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A Perov Version of Fuzzy Metric Spaces and Common Fixed Points for Compatible Mappings

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Abstract: In this paper, we define and study the Perov fuzzy metric space and the topology induced by this space. We prove Banach contraction theorems. Moreover, we devised new results for Kramosil and Michálek fuzzy metric spaces. In the process, some results about multidimensional common fixed points as coupled/tripled common fixed point results are derived from our main results.

Keywords: fuzzy metric space; perov metric space; coincidence point; fuzzy topology; hadžić type t-norm

1. Introduction and preliminaries

Perov [1] introduced the notion of vector-valued metric spaces by replacing real numbers with \mathbb{R}^n and proved some fixed point theorems for contractive mappings between these spaces. After the paper [1], series of articles about vector-valued metric spaces started to appear, see e.g., [2–6].

The concept of fuzzy sets was initially investigated by an Iranian mathematician Lofti Zadeh [7] as a new way to represent vagueness in every life. Subsequently, it was developed extensively by many authors and used in various applications in diverse areas and references cited therein. To use this concept in topology, Kramosil and Michálek in [8] introduced the class of fuzzy metric spaces. Later on, George and Veeramani in [9] gave a stronger form of metric fuzziness. This notion has an evident appeal due to its close relationship with probabilistic metric spaces. In particular, they observed that the class of fuzzy metric spaces in their sense, is “equivalent” to the class of Menger spaces with a continuous t-norm.

In this paper, we introduce the notion of Perov fuzzy metric space that generalize the corresponding notions of fuzzy metric space due to Kramosil and Michálek. Additionally, we give the topology induced by this space. Finally we give a Banach contraction theorem. With the help of these results one can derive some results of multidimensional common fixed point as a coupled/tripled common fixed point results for Perov fuzzy metric spaces and Kramosil and Michálek’ ones.

One of the main ingredients of a fuzzy metric space is the notion of *triangular norm*. In this connection let us denote $\mathbb{I} = [0, 1]$ and let X be a nonempty set.

Definition 1 (Schweizer and Sklar [10]). *A triangular norm (also called a t-norm) is a map $* : \mathbb{I}^2 \rightarrow \mathbb{I}$ that is associative, commutative, nondecreasing in both arguments and has 1 as identity. A t-norm is continuous if it is continuous in \mathbb{I}^2 as mapping. If $a_1, a_2, \dots, a_m \in \mathbb{I}$, then*

$$*_i^m a_i = a_1 * a_2 * \dots * a_m.$$

*For each $a \in \mathbb{I}$, the sequence $\{*_m a\}_{m=1}^\infty$ is defined inductively by $*^1 a = a$ and $*^{m+1} a = (*^m a) * a$ for all $m \geq 1$.*



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It is usual to consider continuous t-norms, mainly because fuzzy metric spaces involve a continuous t-norm. However, there is a wide range of non-continuous t-norms (see [10]).

Remark 1. If $m, n \in \mathbb{N}$, then $*^m(*^n a) = *^{m+n} a$ for all $a \in \mathbb{I}$.

Definition 2 (Hadžić and Pap [11]). A t-norm $*$ is said to be of H-type if the sequence $\{ *^m a \}_{m=1}^\infty$ is equicontinuous at $a = 1$, i.e., for all $\varepsilon \in (0, 1)$, there exists $\eta \in (0, 1)$ such that if $a \in (1 - \eta, 1]$, then $*^m a > 1 - \varepsilon$ for all $m \in \mathbb{N}$.

The most important and well known continuous t-norm of H-type is $* = \min$. Other examples can be found in [11,12].

There exist different notions of fuzzy metric space (see [13]). For our purposes, we will use the following one.

Definition 3 (Kramosil and Michálek [8], Grabiec [14]). A triple $(X, M, *)$ is called a fuzzy metric space (briefly, a FMS) if X is an arbitrary non-empty set, $*$ is a continuous t-norm and $M : X \times X \times [0, \infty) \rightarrow \mathbb{I}$ is a fuzzy set satisfying the following conditions, for each $x, y, z \in X$, and $t, s \geq 0$:

- (KM-1) $M(x, y, 0) = 0$;
- (KM-2) $M(x, y, t) = 1$ for all $t > 0$ if, and only if, $x = y$;
- (KM-3) $M(x, y, t) = M(y, x, t)$;
- (KM-4) $M(x, y, \cdot) : [0, \infty) \rightarrow \mathbb{I}$ is left-continuous;
- (KM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.

In this case, we also say that (X, M) is a FMS under $*$.

Let denote $\mathbb{R}_+^n = [0, +\infty)^n$. Recall the concept of generalized metric in Perov’s sense:

Definition 4. By a vector-valued metric on X we mean a mapping $d : X \times X \rightarrow \mathbb{R}_+^n$ such that

- (i) $\overrightarrow{d(u, v)} \geq \overrightarrow{0}$ for all $u, v \in X$ and if $\overrightarrow{d(u, v)} = \overrightarrow{0}$ then $u = v$;
- (ii) $\overrightarrow{d(u, v)} = \overrightarrow{d(v, u)}$ for all $u, v \in X$;
- (iii) $\overrightarrow{d(u, v)} \leq \overrightarrow{d(u, w)} + \overrightarrow{d(w, v)}$ for all $u, v, w \in X$.

Here, if $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$, $\overrightarrow{x} = (x_1, x_2, \dots, x_n)$, $\overrightarrow{y} = (y_1, y_2, \dots, y_n)$, by $\overrightarrow{x} \leq \overrightarrow{y}$ we mean $x_i \leq y_i$ for $i = 1, 2, \dots, n$. In this sense, $\overrightarrow{x} \geq \overrightarrow{0}$ means $x_i \geq 0$ for $i = 1, 2, \dots, n$. (Similarly, $\overrightarrow{x} > \overrightarrow{0}$ means $x_i > 0$ for $i = 1, 2, \dots, n$). We call the pair (X, d) a Perov metric space. For such a space convergence and completeness are similar to those in usual metric spaces.

Throughout this paper we denote by $M_{n,n}(\mathbb{R}_+)$ the set of all $n \times n$ matrices with nonnegative elements, by Θ the zero $n \times n$ matrix and by I the identity $n \times n$ matrix.

