly $n(n-1)$ times, and therefore, no edge of the path P_n^* is crossed in D. In addition, all vertices t_i of the path P_n^* have to be placed in the same region of the considered good subdrawing of G_{11}. By Lemma 1, we can only suppose planar subdrawings of the graph G_{11} induced by D, that is, $\mathrm{cr}_D(G_{11}) = 0$. If $r = |R_D|$ and $s = |S_D|$, the assumption (5) together with $\mathrm{cr}(K_{5,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ enforces that there are at least $\lceil \frac{n}{2} \rceil$ subgraphs T^i whose edges cross the edges of G_{11} at most once in D. More precisely:

$$\mathrm{cr}_D(G_{11}) + \mathrm{cr}_D(G_{11}, K_{5,n}) \leq 2\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$0 + 0r + 1s + 2(n - r - s) \leq 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{6}$$

This implies that $2r + s \geq 2\lceil \frac{n}{2} \rceil$. Now, we will show that, in all subcases, a contradiction with the assumption (4) can be obtained:

Case 1: We suppose the drawing with the vertex notation of G_{11} in such a way as shown in Figure 2a. Since the set $R_D \cup S_D$ is nonempty and no edge of the path P_n^* is crossed in the drawing D, all vertices t_i of P_n^* are placed in the region of subdrawing $D(G_{11})$ with five vertices v_1, v_2, v_3, v_4, and v_5 of G_{11} on its boundary. By Klešč and Staš [11], it was proved that $\mathrm{cr}(G_0 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$, and thus there are at most $\lfloor \frac{n}{2} \rfloor$ crossings on three edges v_1v_5, v_3v_5, and v_4v_5 in D. This, also with Corollary 1, enforces that $r \geq n - \lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil$, because each subgraph $T^k \notin R_D$ crosses some of these three edges at least once.

As the set R_D is nonempty, our aim is to list all possible rotations $\mathrm{rot}_D(t_i)$ existing in D if no edge of T^i cross any edge of G_{11}. Since there is only one subdrawing of $F^i \setminus v_5$ represented by the subrotation (1234), we have only two ways to obtain the subdrawing of the subgraph F^i depending on which region the edge t_iv_5 is placed in. We denote these two possibilities by $\mathcal{A}_1$ and $\mathcal{A}_3$, and they are represented by the cyclic permutations (12345)

and (12354), respectively (in order to comply with the same notation as in [12]). One can easily determine, in five possible regions of $D(G_{11} \cup T^i)$, that $\mathrm{cr}_D(G_{11} \cup T^i, T^k) \geq 3$ holds for any subgraph T^k, $k \neq i$. Thus, by fixing the subgraph $G_{11} \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(G_{11} + P_n) &= \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, G_{11} \cup T^i) + \mathrm{cr}_D(G_{11} \cup T^i) \\ &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3(n-1) + 0,\end{aligned}$$

where $4\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor + 3(n-1) \geq n(n-1) + 1$ is true only for n even. For n odd, without a loss of generality on based their symmetry, let us also consider that the number of all subgraphs with the configuration $\mathcal{A}_1$ is at least as much as the number of all subgraphs with the configuration $\mathcal{A}_3$, and let $T^i \in R_D$ be such a subgraph with the configuration $\mathcal{A}_1$ of F^i. As $r \geq \lceil \frac{n}{2} \rceil \geq 3$ for n at least 5, there is at least one subgraph $T^j \in R_D$, $j \neq i$ with $\mathrm{rot}_D(t_j) = \mathrm{rot}_D(t_i)$, which yields that $\mathrm{cr}_D(G_{11} \cup T^i, T^j) \geq 0 + 4 = 4$. This allows us to add at least one crossing in the following inequalities

$$\mathrm{cr}_D(G_{11} + P_n) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3(n-1) + 1 \geq n(n-1) + 1.$$

Both subcases confirm a contradiction with the assumption in D.

Case 2: We consider the planar drawing of G_{11} in D given in Figure 2b. As no face is incident to all vertices in $D(G_{11})$, there is no possibility to obtain a subdrawing of $G_{11} \cup T^i$ for a $T^i \in R_D$. As $r = 0$, each subgraph T^i crosses the edges of G_{11} exactly once using the inequality (6). If all vertices t_i of the path P_n^* are placed in the region of $D(G_{11})$ with four vertices v_1, v_2, v_3, and v_5 of G_{11} on its boundary, then the edge v_1v_5 must be crossed by each subgraph $T^i \in S_D$ by Corollary 1.

