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The Reduction of Initial Reserves Using the Optimal Reinsurance Chains in Non-Life Insurance

Galina Horáková, František Slaninka and Zsolt Simonka *

Department of Mathematics & Actuarial Science, Faculty of Economic Informatics, University of Economics in Bratislava, Dolnozemská Cesta 1/b, 852 35 Bratislava, Slovakia; galina.horakova@euba.sk (G.H.); frantisek.slaninka@euba.sk (F.S.)

* Correspondence: zsolt.simonka@euba.sk

Abstract: The aim of the paper is to propose, and give an example of, a strategy for managing insurance risk in continuous time to protect a portfolio of non-life insurance contracts against unwelcome surplus fluctuations. The strategy combines the characteristics of the ruin probability and the values *VaR* and *CVaR*. It also proposes an approach for reducing the required initial reserves by means of capital injections when the surplus is tending towards negative values, which, if used, would protect a portfolio of insurance contracts against unwelcome fluctuations of that surplus. The proposed approach enables the insurer to analyse the surplus by developing a number of scenarios for the progress of the surplus for a given reinsurance protection over a particular time period. It allows one to observe the differences in the reduction of risk obtained with different types of reinsurance chains. In addition, one can compare the differences with the results obtained, using optimally chosen parameters for each type of proportional reinsurance making up the reinsurance chain.

Keywords: compound Poisson process; surplus process; Brownian motion; inverse Gaussian distribution; ruin probability; conditional value at risk; reinsurance chain; optimisation criteria; reserves

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1. Introduction

The role of Solvency II is to create, via capital adequacy, a unified regulatory framework with the aim of protecting European Union policyholders in accordance with consistent rules for managing risk. The aim of IFRS is to create a unified framework for financial reporting by means of transparent and comparable accounting statements. These fundamental changes to external reporting increase even more the opportunity for an insurer to make use of the information gained in the effective management of the company. The aim of this paper is to present a particular mathematical approach to setting the reserves required by an insurer in order to maintain solvency with a specified probability. The approach allows us to analyse a portfolio of non-life insurance contracts, using a combination of the properties of the ruin probability and a relevant conditional value at risk *CVaR*. Based on a combination of the properties of these two risk measures, we propose a strategy for protecting the portfolio against undesirable surplus fluctuations. The approach is complemented by the possibility to reduce the required initial reserves by increasing the amount of capital at times when the surplus is tending to a negative value. It also permits the creation of a number of scenarios for the development of the surplus, allowing for a given reinsurance cover. Using these scenarios, one can not only analyse a portfolio based on a specified structure of reinsurance protection with given parameters, but also to set these optimally.

Our starting point is the well-known definitions of the basic terms and so the contribution of the paper should be the approach shown for the concurrent use of risk measures in combination with a plan of reinsurance. It should be noted that although we use chains made up of proportional reinsurance covers in our example, the approach can also be used with non-proportional covers, including the application of reinsurers' limits.

Generalisation of the basic collective model can be used to follow the development of the surplus in continuous time. Amongst the authors who have grappled with and enhanced the topic of the collective risk model in a single time period, we can mention Bowers, Gerber, Hickman, Jones, Nesbitt [1], Bühlmann [2], Cipra [3] Daykin, Pentikäinen, Pesonen [4], Dickson, Waters [5], Gerber [6], Heilmann [7], Heilmann, Schröter [8], Hogg, Klugman [9], Hossack, Pollard, Zehnwirth [10], Kass, Goovaerts, Dhaene [11], Klugman, Panjer, Willmot [12], Pacáková [13], Panjer, and Willmot [14].

The definition of the collective risk model for a single time period is the basic building block which we use to illustrate our submitted approach for managing insurance risk. This is closely linked to the ruin probability, i.e., the quantification of the situation where the insurer is unable to meet its liabilities. Those of note in the field of ruin theory, who have expanded the theoretical results from Lundberg's era, are Asmussen [15], Picard, Lèfevre [16], Grimmett, Welsh [17], Dickson [18] and Schmidli [19].

Dickson [20], Hossack, Pollard, Zehnwirth [10] and Panjer [21] have dealt with reinsurance at the theoretical level. Analysis of the capital strategy whereby capital is increased when surplus becomes negative was dealt with, for example, by Dickson and Waters [22]. Hindley [23] has dealt with the whole problem of reserves.