Definition 5. A square matrix K with nonnegative elements is said to be convergent to zero if

$$K^p \rightarrow \Theta \text{ as } p \rightarrow \infty.$$

The property of being convergent to zero is equivalent to each of the following conditions from the characterisation lemma below (see [15,16]):

Lemma 1. Let K be a square matrix of nonnegative numbers. The following statements are equivalent:

- (i) K is a matrix convergent to zero;
- (ii) $I - K$ is nonsingular and $(I - K)^{-1} = I + K + K^2 + \dots$;
- (iii) the eigenvalues of K are located inside the unit disc of the complex plane;
- (iv) $I - K$ is nonsingular and $(I - K)^{-1}$ has nonnegative elements.

Please note that according to the equivalence of the statements (i) and (iv), a matrix K is convergent to zero if and only if the matrix $I - K$ is *inverse-positive*.

The following lemma is a consequence of the previous characterisations.

Lemma 2. *Let K be a matrix that is convergent to zero. Then for each matrix P of the same order whose elements are nonnegative and sufficiently small, the matrix $K + P$ is also convergent to zero.*

The matrices convergent to zero were used by A. I. Perov to generalize the contraction principle in the case of metric spaces with a vector-valued distance.

Definition 6. *Let (X, d) be a Perov metric space. An operator $f : X \rightarrow X$ is said to be contractive (with respect to the vector-valued metric d on X) if there exists a convergent to zero (Lipschitz) matrix K such that*

$$\overrightarrow{d}(fu, fv) \leq K\overrightarrow{d}(u, v) \text{ for all } u, v \in X.$$

Theorem 1. *Refs. [1,4] Let (X, d) be a complete Perov metric space and $f : X \rightarrow X$ a contractive operator with Lipschitz matrix K . Then f has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$\overrightarrow{d}(f^k u_0, u^*) \leq K^k (I - K)^{-1} \overrightarrow{d}(u_0, fu_0) \text{ for all } k \in \mathbb{N}.$$

2. Perov Fuzzy Metric Space

We will introduce now the concept of Perov fuzzy metric space and the topology induced by this space. Then we give some properties.

Definition 7. *We will call the triple $(X, M, *)$ Perov fuzzy metric space (briefly, a PFMS) if X is an arbitrary non-empty set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times \mathbb{R}_+^n$ satisfying the following conditions, for each $x, y, z \in X$, and $t, s \geq 0$, where $\overrightarrow{t}, \overrightarrow{s} \in \mathbb{R}_+^n$,*

- (GM-1) $M(x, y, \overrightarrow{0}) = 0$;
- (GM-2) $M(x, y, \overrightarrow{t}) = 1$ for all $\overrightarrow{t} > \overrightarrow{0}$ if, and only if, $x = y$;
- (GM-3) $M(x, y, \overrightarrow{t}) = M(y, x, \overrightarrow{t})$;
- (GM-4) $M(x, y, \cdot) : \mathbb{R}_+^n \rightarrow \mathbb{I}$ is continuous;
- (GM-5) $M(x, y, \overrightarrow{t}) * M(y, z, \overrightarrow{s}) \leq M(x, z, \overrightarrow{t + s})$.

In this case, we also say that (X, M) is a PFMS under $*$.

We will restrict to the case that $\lim_{\overrightarrow{t} \rightarrow \overrightarrow{\infty}} M(x, y, \overrightarrow{t}) = 1$ for all $x, y \in X$, where $\overrightarrow{\infty} = (\infty, \infty, \dots, \infty)$.

Example 1. *Let $X = \mathbb{R}$, $a * b = ab$ and $M : \mathbb{R}^2 \times \mathbb{R}_+^n \rightarrow \mathbb{I}$ defined by*

$$M(x, y, \overrightarrow{t}) = e^{-\frac{|x-y|}{\|\overrightarrow{t}\|}},$$

*if $x \neq y$ and $\overrightarrow{t} > \overrightarrow{0}$; $M(x, y, \overrightarrow{0}) = 0$ and finally $M(x, x, \overrightarrow{t}) = 1$. Then $(X, M, *)$ is a PFMS.*

Example 2. Let $X = \mathbb{N} = \{1, 2, \dots\}$, $a * b = ab$ and $M : \mathbb{N}^2 \times \mathbb{R}_+^n \rightarrow \mathbb{I}$ defined by

$$M(x, y, \vec{t}) = \begin{cases} x/y & \text{if } x \leq y \\ y/x & \text{if } x \geq y \end{cases}.$$

Then $(X, M, *)$ is a PFMS.

Lemma 3. If (X, M) is a PFMS under some t -norm and $x, y \in X$, then $M(x, y, \cdot)$ is a non-decreasing function on \mathbb{R}_+^n .

Proof. Assume that $M(x, y, \vec{t}) > M(x, y, \vec{s})$ for $\vec{s} > \vec{t} > \vec{0}$. Then by (GM-2) and (GM-5),

$$M(x, y, \vec{t}) * 1 = M(x, y, \vec{t}) * M(y, y, \vec{s} - \vec{t}) \leq M(x, y, \vec{s}) < M(x, y, \vec{t}).$$

This is a contradiction. \square

To construct a suitable topology on a PFMS $(X, M, *)$, we consider the natural balls:

Definition 8. Let $(X, M, *)$ be a PFMS. For $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$, the open ball $B(x, r, \vec{t})$ with center $x \in X$, radius \vec{t} and fuzziness parameter $r \in (0, 1)$ is defined by

$$B(x, r, \vec{t}) = \{y \in X : M(x, y, \vec{t}) > 1 - r\}.$$

As with the proof of Results 3.2, 3.3 and Theorem 3.11 of [9], one can show the following results.

Theorem 2. Let $(X, M, *)$ be a PFMS. Define

$$\tau = \{A \subset X : x \in A \text{ iff there exist } r \in (0, 1), \vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}, \text{ s.t. } B(x, r, \vec{t}) \subset A\}.$$

Then τ is a topology on X .

In this topology, we may consider the following notions.

- A sequence $\{x_m\}_{m \geq 0} \subset X$ is *Cauchy* if for any $\varepsilon > 0$ and $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$, there exists $m_0 \in \mathbb{N}$ such that $M(x_m, x_{m+p}, \vec{t}) > 1 - \varepsilon$ for all $m \geq m_0$ and all $p \geq 1$.
- A sequence $\{x_m\}_{m \geq 0} \subset X$ is *convergent* (or *M-convergent*) to $x \in X$, denoted by $\lim_{m \rightarrow \infty} x_m = x$ or $\{x_m\} \xrightarrow{M} x$, if for any $\varepsilon > 0$ and $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$, there exists $m_0 \in \mathbb{N}$ such that $M(x_m, x, \vec{t}) > 1 - \varepsilon$, for all $m \geq m_0$.
- A PFMS in which every Cauchy sequence is convergent is called *complete*.