This contradicts the fact that there are, at most, $\lfloor \frac{n}{2} \rfloor$ crossings on the edge v_1v_5 using the already well-known result from the previous case by [11]. Finally, if all vertices t_i are placed in the region of $D(G_{11})$ with four vertices v_1, v_2, v_4, and v_5 of G_{11} on its boundary, then only one of the edges v_1v_5 and v_2v_5 can be crossed by any subgraph $T^i \in S_D$ again by Corollary 1. The authors in [10,11] proved that $\mathrm{cr}(G_5 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ and $\mathrm{cr}(G_7 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$, respectively, and thus there are at most $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - 1$ crossings on the pair of edges v_1v_5 and v_2v_5. These facts imply a contradiction with $s = n$.

We have shown, in all cases, that there are at least $n(n-1) + 1$ crossings in each good drawing D of the graph $G_{11} + P_n$. The proof of Theorem 2 is done. □

4. The Crossing Number of $G_{11} + C_n$

Let S_m denote the star on $m + 1$ vertices. Using the results of Klešč et al. [9], the crossing numbers of the graphs $S_m + C_n$ for $m = 3, 4, 5$ and $n \geq 3$ were established. Hence, the exact value for the crossing number of the graph $G_2 + C_n$ is given by $n(n-1) + 2$. Given the use of arguments similar to those in the proof of Lemma 1, the proofs of Lemma 3 and Corollary 2 can be omitted.

Lemma 3. *For $n \geq 3$, if D is any good drawing of the join product $G_{11} + C_n$ with $\mathrm{cr}_D(G_{11}) \geq 1$, then there at least $n(n-1) + 3$ crossings in D.*

Corollary 2. *Let D be any good drawing of the join product $G_{11} + C_n$, $n \geq 3$, with $\mathrm{cr}_D(G_{11}) = 0$ and also with the vertex notation of G_{11} given in Figure 2a or Figure 2b. If any of the edges v_1v_2 or v_2v_3 is crossed in D, then there are at least $n(n-1) + 3$ crossings in the drawing D.*

Two vertices t_i and t_j of the graph $G_{11} + D_n$ are said to be *antipodal* in a drawing of $G_{11} + D_n$ if the considered subgraphs T^i and T^j do not cross. A drawing with no antipodal vertices is *antipode-free*. Clearly, this antipode-free property is also retained in all drawings of the graph $G_{11} + C_n$.

Lemma 4. *For $n > 2$, let D be a good and antipode-free drawing of the join product $G_{11} + D_n$ with $\mathrm{cr}_D(G_{11}) = 0$ and also with the vertex notation of the graph G_{11} given in Figure 2a. Let $T^i \in R_D$ be a subgraph such that F^i has the configuration $\mathcal{A}_1$, i.e., $\mathrm{rot}_D(t_i) = (12345)$, and let there be no crossing on the edges v_1v_2 and v_2v_3 in D. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, then*

(a) $\mathrm{cr}_D(G_{11} \cup T^i \cup T^k, T^l) \geq 6$ *holds for any subgraph $T^l \in R_D$, $l \neq i$; and*

(b) $\mathrm{cr}_D(G_{11} \cup T^i \cup T^k, T^l) \geq 6$ *holds for any subgraph $T^l \notin R_D$, $l \neq k$ such that the edge v_1v_5 of G_{11} is not crossed by the edges of T^l.*

Proof. Let us consider the configuration $\mathcal{A}_1$ of the subgraph F^i. If there is a subgraph $T^k \in S_D$ such that $\mathrm{cr}_D(T^i, T^k) = 2$, then the considered vertex t_k has to be placed in the quadrangular region of $D(G_{11} \cup T^i)$ with exactly three vertices v_3, v_4, and v_5 of G_{11} on its boundary. This enforces that the edges v_3v_5 or v_4v_5 of the graph G_{11} must be crossed by the edges t_kv_2 or t_kv_1, respectively. For more, see also the two mentioned subdrawings of the graph $G_{11} \cup T^i \cup T^k$ in Figure 4.