Using the above literature, the text Teória rizika v poistení (Risk theory in insurance) by Horáková, Páleš and Slaninka [24] was published. It unifies and adapts the notation in this field and amplifies the proofs of the assertions needed for the desired analyses. This forms the basis for combining a generalisation of the collective risk model for longer time periods and the ruin probability with a framework for optimising reinsurance chains containing both proportional and non-proportional reinsurance cover. In accordance with the requirements of the Slovak Society of Actuaries, a complex model was created, based on which it is possible to analyse risk in non-life insurance from various points of view. This mathematical framework forms the starting point for this paper in which we modify it to cover the idea of revising the capital needed to cover the risks taken on.

Formulation of the solution to the problem: The insurer's aim is to have sufficient reserves to cover its accepted liabilities. Our aim is to illustrate the possibility of setting an optimal level of initial reserves together with reinsurance, where we allow for the injection of further capital at times when the insurer's surplus is tending towards negative values. The results will provide the insurer with additional information based on which it will maintain or even change its chosen risk management path.

We follow the insurer's surplus over time given a specified probability that the insurer is unable to cover the risks it has taken on and that the insurer has reduced its risk, using a given type of reinsurance. We use a compound process for the total claims to determine the risk measures, namely, the ruin probability and the value *CVaR* in various time periods. We incorporate into our approach types of reinsurance based on a proportional split of the risk between the insurer and the reinsurer or reinsurers. We make use of the optimisation criteria for minimising the ruin probability and minimising the variance given the expected surplus. Based on the obtained results, we proceed to an analysis of the followed risk resulting in a proposed approach for reducing the initial reserves required to cover the risk with an adequate predetermined probability.

We present a set of mathematical methods and approaches, which we use to obtain the results in the application part of this paper. The generalised collective model is used to follow the development of the surplus in continuous time. Analyses of strategies for increasing the insurer's capital in the literature relating to insurance risk models, e.g., Dickson and Waters [5], usually assume that whenever the surplus falls into the negative territory, an adequate amount of capital will be injected so that the company can continue reliably to operate.

The timing of the capital injections and their amount can be modelled by means of our proposed approach, using a compound Poisson process and its approximation by

Brownian motion in association with the inverse Gaussian distribution. Using a generalised basic collective risk model in continuous time, we determine the ruin probability for a given initial level of the reserves and we use the fact that the random time to the point of ruin, in the case that this occurs, has an inverse Gaussian distribution. We determine the conditional value at risk of this random variable, given that it belongs to the exponential reproductive dispersion class of distributions. Using the results obtained, we can determine the time period in which we need to apply an algorithm to estimate the additional capital required at that time so that the surplus returns to an acceptable level. The approach is illustrated using the mathematical framework set out in Horákova, Slaninka, Páleš [24] whereby, as already stated, this publication arose as a specific mathematical base precisely for use in studying the impact of risk on an insurer's results.

2. Mathematical Framework for Following an Insurer's Surplus over Time

Given the goal we have set, we use the advantages of the risk measures—ruin probability and the relevant *VaR* and *CVaR*—whereby when comparing the results from these measures, we develop scenarios for how the insurer deals with the risk investigated. We therefore set out, in brief, the essential assertions, facts and tools needed to achieve the approach proposed for reducing the initial capital requirement.

We make use of the distribution laws of the random variable representing the total claim amounts at a particular time, its characteristics, including the value-at-risk (*VaR*) and the conditional-value-at-Risk (*CVaR*). VaR_p is the 100p percentile of the distribution of the total claim amounts and $CVaR_p$ is the expected loss, given that the loss exceeds the 100p percentile of the distribution of the total claim amounts.

2.1. Collective Risk Model for Longer Time Periods

The continuous time ruin theory deals with collective risk models for longer time periods, with the help of which it is possible to analyse the progress of the surplus over a period of years. The insurer's surplus U_t at time t is determined by the initial reserves, i.e., the amount of the surplus U_0 at time $t = 0$, by the amount of the premiums accepted up to time t and by the amount of the claims paid up to that time, whereby the expenses of meeting the claims are a random variable. In order to describe the model mathematically, we introduce the following notation.