The limit of a convergent sequence in a PFMS is unique.

Given any t -norm $*$, it is easy to prove that $* \leq \min$. Therefore, if (X, M) is a PFMS under \min , then (X, M) is a PFMS under any (continuous or not) t -norm.

Proposition 1. Let $(X, M, *)$ be a PFMS. Then M is a continuous function on $X \times X \times \mathbb{R}_+^n$.

Proof. Let $x, y \in X$ and $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$ and let $\{(x_m, y_m, \vec{t}_m)\}$ be a sequence in $X \times X \times \mathbb{R}_+^n$ that converges to (x, y, \vec{t}) . Since $\{M(x_m, y_m, \vec{t}_m)\}$ is a sequence in $(0, 1]$ and hence $\{M(x_m, y_m, \vec{t}_m)\}$ converges to some point of $[0, 1]$ up to a subsequence.

Consider a subsequence $(x_{m_k}, y_{m_k}, \vec{t}_{m_k})$ of (x_m, y_m, \vec{t}_m) such that

$$\lim_{k \rightarrow \infty} M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}) = \liminf_{m \rightarrow \infty} M(x_m, y_m, \vec{t}_m)$$

(that always exists by definition of limit inferior).

Fix $\vec{\delta} \in \mathbb{R}_+^n, \vec{\delta} > \vec{0}$ such that $2\|\vec{\delta}\| < \|\vec{t}\|$. Then there is $m_0 \in \mathbb{N}$ such that $\|\vec{t}_{m_k} - \vec{t}\| < \|\vec{\delta}\|$ for all $m_k \geq m_0$. Hence,

$$M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}) \geq M(x_{m_k}, x, \vec{\delta}/2) * M(x, y, \vec{t} - 2\vec{\delta}) * M(y, y_{m_k}, \vec{\delta}/2)$$

and

$$M(x, y, \vec{t} + \vec{\delta}) \geq M(x, x_{m_k}, \vec{\delta}/2) * M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}) * M(y, y_{m_k}, \vec{\delta}/2),$$

for all $m_k \geq m_0$. By taking limits when $k \rightarrow \infty$, we obtain

$$\liminf_{m \rightarrow \infty} M(x_m, y_m, \vec{t}_m) = \lim_k M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}) \geq 1 * M(x, y, \vec{t} - 2\vec{\delta}) * 1 = M(x, y, \vec{t} - 2\vec{\delta}).$$

Now, consider another subsequence $(x_{m_k}, y_{m_k}, \vec{t}_{m_k})$ of (x_m, y_m, \vec{t}_m) , this time such that

$$\lim_{k \rightarrow \infty} M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}) = \limsup_{m \rightarrow \infty} M(x_m, y_m, \vec{t}_m).$$

Then

$$M(x, y, \vec{t} + \vec{\delta}) \geq 1 * \lim_k M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}) * 1 = \lim_k M(x_{m_k}, y_{m_k}, \vec{t}_{m_k}).$$

Sending $\vec{\delta} \rightarrow 0$ one concludes the proof. \square

Now we are going to introduce fuzzy balls:

Definition 9. Let $(X, M, *)$ be a PFMS. The fuzzy open ball $\mathbb{B}(x, \vec{t}) : X \rightarrow \mathbb{I}$ with center $x \in X$ and radius $\vec{t} \in \mathbb{R}_+^n$ is a fuzzy set defined by

$$\mathbb{B}(x, \vec{t})(y) = 1 - M(x, y, \vec{t}).$$

Proposition 2.

- $\mathbb{B}(x, \vec{t})(y) = \mathbb{B}(y, \vec{t})(x)$
- $\mathbb{B}(x, \vec{t})_r = \mathbb{B}(x, 1 - r, \vec{t})$, for every $r \in (0, 1)$, where

$$A_r := \{y \in X : A(y) \geq r\}, \quad \bar{A}(y) = 1 - A(y),$$

for every fuzzy set $A : X \rightarrow \mathbb{I}$.

- Fuzzy open balls reduce in the crisp case to open balls.

To begin with, we consider the standard intersection $\cap_{i \in N} A_i$ of fuzzy sets A_i on $X, i \in N$ with N a finite set. It is defined by the membership function $(\cap_{i \in N} A_i)(x) = \min_{i \in N} A_i(x), x \in X,$

The operations of fuzzy sets A_1 and A_2 are listed as follows:

$$(A_1 \cap A_2)(x) := \min\{A_1(x), A_2(x)\},$$

$$(A_1 \cup A_2)(x) := \max\{A_1(x), A_2(x)\}$$

Proposition 3. Let $\{t_m\}_{m \in Y}, \{s_m\}_{m \in Z}, t, s \in \mathbb{R}_+^n, s, t$

$$\sup t_m = t, \quad \inf s_m = s > 0$$

- $\mathbb{B}(x, t) = \bigcup_{m \in Y} \mathbb{B}(x, t_m)$
- $\mathbb{B}(x, s) \subset \bigcap_{m \in Z} \mathbb{B}(x, s_m)$ and equality holds when Z is finite.

3. Main Results

We start this section with an auxiliary result.

Lemma 4. Let $(X, M, *)$ be a PFMS such that $*$ is a t -norm of H -type, let $K \in M_{n,n}(\mathbb{R}_+)$, $K \rightarrow \Theta$, and $\{x_m\}_{m \geq 0}$ be a sequence of X such that for all $m \geq 1$ and all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$,

$$M(x_m, x_{m+1}, K\vec{t}) \geq M(x_{m-1}, x_m, \vec{t}). \tag{1}$$

Then $\{x_m\}_{m \geq 0}$ is a Cauchy sequence.