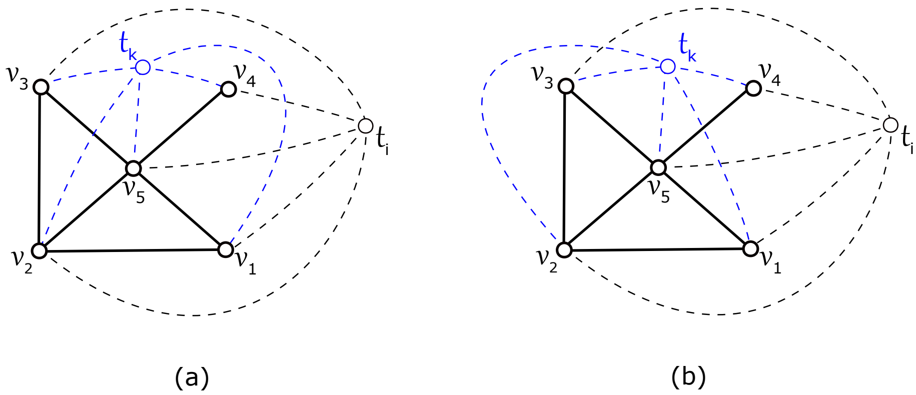

Figure 4. Two possible subdrawings of $G_{11} \cup T^i \cup T^k$ for $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 2$, where $T^i \in R_D$ with the configuration $\mathcal{A}_1$ of F^i. (**a**) The subdrawing in which the edge v_3v_5 of G_{11} is crossed by the edge t_kv_2 ; (**b**) the subdrawing in which the edge v_4v_5 of G_{11} is crossed by the edge t_kv_1.

(a) Let $T^k \in S_D$ be a subgraph with $\mathrm{cr}_D(T^i, T^k) = 2$. If we suppose the drawing of subgraph T^k as shown in Figure 4a, then $\mathrm{rot}_D(t_k) = (13254)$. For $T^l \in R_D$ with $l \neq i$, the possible configurations $\mathcal{A}_1$ and $\mathcal{A}_3$ are uniquely represented by the cyclic permutations (12345) and (12354), respectively. Using the distances between two cyclic permutations, we are able to determine the minimum numbers of crossings of T^l with the subgraphs T^i and T^k in the first two columns of Table 1. The smallest value in the last column of Table 1 gives the required minimum number of crossings. Of course, the same idea for the case of $\mathrm{rot}_D(t_k) = (14235)$ forces the same result.

Table 1. All possibilities of the subgraph F^l for $T^l \in R_D, l \neq i$ with $\mathrm{cr}_D(T^i, T^k) = 2$ and $T^k \in S_D$.

$\mathbf{conf}(F^l)$	$\mathbf{cr}_D(T^i, T^l)$	$\mathbf{cr}_D(T^k, T^l)$	$\mathbf{cr}_D(T^i \cup T^k, T^l)$	$\mathbf{cr}_D(G_{11} \cup T^i \cup T^k, T^l)$
$\mathcal{A}_1$	4	2	6	6
$\mathcal{A}_3$	3	3	6	6

(b) For $l \neq k$, let $T^l \notin R_D$ be a subgraph with respect to the restriction that the edges of T^l does not cross the edges v_1v_2, v_2v_3, and v_1v_5 of the graph G_{11}. Since the considered drawing D is antipode-free and T^l can cross only some of edges v_2v_5, v_3v_5, and v_4v_5 of G_{11}, one can easily determine, in all possible regions of the subdrawing $D(G_{11} \cup T^i \cup T^k)$, that $\mathrm{cr}_D(G_{11} \cup T^i \cup T^k, T^l) \geq 6$ is fulfilling for such a subgraph T^l.

□

Again, using the algorithm on the website http://crossings.uos.de/ accessed on 10 October 2020, we can also determine the crossing numbers of two small graphs in Lemma 5.

Lemma 5. $\mathrm{cr}(G_{11}+C_3)=9$ *and* $\mathrm{cr}(G_{11}+C_4)=15$.

Theorem 3. $\mathrm{cr}(G_{11}+C_n)=4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+3=n(n-1)+3$ *for* $n\geq 3$.

Proof. Figure 5 shows the drawing of the graph $G_{11}+C_n$ with exactly $n(n-1)+3$ crossings. Thus, $\mathrm{cr}(G_{11}+C_n)\leq n(n-1)+3$. By Lemma 5, the result holds for $n=3$ and $n=4$. We prove the reverse inequality by induction on n. Now, let us suppose that, for some $n\geq 5$, there is a drawing D for which

$$\mathrm{cr}_D(G_{11}+C_n)<n(n-1)+3, \tag{7}$$

and that

$$\mathrm{cr}(G_{11}+C_m)=m(m-1)+3 \text{ for any integer } 3\leq m<n. \tag{8}$$

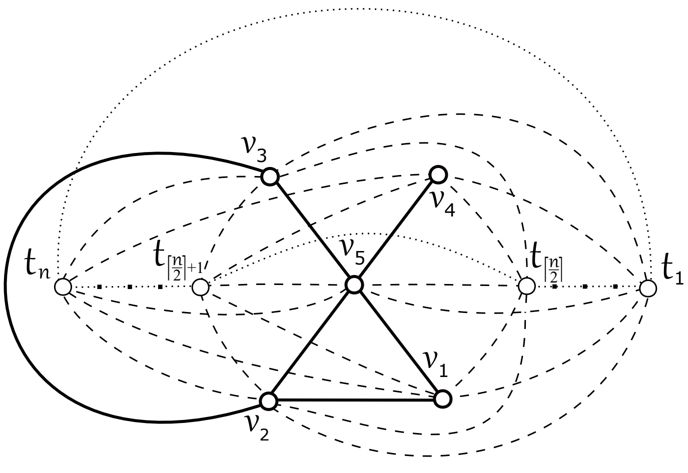

Figure 5. The good drawing of $G_{11}+C_n$ with $n(n-1)+3$ crossings.

Since the graph $G_{11}+D_n$ is also a subgraph of $G_{11}+C_n$, also by Theorem 1, the edges of $G_{11}+C_n$ are crossed at least $n(n-1)$ times. Therefore, at most two edges of the cycle C_n^* can be crossed in D, and this also implies that the vertices t_i of C_n^* have to be placed at most in two different regions of $D(G_{11})$. Moreover, by Theorem 2, there is at most one crossing on each edge of C_n^*. By Lemma 3, we can only suppose two possible planar subdrawings of the graph G_{11} induced by D. All our assumptions on D with $\mathrm{cr}(K_{5,n})=4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ enforce that

$$\mathrm{cr}_D(G_{11})+\mathrm{cr}_D(G_{11},K_{5,n})\leq 2\left\lfloor\frac{n}{2}\right\rfloor+2,$$

i.e.,

$$0+0r+1s+2(n-r-s)\leq 2\left\lfloor\frac{n}{2}\right\rfloor+2, \tag{9}$$

if we use the notation $r=|R_D|$ and $s=|S_D|$ again. This forces that $2r+s\geq 2\lceil\frac{n}{2}\rceil-2$, and if $r=0$, then $s\geq 2\lceil\frac{n}{2}\rceil-2$. Again, we will suppose all possibilities of obtaining some subgraph $T^i\in R_D\cup S_D$ in order to obtain a contradiction with the assumption (7) in all considered subcases in D:

Case 1: We consider the planar drawing of G_{11} in D with the vertex notation in such a way as shown in Figure 2a. We claim that the drawing D must be antipode-free. For a contradiction, suppose that $\mathrm{cr}_D(T^k,T^l)=0$ for two different subgraphs T^k and

T^l. If at least one of T^k and T^l, say T^k, does not cross G_{11}, it is not difficult to check in Figure 2a that the subgraph T^l must cross the edges of $G_{11} \cup T^k$ at least twice, that is, $\mathrm{cr}_D(G_{11}, T^k \cup T^l) = \mathrm{cr}_D(G_{11}, T^l) \geq 2$. Moreover, the Kleitman's result [3] for $\mathrm{cr}(K_{5,3}) = 4$ implies that each T^m, $m \neq k, l$ crosses the edges of the subgraph $T^k \cup T^l$ at least four times. Consequently, for the number of crossings in D holds:

$$\mathrm{cr}_D(G_{11} + C_n) = \mathrm{cr}_D(G_{11} + C_{n-2}) + \mathrm{cr}_D(K_{5,n-2}, T^k \cup T^l) + \mathrm{cr}_D(G_{11}, T^k \cup T^l)$$
$$+\mathrm{cr}_D(T^k \cup T^l) \geq (n-2)(n-3) + 3 + 4(n-2) + 2 + 0 = n(n-1) + 3.$$