Proof. For all $m, p \geq 0$ and all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$, define

$$\beta_{m,p}(\vec{t}) = M(x_m, x_{m+p}, \vec{t}).$$

Since $*$ is non-decreasing on each argument and each $M(x, y, \cdot)$ is non-decreasing, whatever $x, y \in X$, then every $\beta_{m,p}$ is a non-decreasing function on \mathbb{R}_+^n . Repeating (1), for all $m, p \geq 1$ and all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$

$$\begin{aligned} M(x_{m+p}, x_{m+p+1}, K^{m+p}\vec{t}) &\geq M(x_{m+p-1}, x_{m+p}, K^{m+p-1}\vec{t}) \geq \dots \geq M(x_m, x_{m+1}, K^p\vec{t}) \\ &\geq \dots \geq M(x_0, x_1, \vec{t}), \end{aligned}$$

which means that

$$\beta_{m+p,1}(K^{m+p}\vec{t}) \geq \beta_{m+p-1,1}(K^{m+p-1}\vec{t}) \geq \dots \geq \beta_{m,1}(K^m\vec{t}) \geq \beta_{0,1}(\vec{t}). \tag{2}$$

For a fixed \vec{t} we can assume that $(I - K)\vec{t}$ admits positive componets. In other case, there exists a power K^p such that $(I - K^p)\vec{t}$ admits positive componets and then arguing with that power in place of just K .

Since K is a matrix convergent to zero, $I - K$ is non-singular and

$$(I - K)^{-1} = \sum_{q=0}^{\infty} K^q,$$

then, for all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$ and all $p \geq 1$,

$$\vec{t} = (I - K)^{-1}(I - K)\vec{t} = \left(\sum_{q=0}^{\infty} K^q \right) (I - K)\vec{t} > (I + K + \dots + K^{p-1})(I - K)\vec{t}.$$

Hence

$$\begin{aligned} \beta_{m,p}(\vec{t}) &= M(x_m, x_{m+p}, \vec{t}) \geq M(x_m, x_{m+p}, (I + K + \dots + K^{p-1})(I - K)\vec{t}) \\ &\geq M(x_m, x_{m+1}, (I - K)\vec{t}) * \dots * M(x_{m+p-1}, x_{m+p}, K^{p-1}(I - K)\vec{t}) \\ &= \underset{r=0}{*}^{p-1} M(x_{m+r}, x_{m+r+1}, K^r(I - K)\vec{t}) = \underset{r=0}{*}^{p-1} \beta_{m+r,1}(K^r(I - K)\vec{t}). \end{aligned}$$

Applying (1)

$$\begin{aligned} M(x_{m+r}, x_{m+r+1}, K^r(I - K)\vec{t}) &\geq M(x_{m+r-1}, x_{m+r}, K^{r-1}(I - K)\vec{t}) \geq \dots \\ &\geq M(x_m, x_{m+1}, (I - K)\vec{t}) = \beta_{m,1}((I - K)\vec{t}) \end{aligned}$$

for all $r \in \{0, 1, \dots, p - 1\}$ and all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$. Joining the previous inequalities

$$\beta_{m,p}(\vec{t}) \geq \prod_{r=0}^{p-1} M(x_m, x_{m+1}, (I - K)\vec{t}) \geq \prod_{r=0}^{p-1} (\beta_{m,1}((I - K)\vec{t})) = *^p \beta_{m,1}((I - K)\vec{t}).$$

Therefore, by (2),

$$\beta_{m,p}(K^m(I - K)^{-1}\vec{t}) \geq *^p \beta_{m,1}(K^m\vec{t}) \geq *^p \beta_{0,1}(\vec{t}).$$

Fix $\vec{t} > \vec{0}$ and $\varepsilon > 0$. Since $*$ is of H -type, there exists $\delta > 0$ such that for all $s \in (1 - \delta, 1]$, one has $*^m s > 1 - \varepsilon$ for all $m \in \mathbb{N}$. Since $\sup_{\vec{t} > \vec{0}} \beta_{0,1}(\vec{t}) = 1$, there exists $\vec{t}_0 > \vec{0}$ such that $\beta_{0,1}(\vec{t}_0) > 1 - \delta$. Thus $*^p \beta_{0,1}(\vec{t}_0) > 1 - \varepsilon$ for all $p \in \mathbb{N}$. Since K is a matrix convergent to zero, there exists $\vec{s} > \vec{t}_0, \vec{s} \in [0, \infty)^n$, such that $\lim_{m \rightarrow \infty} K^m \vec{s} = \vec{0}$. Thus, there exists $m_0 \in \mathbb{N}$ such that $K^n(I - K)\vec{s} < \vec{t}$ for all $m > m_0$. Then we have

$$\beta_{m,p}(\vec{t}) \geq \beta_{m,p}(K^m(I - K)^{-1}\vec{t}) \geq *^p \beta_{0,1}(\vec{t}) > 1 - \varepsilon,$$

for all $m > m_0$.

Hence we can conclude that

$$\lim_{m \rightarrow \infty} \beta_{m,p}(\vec{t}) = 1 \quad \text{for all } \vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0} \text{ and all } p \geq 1.$$

We are going to prove that $\{x_m\}$ is Cauchy. Indeed, let $\varepsilon > 0$ and $\vec{t} > \vec{0}$ arbitrary. Since $*$ is of H -type, there exists $\eta \in (0, 1)$ such that if $a \in (1 - \eta, 1]$, then $*^m a > 1 - \varepsilon$ for all $m \in \mathbb{N}$. Since $\lim_{m \rightarrow \infty} \beta_{m,1}((I - K)\vec{t}) = 1$, there exists $m_0 \in \mathbb{N}$ such that if $m \geq m_0$, then $\beta_{m,1}((I - K)\vec{t}) > 1 - \eta$. Therefore, if $m \geq m_0$ and $p \geq 1$

$$\beta_{m,1}((I - K)\vec{t}) > 1 - \eta \quad \Rightarrow \quad *^p \beta_{m,1}((I - K)\vec{t}) > 1 - \varepsilon.$$

It follows that

$$M(x_m, x_{m+p}, \vec{t}) \geq \beta_{m,p}(\vec{t}) \geq *^p \beta_{m,1}((I - K)\vec{t}) > 1 - \varepsilon.$$

for all $m \geq m_0$ and all $p \in \mathbb{N}$. This means that the sequence $\{x_m\}$ is Cauchy. \square

To avoid the commutativity condition between f and g , we introduce the concept of compatible mappings in PFMS

Definition 10. Let $(X, M, *)$ be a PFMS. Two mappings $f : X \rightarrow X$ and $g : X \rightarrow X$ are said to be compatible if, for any sequence $\{x_m\}_{m \geq 0}$ such that there exists $\lim_{m \rightarrow \infty} f x_m = \lim_{m \rightarrow \infty} g x_m \in X$, we have that

$$\lim_{m \rightarrow \infty} M(g f x_m, f g x_m, \vec{t}) = 1 \quad \text{for all } \vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}.$$

Obviously, if f and g are commuting, then they are compatible, but the converse does not hold. We state and prove some fixed point results for compatible mappings.