This contradiction with (7) confirms that D is antipode-free. The authors in [11] also proved that $\mathrm{cr}(G_0 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$, and therefore there are at most $\lfloor \frac{n}{2} \rfloor + 1$ crossings on three edges v_1v_5, v_3v_5, and v_4v_5 in D. This, also with Corollary 2, implies that $r \geq n - \left(\lfloor \frac{n}{2} \rfloor + 1\right) = \lceil \frac{n}{2} \rceil - 1$ provided by each subgraph $T^k \notin R_D$ crosses some of these three edges at least once. Thus, for any $T^i \in R_D$, the vertex t_i have to be placed in the region of $D(G_{11})$ with all five vertices of the graph G_{11} on its boundary.

Let us turn to the possibility of an existence of vertex t_j of the cycle C_n^* in some region of $D(G_{11})$ with three vertices of G_{11} on its boundary, that is, two different edges of C_n^* cross one of the edges v_1v_5 or v_3v_5 in D again by Corollary 2. Since there are two additional crossings on one of these two edges of the graph G_{11}, the mentioned result [11] enforces $r \geq \lceil \frac{n}{2} \rceil + 1 \geq 4$ for n at least 5. Let D' be the subdrawing of $G_{11} + D_n$ induced by D without the edges of C_n^*.

Clearly, the subdrawing D' is some optimal drawing of the graph $G_{11} + D_n$ with exactly $n(n-1)$ crossings. Therefore, we can apply the similar idea as in the proof of Theorem 2, because $\mathrm{cr}_{D'}(G_{11} \cup T^i, T^k) \geq 3$ holds for any two different subgraphs T^i, T^k with $T^i \in R_D$. Again, without a loss of generality, let us also consider that the number of all subgraphs with the configuration $\mathcal{A}_1$ is at least as much as the number of all subgraphs with the configuration $\mathcal{A}_3$, and let $T^i \in R_D$ be such a subgraph with this configuration $\mathcal{A}_1$ of F^i. Then, by fixing the subgraph $G_{11} \cup T^i$, we have

$$\mathrm{cr}_{D'}(G_{11} + D_n) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3(n-1) + 1 \geq n(n-1) + 1.$$

This contradiction with the optimality of the subdrawing D' of $G_{11} + D_n$ confirms that all vertices t_i of the cycle C_n^* are placed in the region of $D(G_{11})$ with five vertices v_1, v_2, v_3, v_4, and v_5 of G_{11} on its boundary. By [11], we already know that there are, at most, $\lfloor \frac{n}{2} \rfloor + 1$ crossings on the three edges v_1v_5, v_3v_5, and v_4v_5 in D. In the rest of the paper, based on their symmetry, let the edge v_1v_5 be crossed, at most, as many times as the edge v_3v_5, that is, there are at most $\lfloor \frac{\lfloor \frac{n}{2} \rfloor + 1}{2} \rfloor$ crossings on the edge v_1v_5 in D. We denote, by $\mathcal{M}_D$, the nonempty subset of $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_3\}$ consisting of all configurations existing in D. Now, two possible subcases may occur:

(a) $\mathcal{A}_1 \in \mathcal{M}_D$. For $T^i \in R_D$ with the configuration $\mathcal{A}_1$ of F^i, there is the possibility of obtaining a subdrawing of $G_{11} \cup T^i \cup T^k$ in which $\mathrm{cr}_D(T^i, T^k) = 2$ holds for some $T^k \in S_D$. For this case by Lemma 4, the edges of the graph $G_{11} \cup T^i \cup T^k$ are crossed by each subgraph T^l, $l \neq i, k$ at least six times except in cases where the edge v_1v_5 of G_{11} is crossed by the edges of T^l. Thus, by fixing the subgraph $G_{11} \cup T^i \cup T^k$, we have

$$\mathrm{cr}_D(G_{11} + C_n) \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(n-2) - \left\lfloor \frac{\lfloor \frac{n}{2} \rfloor + 1}{2} \right\rfloor + 3 \geq n(n-1) + 3.$$