Theorem 3. Let $(X, M, *)$ be a complete PFMS such that $*$ is a t -norm of H -type. Let $f : X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that $f(X) \subseteq g(X)$ and g is continuous and compatible with f . Assume that there exists a matrix $K \in M_{n,n}(\mathbb{R}_+), K \rightarrow \Theta$, such that

$$M(fx, fy, K\vec{t}) \geq M(gx, gy, \vec{t}) \tag{3}$$

for all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$ and all $x, y \in X$. Then f and g have a coincidence fixed point (that is, there is a unique $z \in X$ such that $fz = gz$).

Proof. Let $x_0 \in X$. There exists a sequence $\{x_m\}_{m \geq 0}$ such that $gx_{m+1} = fx_m$ for all m . For all $\vec{t} \in \mathbb{R}_+^n$ and all m ,

$$M(gx_{m+1}, gx_{m+2}, K\vec{t}) = M(fx_m, fx_{m+1}, K\vec{t}) \geq M(gx_m, gx_{m+1}, \vec{t}).$$

Lemma 4 guarantees that $\{gx_m\}_{m \geq 0}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $x \in X$ such that $x = \lim_{m \rightarrow \infty} gx_m = \lim_{m \rightarrow \infty} fx_m$.

As g is continuous,

$$\lim_{m \rightarrow \infty} gfx_m = \lim_{m \rightarrow \infty} ggx_{m+1} = gx. \tag{4}$$

Since f and g are compatible, we have that

$$\lim_{m \rightarrow \infty} M(ggx_{m+1}, fgx_m, \vec{t}) = \lim_{m \rightarrow \infty} M(fgx_m, fgx_m, \vec{t}) = 1. \tag{5}$$

Hence,

$$M(fx, gx, t) \geq M(fx, ggx_m, K\vec{t}) * M(ggx_m, gx, (I - K)\vec{t}).$$

Taking $m \rightarrow \infty$ on the both side of the above inequality and using Proposition 1,

$$\begin{aligned} M(fx, gx, \vec{t}) &\geq \lim_{m \rightarrow \infty} M(fx, ggx_m, K\vec{t}) * M(ggx_m, gx, (I - K)\vec{t}) \\ &= M(fx, \lim_{m \rightarrow \infty} fgx_m, K\vec{t}) * M(\lim_{m \rightarrow \infty} ggx_m, gx, (I - K)\vec{t}) \\ &\geq \lim_{m \rightarrow \infty} M(fx, fgx_m, K\vec{t}) * M(gx, gx, (I - K)\vec{t}) \\ &\geq \lim_{m \rightarrow \infty} M(fx, fgx_m, K\vec{t}), \end{aligned}$$

for all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$. From the above, using (3), for all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$, we have

$$M(fx, gx, \vec{t}) \geq \lim_{m \rightarrow \infty} M(fx, fgx_m, K\vec{t}) \geq \lim_{m \rightarrow \infty} M(gx, ggx_m, \vec{t}) \geq M(gx, gx, \vec{t}) = 1.$$

Therefore, $gx = fx$, i.e., f and g have a coincidence point. \square

Corollary 1. Under the hypothesis of Theorem 3, if x is a coincidence fixed point of f and g , then $z = gx$ is also a coincidence point of f and g .

Proof. Call $\{u_m\} = \{gx_m\}$ and $z = fx = gx$. First $\{fu_m\}_m = \{fgx_m\}_m \rightarrow fx = z$ by (5) and g is continuous

$$\lim_{m \rightarrow \infty} gfx_m = \lim_{m \rightarrow \infty} ggu_{m+1} = gz$$

Since, $\{gu_m\}_m = \{ggx_m\}_m \rightarrow gx = z$ and f and g are compatible

$$\lim_{m \rightarrow \infty} M(ggu_m, fggu_m, \vec{t}) = \lim_{m \rightarrow \infty} M(fgu_m, fggu_m, \vec{t}) = 1.$$

Then,

$$M(fz, gz, \vec{t}) \geq M(fz, ggu_m, K\vec{t}) * M(ggu_m, gz, (I - K)\vec{t}).$$

Taking $m \rightarrow \infty$ on the both side of the above inequality and using Proposition 1,

$$\begin{aligned}
 M(fz, gz, \vec{t}) &\geq \lim_{m \rightarrow \infty} M(fz, ggu_m, K \vec{t}) * M(ggu_m, gz, (I - K) \vec{t}) \\
 &= M(fz, \lim_{m \rightarrow \infty} fgu_m, K \vec{t}) * M(\lim_{m \rightarrow \infty} ggu_m, gz, (I - K) \vec{t}) \\
 &\geq \lim_{m \rightarrow \infty} M(fz, fgu_m, K \vec{t}) * M(gz, gz, (I - K) \vec{t}) \\
 &\geq \lim_{m \rightarrow \infty} M(fz, fgu_m, K \vec{t}),
 \end{aligned}$$

for all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$. From the above, using (3), for all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$, we have

$$M(fz, gz, \vec{t}) \geq \lim_{m \rightarrow \infty} M(fz, fgu_m, K \vec{t}) \geq \lim_{m \rightarrow \infty} M(gz, ggu_m, \vec{t}) \geq M(gz, gz, \vec{t}) = 1.$$

Therefore, $gz = fz$, i.e., $fx = gx = z$ is also a coincidence point of f and g . □

Theorem 4. Under the hypotheses of Theorem 3, f and g have a unique common fixed point (that is, there is a unique $z \in X$ such that $fz = gz = z$). In fact, if $x \in X$ is any coincidence point of f and g , then $z = fx = gx$ is their only common fixed point.