This also contradicts the assumption of D, and therefore, in the next part, suppose that $\mathrm{cr}_D(G_{11} \cup T^i, T^k) \geq 4$ holds for each $T^k \in S_D$. Notice that if $r \geq 3$ and there are two different subgraphs T^i, $T^j \in R_D$ such that F^i and F^j have configurations $\mathcal{A}_1$ and $\mathcal{A}_3$, respectively, then $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq 3 + 4 = 7$ is fulfilling for any $T^k \in R_D$, $k \neq i, j$ and $\mathrm{cr}_D(G_{11} \cup T^i \cup T^j, T^k) \geq 5$ holds for any $T^k \notin R_D$. Therefore, in such a contemplated case, by fixing the graph $G_{11} \cup T^i \cup T^j$, we receive the following contradiction with the assumption in D

$$\mathrm{cr}_D(G_{11}+C_n) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 7(r-2) + 5(n-r) + 3$$

$$\geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 5n + 2r - 11 \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor + 5n + 6 - 11 \geq n(n-1) + 3.$$

Further, if there is a subgraph $T^k \notin R_D \cup S_D$ such that $\mathrm{cr}_D(G_{11} \cup T^i, T^k) = 3$, then the edges v_3v_5 and v_4v_5 of the graph G_{11} are crossed by the edges t_kv_2 and t_kv_1, respectively, which yields by the result in [11] that $r \geq \lceil \frac{n}{2} \rceil \geq 3$ for n at least 5. Finally, if either $\mathcal{M}_D = \{\mathcal{A}_1, \mathcal{A}_3\}$ and $r = 2$ or $\mathcal{M}_D = \{\mathcal{A}_1\}$, by fixing the subgraph $G_{11} \cup T^i$, we have

$$\mathrm{cr}_D(G_{11}+C_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3(n-1) + 1 + 1 + 1 \geq n(n-1) + 3.$$

(b) $\mathcal{M}_D = \{\mathcal{A}_3\}$. Let T^i be any subgraph from the nonempty set R_D. Then, $\mathrm{cr}_D(T^i, T^k) \geq 4$ holds for each subgraph $T^k \in R_D$, $k \neq i$ provided by $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_k)$. Moreover, we can easily verify in five possible regions of $D(G_{11} \cup T^i)$ that $\mathrm{cr}_D(G_{11} \cup T^i, T^k) \geq 4$ is fulfilling for any $T^k \notin R_D$, if the edge v_1v_5 of G_{11} is not crossed by the edges of T^k. Thus, by fixing the subgraph $G_{11} \cup T^i$, we have

$$\mathrm{cr}_D(G_{11}+C_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4(n-1) - \left\lfloor\frac{\lfloor\frac{n}{2}\rfloor + 1}{2}\right\rfloor + 0 \geq n(n-1) + 3.$$

All these subcases confirm a contradiction with the assumption in D.

Case 2: We assume the planar subdrawing of G_{11} with the vertex notation given in Figure 2b. The set R_D is empty, and therefore there are at least $2\lceil \frac{n}{2} \rceil - 2$ subgraphs $T^i \in S_D$ using the inequality (9). The authors in [10,11] also proved that $\mathrm{cr}(G_5 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ and $\mathrm{cr}(G_7 + C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$, and therefore there are at most $\lfloor \frac{n}{2} \rfloor$ crossings on each of the edges v_1v_5 and v_2v_5, respectively. Since each subgraph T^i crosses some edge of the cycle $v_1v_2v_5v_1$ at least once in $D(G_{11} \cup T^i)$, and none of the edges v_1v_2 and v_2v_3 can be crossed in D due to Corollary 2, each of the edges v_1v_5 and v_2v_5 is crossed exactly $\lfloor \frac{n}{2} \rfloor$ times. This also enforces that n must be even and all vertices t_i of the cycle C_n^* are placed in the region of $D(G_{11})$ with four vertices v_1, v_2, v_4, and v_5 of G_{11} on its boundary.

Now, let us turn to list all possible rotations $\mathrm{rot}_D(t_i)$ that can appear in the drawing D if the edges of the graph G_{11} are crossed by the edges of T^i just once. For $T^i \in S_D$, based on the previous discussion, there is only one possible subdrawing of $F^i \setminus \{v_3, v_5\}$ represented by the subrotation (142). This offers four ways of obtaining the subdrawing of F^i depending on which of two edges of the graph G_{11} can be crossed by the edge t_iv_3 and in which region of $D(F_i \setminus v_5)$ the edge t_iv_5 is placed.

We denote these four possibilities by $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, and $\mathcal{B}_4$ with the corresponding cyclic permutations (13542), (14532), (13452), and (15432), respectively. For any $T^i \in S_D$ with the configuration of either $\mathcal{B}_3$ or $\mathcal{B}_4$ of F^i, the reader can easily verify in five possible regions of $D(G_{11} \cup T^i)$ that $\mathrm{cr}_D(G_{11} \cup T^i, T^k) \geq 3$ holds for each subgraph T^k with $k \neq i$. Moreover, $\mathrm{cr}_D(G_{11} \cup T^i, T^k) \geq 4$ is fulfilling for each subgraph T^k, $k \neq i$ if the edges t_iv_3 and t_kv_3 cross the same edge of G_{11}. Thus, by fixing the subgraph $G_{11} \cup T^i$ having the configuration either $\mathcal{B}_3$ or $\mathcal{B}_4$, we obtain

$$\mathrm{cr}_D(G_{11}+C_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right) + 3\left\lfloor\frac{n}{2}\right\rfloor + 1 \geq n(n-1) + 3.$$

This also contradicts the assumption (7) of D. Finally, suppose that there is no subgraph $T^i \in S_D$ with the configuration $\mathcal{B}_3$ and $\mathcal{B}_4$ of F^i. As $s \geq n-2 \geq 4$ for n even of at least 6, there are two different subgraphs $T^i, T^j \in S_D$ such that F^i and F^j have the configurations $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The minimum number of interchanges of adjacent elements of (13542) required to produce the cyclic permutation $\overline{(14532)} = (12354)$ is one. Thus, the subgraph T^j must cross the edges of T^i at least $1 + 2m$ times for some nonnegative integer m and $\mathrm{cr}_D(T^i \cup T^j, T^k) \geq \lfloor \frac{5}{2} \rfloor \left\lfloor \frac{5-1}{2} \right\rfloor - 1 = 3$ is also fulfilling for each subgraph T^k, $k \neq i, j$, for more see Woodall's results [23]. These properties of the cyclic permutations imply that $\mathrm{cr}_D(G_{11} \cup T^i \cup T^j, T^k) \geq 1 + 1 + 4 = 6$ holds for any $T^k \in S_D$, $k \neq i, j$, and $\mathrm{cr}_D(G_{11} \cup T^i \cup T^j, T^k) \geq 2 + 3 = 5$ is also true for any $T^k \notin S_D$. Hence, by fixing the subgraph $G_{11} \cup T^i \cup T^j$, we have

$$\mathrm{cr}_D(G_{11} + C_n) \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(s-2) + 5(n-s) + 2 + 1 = 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

$$+5n + s - 9 \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 5n + (n-2) - 9 \geq n(n-1) + 3.$$

We have shown, in all cases, that there are at least $n(n-1)+3$ crossings in each good drawing D of the graph $G_{11} + C_n$. This completes the proof of Theorem 3. □

5. Some Consequences of the Main Result

In Figure 6, let G_{14} be the connected graph of order five obtained from G_{11} by adding the edge v_3v_4 in the subdrawing in Figure 2a. Since we can add this edge v_3v_4 to the graph G_{11} without additional crossings in Figures 3 and 5, the drawings of the graphs $G_{14} + P_n$ and $G_{14} + C_n$ with exactly $n(n-1)+1$ and $n(n-1)+3$ crossings are obtained, respectively. Further, the graph G_{11} is some subgraph of G_{14}, and therefore, $\mathrm{cr}(G_{14} + P_n) \geq \mathrm{cr}(G_{11} + P_n)$ and $\mathrm{cr}(G_{14} + C_n) \geq \mathrm{cr}(G_{11} + C_n)$. Therefore, the following results are obvious.