Proof. Step 1. Existence. Let x be a coincidence point of f and g and $z = gx$ is another one. Next, we claim that $gz = z$. Indeed, fix $\varepsilon > 0$ and $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$ arbitrary. We know that $\lim_{\vec{s} \rightarrow \vec{\infty}} M(z, gz, \vec{s}) = \lim_{\vec{s} \rightarrow \vec{\infty}} M(gx, gz, \vec{s}) = 1$, so there exists $\vec{t}_0 > \vec{0}$ such that $M(z, gz, \vec{t}_0) > 1 - \eta$.

We notice that

$$M(gz, z, K \vec{t}) = M(fz, fx, K \vec{t}) \geq M(gz, gx, \vec{t}) = M(gz, z, \vec{t}).$$

Repeating this argument, it can be possible to prove, by induction, that

$$M(gz, z, K^m \vec{t}) \geq M(gz, z, \vec{t}) \quad \text{for all } m \in \mathbb{N}. \tag{6}$$

As $K \rightarrow \Theta$, then $\{K^m \vec{t}\} \rightarrow 0$. Additionally, as $\vec{t} > \vec{0}$, there is $m_0 \in \mathbb{N}$ such that $K^{m_0} \vec{t}_0 < \vec{t}$. It follows from (6) and Lemma 3 that

$$M(gz, z, \vec{t}) \geq M(gz, z, K^{m_0} \vec{t}_0) \geq M(gz, z, \vec{t}_0) > 1 - \varepsilon.$$

Taking into account that ε and $\vec{t} > \vec{0}$ are arbitrary, we deduce that $M(gz, z, \vec{t}) = 1$ for all $\vec{t} \in \mathbb{R}_+^n, \vec{t} > \vec{0}$, i.e., $gz = z$. This proves that $fz = gz = z$, so z is a common fixed point of f and g .

Step 2. Uniqueness. To prove the uniqueness, let $y \in X$ be another common fixed point of f and g , i.e., $fy = gy = y$. Fix $\varepsilon > 0$ and $\vec{t} > \vec{0}$ arbitrary. We know that $\lim_{\vec{s} \rightarrow \vec{\infty}} M(z, y, \vec{s}) = 1$, so there exists $\vec{t}_0 > \vec{0}$ such that $M(z, y, \vec{t}_0) > 1 - \eta$. We notice that

$$M(z, y, K \vec{t}_0) = M(fz, fy, K \vec{t}_0) \geq M(gz, gy, \vec{t}_0) = M(z, y, \vec{t}_0).$$

Repeating this argument, it can also be possible to prove, by induction, that

$$M(z, y, K^m \vec{t}_0) \geq M(z, y, \vec{t}_0) \quad \text{for all } m \in \mathbb{N}.$$

As $K \rightarrow \Theta$, then $\{K^m \vec{t}_0\} \rightarrow 0$. Additionally, as $\vec{t} > \vec{0}$, there is $m_0 \in \mathbb{N}$ such that $K^{m_0} \vec{t}_0 < \vec{t}$. It follows that

$$M(z, y, \vec{t}) \geq M(z, y, K^{m_0} \vec{t}_0) \geq M(z, y, \vec{t}_0) > 1 - \varepsilon.$$

Taking into account that $\varepsilon > 0$ and $\vec{t} > \vec{0}$ are arbitrary, we deduce that $M(z, y, \vec{t}) = 1$ for all $\vec{t} > \vec{0}$, i.e., $z = y$. This proves that f and g have a unique common fixed point. \square

The following corollary is a fixed point result, particularizing Theorem 3 to the case in which g is the identity mapping on X .

Corollary 2. *Let $(X, M, *)$ be a complete PFMS such that $*$ is a t -norm of H -type. Let $f : X \rightarrow X$ be a mapping such that there exists a matrix $K \in M_{n,n}(\mathbb{R}_+)$, $K \rightarrow \Theta$, with*

$$M(fx, fy, K \vec{t}) \geq M(x, y, \vec{t}) \tag{7}$$

for all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$ and all $x, y \in X$. Then f has a unique fixed point.

4. The Case of Product of Perov Fuzzy Metric Spaces

One of the newest branches of fixed point theory is devoted to the study of coupled fixed points, introduced by Guo and Lakshmikantham [17] in 1987. Thereafter, their results were extended and generalized by several authors in the last few years; see [12,18] and the references cited therein. Recently, Roldán et al. [18] introduced the notion of coincidence point between mappings in any number of variables, and several special extended to multidimensional case appeared in the literature; see, for example [19–23], respectively. Many of the presented high-dimensional results become simple consequences of their corresponding unidimensional versions (see [24]).

The following results are given to show how coupled/tripled notions and the compatibility can be reduced to the unidimensional case using the following mappings. Given $N \in \{2, 3\}$ and $F : X^N \rightarrow X$ and $g : X \rightarrow X$, let denote by $T_F^N, G^N : X^N \rightarrow X^N$ the mappings

$$\begin{cases} N = 2, & T_F^2(x, y) = (F(x, y), F(y, x)), \\ N = 3, & T_F^3(x, y, z) = (F(x, y, z), F(y, x, y), F(z, y, x)), \end{cases} \tag{8}$$

$$\begin{cases} N = 2, & G^2(x, y) = (gx, gy), \\ N = 3, & G^3(x, y, z) = (gx, gy, gz), \end{cases} \tag{9}$$

For instance, the following lemma guarantees that multidimensional notions of common/fixed/coincidence points can be interpreted in terms of T_F^N and G^N .

Lemma 5. *Given $N \in \{2, 3\}$, $F : X^N \rightarrow X$ and $g : X \rightarrow X$, a point $(x_1, x_2, \dots, x_N) \in X^N$ is:*

1. *a coupled/tripled fixed point of F if, and only if, it is a fixed point of T_F^N ;*
2. *a coupled/tripled coincidence point of F and g if, and only if, it is a coincidence point of T_F^N and G^N ;*
3. *a coupled/tripled common fixed point of F and g if, and only if, it is a common fixed point of T_F^N and G^N .*

Definition 11. *Let $(X, M, *)$ be a PFMS. Two mappings $F : X^n \rightarrow X$ and $g : X \rightarrow X$ are said to be Φ -compatible if, for all sequences $\{x_m^1\}_{m \geq 0}, \{x_m^2\}_{m \geq 0}, \dots, \{x_m^n\}_{m \geq 0} \subset X$ such that*

$$\exists \lim_{m \rightarrow \infty} F(x_m^1, x_m^2, \dots, x_m^n) = \lim_{m \rightarrow \infty} gx_m^i \in X \text{ for all } i,$$

we have that

$$\lim_{m \rightarrow \infty} M(gF(x_m^1, x_m^2, \dots, x_m^n), F(gx_m^1, gx_m^2, \dots, gx_m^n), \vec{t}) = 1$$

for all $\vec{t} > \vec{0}$, $t \in \mathbb{R}_+^n$ and all i .