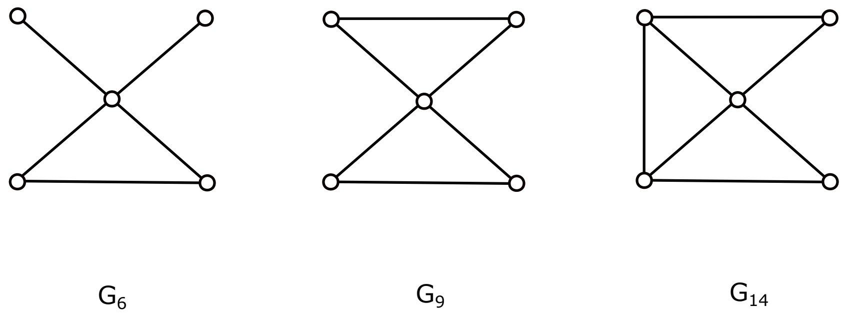

Figure 6. Three graphs G_6, G_9, and G_{14}.

Corollary 3. $\mathrm{cr}(G_{14} + P_n) = n(n-1) + 1$ *for* $n \geq 2$.

Corollary 4. $\mathrm{cr}(G_{14} + C_n) = n(n-1) + 3$ *for* $n \geq 3$.

Similarly, in Figure 6, let G_9 be the graph obtained from G_{11} by adding the edge v_3v_4 and by removing the edge v_2v_3 from the subdrawing in Figure 2a, which yields the good drawing of $G_9 + C_n$ with exactly $n(n-1)+2$ crossings from the optimal drawing of $G_{11} + C_n$ in Figure 5. As G_2 is a subgraph of the graph G_9, we have $\mathrm{cr}(G_9 + C_n) \geq \mathrm{cr}(G_2 + C_n) = n(n-1) + 2$ due to the result by Klešč et al. [9]. Let G_6 be the graph obtained from G_{11} by removing the edge v_2v_3 from the subdrawing in Figure 2a, that is, $\mathrm{cr}(G_9 + C_n) \geq \mathrm{cr}(G_6 + C_n) \geq \mathrm{cr}(G_2 + C_n)$.

Corollary 5. $\mathrm{cr}(G_6 + C_n) = n(n-1) + 2$ *for* $n \geq 3$.

Corollary 6. $\mathrm{cr}(G_9 + C_n) = n(n-1) + 2$ *for* $n \geq 3$.

Notice that Staš [12] also established the results of $\mathrm{cr}(G_6 + P_n) = \mathrm{cr}(G_9 + P_n) = n(n-1)$ as some consequences of $\mathrm{cr}(G_2 + D_n) = n(n-1)$. Finally, Staš and Valiska [14] conjectured that the crossing numbers of $W_m + P_n$ are given by $(Z(m) - 1)\lfloor \frac{n}{2} \rfloor + Z(m + 1)Z(n) + n + 1$, for all $m \geq 3$ and $n \geq 2$, and where W_m denotes the wheel on $m + 1$ vertices and the Zarankiewicz's number $Z(n) = \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is defined for all positive integers n. Recently, this conjecture was proven for $W_3 + P_n$ and $W_4 + P_n$ by Klešč and Schrötter [6] and by Staš and Valiska [14], respectively.

On the other hand, the graphs $W_m + P_2$ and $W_m + P_3$ are isomorphic to the join product of the cycle C_m with the cycle C_3 and with the graph $K_4 \setminus e$ obtained by removing one edge from K_4, respectively. The exact values for the crossing numbers of the graphs $C_m + C_n$ and $K_4 \setminus e + C_m$ are given by Klešč [4,5], respectively, and so the graphs $W_m + P_2$ and $W_m + P_3$ confirm the validity of this conjecture. Since the graph $W_m + P_4$ is isomorphic to the graph $G_{14} + C_m$, we establish the validity of this conjecture also for the graph $W_m + P_4$.

Corollary 7. $\mathrm{cr}(W_m + P_4) = m(m-1) + 3$ *for* $m \geq 3$.

6. Conclusions

We suppose that similar forms of discussions can be used to estimate the unknown values of the crossing numbers of the remaining graphs on five vertices with a much larger number of edges in the join products with the paths, and also with the cycles. We expect the same for other symmetric graphs of order six. Berežný and Staš [24] determined the crossing number of $W_5 + D_n$. Using this result, it would also be useful to confirm the conjecture mentioned in Section 5 for the graph $W_5 + P_n$ in the form $\mathrm{cr}(W_5 + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 3\lfloor \frac{n}{2} \rfloor + 1$ for $n \geq 2$.

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