Lemma 6. Given $N \in \{2, 3\}$, two mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are compatible if, and only if, T_F^N and G^N are compatible.

Next, we show how to use Theorem 3 in order to deduce coupled and tripled common fixed point results. We only have to particularize our main result to the case X^N , where $N \in \{2, 3\}$. We can deduce a multidimensional result similarly.

Corollary 3. Let $(X, M, *)$ be a complete PFMS such that $*$ is a t -norm of H -type. Let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be compatible mappings. Assume that g is continuous and there exists a matrix $K \in M_{n,n}(\mathbb{R}_+)$, $K \rightarrow \Theta$, such that

$$M(F(x, y), F(u, v), K \vec{t}) \geq M(gx, gu, \vec{t}) * M(gy, gv, \vec{t}) \tag{10}$$

for all $x, y, u, v \in X$ and all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$.

Then F and g have a unique coupled common fixed point ω of the form $\omega = (z, z)$, where $z \in X$.

Scheme of the proof. Check that $(X^2, M^2, *)$ is a complete PFMS. By Lemma 6, T_F^2 and G^2 are compatible. Contractivity condition (10) yields contractivity condition in Theorem 3. \square

For 3-case, we can deduce also

Corollary 4. Let $(X, M, *)$ be a complete PFMS such that $*$ is a t -norm of H -type. Let $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ be compatible mappings. Assume that g is continuous and there exists a matrix $K \in M_{n,n}(\mathbb{R}_+)$, $K \rightarrow \Theta$, such that

$$M(F(x, y, z), F(u, v, w), K \vec{t}) \geq M(gx, gu, t) * M(gy, gv, \vec{t}) * M(gz, gw, \vec{t})$$

for all $x, y, z, u, v, w \in X$ and all $\vec{t} \in \mathbb{R}_+^n$, $\vec{t} > \vec{0}$. Assume also that there exist $x_0, y_0, z_0 \in X$ such that $\lim_{\vec{t} \rightarrow \infty} M(gx_0, F(x_0, y_0, z_0), \vec{t}) = \lim_{\vec{t} \rightarrow \infty} M(gy_0, F(y_0, x_0, y_0), \vec{t}) = \lim_{\vec{t} \rightarrow \infty} M(gz_0, F(z_0, y_0, x_0), \vec{t}) = 1$. Then F and g have a tripled fixed point.

Furthermore, assume that for all pairs of tripled fixed points, (x, y, z) and (u, v, w) , $\lim_{\vec{t} \rightarrow \infty} M(gx, gu, \vec{t}) = \lim_{\vec{t} \rightarrow \infty} M(gy, gv, \vec{t}) = \lim_{\vec{t} \rightarrow \infty} M(gz, gw, \vec{t}) = 1$, then F and g have a unique tripled common fixed point ω of the form $\omega = (z, z, z)$, where $z \in X$.

5. The Case of Product of Fuzzy Metric Spaces

As we have pointed out before, many of the high-dimensional results become simple consequences of their corresponding unidimensional versions. In this section, with a similar approach, we obtain new high-dimensional results, but in fuzzy metric spaces context.

We begin showing some basic results that we will need in the main section. We start this section introducing a generalized fuzzy structure on the product space X^N .

Lemma 7. Let $(X, M, *)$ be a FMS and let $N \in \mathbb{N}$. Consider the product space $X^N = X \times X \times \dots \times X$ of N identical copies of X . Let define $M^N : X^N \times X^N \times [0, \infty)^N \rightarrow \mathbb{I}$ given by:

$$M^N(A, B, \vec{t}) = \prod_{i=1}^N M(a_i, b_i, t_i) \tag{11}$$

for all $A = (a_1, a_2, \dots, a_N), B = (b_1, b_2, \dots, b_N) \in X^N$ and all $\vec{t} = (t_1, \dots, t_N) \geq \vec{0}$. Then the following properties hold.

1. $(X^N, M^N, *)$ is also a PFMS.
2. Let $\{A_n = (a_n^1, a_n^2, \dots, a_n^N)\}$ be a sequence on X^N and let $A = (a_1, a_2, \dots, a_N) \in X^N$. Then $\{A_n\} \xrightarrow{M^N} A$ if, and only if, $\{a_n^i\} \xrightarrow{M} a_i$ for all $i \in \{1, 2, \dots, N\}$.

3. If $\{A_n = (a_n^1, a_n^2, \dots, a_n^N)\}$ is a sequence on X^N , then $\{A_n\}$ is M^N -Cauchy if, and only if, $\{a_n^i\}$ is M -Cauchy for all $i \in \{1, 2, \dots, N\}$.
4. $(X, M, *)$ is complete if, and only if, $(X^N, M^N, *)$ is complete.

Proof. (1) All properties are trivial taking into account that $*$ is a continuous mapping.

(2) Notice that for all $n \in \mathbb{N}$ and all $j \in \{1, 2, \dots, N\}$,

$$M^N(A_n, A, \vec{t}) = \underset{i=1}{*}^N M(a_n^i, a_i, t_i) \leq 1 * \dots * 1 * M(a_n^j, a_j, t_j) * 1 * \dots * 1 = M(a_n^j, a_j, t_j) \leq 1$$

Therefore, if $\{A_n\} \xrightarrow{M^N} A$, then $\{a_n^j\} \xrightarrow{M} a_j$ for all $j \in \{1, 2, \dots, N\}$. Conversely, assume that $\{a_n^i\} \xrightarrow{M} a_i$ for all $i \in \{1, 2, \dots, N\}$. As $*$ is a continuous mapping, then, for all $\vec{t} > \vec{0}$,

$$\lim_{n \rightarrow \infty} M^N(A_n, A, \vec{t}) = \lim_{n \rightarrow \infty} \left(\underset{i=1}{*}^N M(a_n^i, a_i, t_i) \right) = \underset{i=1}{*}^N \left(\lim_{n \rightarrow \infty} M(a_n^i, a_i, t_i) \right) = *^N 1 = 1,$$

which means that $\{A_n\} \xrightarrow{M^N} A$.

(3) Similarly, it can be proved that for all $n, k \in \mathbb{N}$, all $j \in \{1, 2, \dots, N\}$ and all $\vec{t} > \vec{0}$,

$$M^N(A_n, A_k, \vec{t}) \leq M(a_n^j, a_k^j, t_j) \leq 1.$$

Therefore, if $\{A_n\}$ is a M^N -Cauchy sequence, then $\{a_n^j\}$ is a M -Cauchy sequences for all $j \in \{1, 2, \dots, N\}$. The converse is similar.

(4) It follows from the last two items. \square

Trivially, we can prove

Lemma 8. Given $N \in \{2, 3\}$, two mappings $F : X^N \rightarrow X$ and $g : X \rightarrow X$ are compatible (in the PFMS's sense) if, and only if, T_F^N and G^N are compatible (in FMS's sense).

We particularize the main result to the coupled and tripled cases and obtain new kind of results (compare with Theorem 3.2. in [25], Theorem 1 in [26]).

Corollary 5. Let $(X, M, *)$ be a complete FMS such that $*$ is a t -norm of H -type. Let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be compatible mappings. Assume that g is continuous and there exists a matrix

$$K = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{2,1} & k_{2,2} \end{pmatrix} \in M_{2,2}(\mathbb{R}_+), K \rightarrow \Theta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ such that}$$

$$M(F(x, y), F(u, v), k_{1,1}t_1 + k_{1,2}t_2) \geq M(gx, gu, t_1) \tag{12}$$

$$M(F(y, x), F(v, u), k_{2,1}t_1 + k_{2,2}t_2) \geq M(gy, gv, t_2) \tag{13}$$

for all $x, y, u, v \in X$ and all $t_1, t_2 > 0$.

Assume also that there exist $x_0, y_0 \in X$ such that

$$\lim_{t \rightarrow \infty} M(gx_0, F(x_0, y_0), t) = \lim_{t \rightarrow \infty} M(gy_0, F(y_0, x_0), t) = 1.$$

Then F and g have a coupled fixed point.

Furthermore, assume that for all pairs of coupled fixed points, (x, y) and (u, v) ,

$$\lim_{t \rightarrow \infty} M(gx, gu, t) = \lim_{t \rightarrow \infty} M(gy, gv, t) = 1,$$

then F and g have a unique coupled common fixed point ω of the form $\omega = (z, z)$, where $z \in X$.

Proof. By items 1 and 4 of Lemma 7, $(X^2, M^2, *)$ is a complete PFMS. By Lemma 8, T_F^2 and

G^2 are compatible. Finally, contractivity conditions (12) and (13) yield, for $\vec{t} = (t_1, t_2)$:

$$\begin{aligned} M^2(T_F^2(x, y), T_F^2(u, v), K \vec{t}) &= M^2((F(x, y), F(y, x)), (F(u, v), F(v, u)), K \vec{t}) \\ &= M(F(x, y), F(u, v), k_{1,1}t_1 + k_{1,2}t_2) * M(F(y, x), F(v, u), k_{2,1}t_1 + k_{2,2}t_2) \\ &\geq M(gx, gu, t_1) * M(gy, gv, t_2) \\ &= M^2((gx, gy), (gu, gv), \vec{t}) = M^2(G^2(x, y), G^2(u, v), \vec{t}). \end{aligned}$$

Applying Theorem 3, T_F^2 and G^2 have a unique common fixed point, i.e., a point $\omega = (\omega_1, \omega_2) \in X^2$ such that $T_F^2\omega = G^2\omega = \omega$. ω is the unique coupled common fixed point of F and g . Following point by point the arguments of the proof of Theorem 3, it is possible to prove that $M(\omega_1, \omega_2, t) > 1 - \varepsilon$ for all $\varepsilon, t > 0$, so $\omega_1 = \omega_2$ and ω is of the form (z, z) . \square

In the previous result, we have new kind of contractivity conditions and moreover the condition $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ usually used is weakened here. Similarly, we can deduce the tripled one.

Corollary 6. *Let $(X, M, *)$ be a complete FMS such that $*$ is a t -norm of H -type. Let $F : X^3 \rightarrow X$ and $g : X \rightarrow X$ be compatible mappings. Assume that g is continuous and there exists a matrix*

$$K = \begin{pmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \\ k_{3,1} & k_{3,2} & k_{3,3} \end{pmatrix} \in M_{3,3}(\mathbb{R}_+), K \rightarrow \Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ such that}$$

$$M(F(x, y, z), F(u, v, w), k_{1,1}t_1 + k_{1,2}t_2 + k_{1,3}t_3) \geq M(gx, gu, t_1) \tag{14}$$

$$M(F(y, x, y), F(v, u, v), k_{2,1}t_1 + k_{2,2}t_2 + k_{2,3}t_3) \geq M(gy, gv, t_2) \tag{15}$$

$$M(F(z, y, x), F(w, v, u), k_{3,1}t_1 + k_{3,2}t_2 + k_{3,3}t_3) \geq M(gz, gw, t_3) \tag{16}$$

for all $x, y, z, u, v, w \in X$ and all $t_1, t_2, t_3 > 0$.

Assume also that there exist $x_0, y_0, z_0 \in X$ such that

$$\lim_{t \rightarrow \infty} M(gx_0, F(x_0, y_0, z_0), t) = 1$$

$$\lim_{t \rightarrow \infty} M(gy_0, F(y_0, x_0, y_0), t) = 1$$

$$\lim_{t \rightarrow \infty} M(gz_0, F(z_0, y_0, x_0), t) = 1.$$

Then F and g have a tripled fixed point.

Furthermore, assume that for all pairs of coupled fixed points, (x, y, z) and (u, v, w) , $\lim_{t \rightarrow \infty} M(gx, gu, t) = \lim_{t \rightarrow \infty} M(gy, gv, t) = \lim_{t \rightarrow \infty} M(gz, gw, t) = 1$, then F and g have a unique tripled common fixed point ω of the form $\omega = (z, z, z)$, where $z \in X$.

6. Discussion

The new concept of Perov fuzzy metric space, which is a generalization of fuzzy metric space has been introduced. Moreover, some properties of this concept have been discussed. In addition, we obtained several new common fixed point results. Ultimately, to illustrate the usability of the main theorem, the existence of a new results in fuzzy metrics is proved.

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