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with parameter λ, and let $\{X_i\}_{i=1}^{\infty}$ be a series of independent identically distributed random variables independent of the N_t for $t \geq 0$, with the identical distribution function $F_X(x)$. We can then define the process $\{S_t\}_{t\geq 0}$ as follows:

$$S_t = X_1 + X_2 + \ldots + X_{N_t} = \sum_{i=1}^{N_t} X_i \tag{1}$$

By analogy with the compound distribution for one time period, we assume that the level of an individual claim is a positive valued random variable. $\{S_t\}_{t\geq 0}$ is a compound Poisson process with parameter λ. For $t > 0$, the random variable S_t has a compound distribution with Poisson parameter $\lambda \cdot t$, with the probability laws and properties applicable in the collective risk model. In the context of the compound Poisson distribution, we denote the appropriate surplus process as $\{U_t\}_{t\geq 0}$ for which we have the following:

$$U_t = U_0 + c \cdot t - S_t,\ t \geq 0 \tag{2}$$

where U_0 is the value of the insurer's reserves at the start of the investigated time period and c is the constant intensity of receipt of premiums in a unit time interval. The total premiums received up to time t can then be expressed as $c \cdot t$ for $t > 0$. The surplus increases continuously as premiums are received and is subject to decrements as claims arise with individual claim amounts $\{X_1, X_2, \ldots\}$ at random times $\{T_1, T_2, \ldots\}$. We assume that in each unit time interval the premiums received exceed the expected claim amounts in that interval, so for example, in the case of the compound Poisson distribution and calculation of the premium using the mean value principle, we have the following:

$$c = \lambda \cdot (1+\theta) \cdot \nu_1(X) \tag{3}$$

where θ is the risk margin and $\nu_1(X)$ is the initial first order moment of the individual claim amount, which we assume exists and is finite in which case U_t is the random variable describing the insurer's surplus at the end of the time period $[0; t]$.

We will denote by $\Psi(U_0; \tau)$ the ruin probability in the bounded continuous time period $[0; \tau]$—the continuous time finite horizon ruin probability—which we define as follows:

$$\Psi(U_0; \tau) = P(U_t < 0 \text{ for some } t \leq \tau | U_0) \tag{4}$$

So $\Psi(U_0; \tau)$ represents the probability that the insurer's surplus falls below zero at some point in the time period and is dependent on the initial surplus U_0. The time of ruin $T(U_0)$ dependent on the level of the initial reserves is defined in continuous time as follows:

$$T(U_0) = \min\{t \geq 0 : U_t < 0\} \tag{5}$$

For given initial reserves U_0 we denote by $\Psi(U_0)$ the probability of ruin at a distant point in the future (the continuous time infinite horizon ruin probability). The insurer's surplus, at that time, becomes negative. We can also express this by means of the random variable T representing the time to ruin:

$$\begin{gathered} \Psi(U_0) = P(T < \infty) \\ \Psi(U_0) = 1 - P(U_t \geq 0;\ \forall t \geq 0 | U_0) \end{gathered} \tag{6}$$

Equation (6) thus expresses the probability that the claim amounts paid will at some time in the distant future exceed the initial reserves plus the received premiums calculated according to the mean value principle up to that time. We can estimate the ruin probability $\Psi(U_0)$, using Lundberg's inequality using knowledge of the adjustment coefficient [25]. The adjustment coefficient R expresses the level of risk of the surplus process. This is dependent on two factors, namely, the total claim amounts and the received premiums. If the aggregate claims represent a Poisson process, the adjustment coefficient is defined using the Poisson parameter λ, the moment generating function of the individual claims $m_X(z)$ and the amount of the premiums received in unit time. We can express it in the case of both discrete and continuous individual claim amounts. In the classical risk process, coefficient R is defined as the sole positive root of the following equation:

$$m_{S_1 - c}(r) = 1 \tag{7}$$

based on the moment generating function of the compound Poisson distribution for $t = 1$. Given that derivation of the coefficient R is not always possible, its accurate value can be replaced by upper and lower limits. Using the Taylor expansion, we have the following [24,26]:

$$\frac{\ln(1+\theta)}{M} \leq R \leq 2\theta \frac{E(X)}{E(X^2)} \tag{8}$$

where $\nu_1(X) = E(X)$ and $\nu_2(X) = E(X^2)$ are the moments of the random variable representing an individual claim amount, θ is the risk addition and M is the upper limit of an individual claim amount. The coefficient R, despite being defined in relation to the expected number of claims in one time period, is finally expressed as not dependent on the parameter λ.

2.2. Brownian Motion with Drift and the Ruin Probability

We use Brownian motion to approximate the surplus process $\{U_t\}_{t\geq 0}$, which is related to the compound Poisson process. Based on the definition given in [12], the stochastic process $\{W_t\}_{t\geq 0}$ is an example of Brownian motion (see Figure 1), if the following is true:

1. $W_0 = 0$;
2. $\{W_t\}_{t\geq 0}$ has stationary and independent increments;
3. For each time $t > 0$, W_t has a normal distribution with mean 0 and variance $\sigma^2 t$.

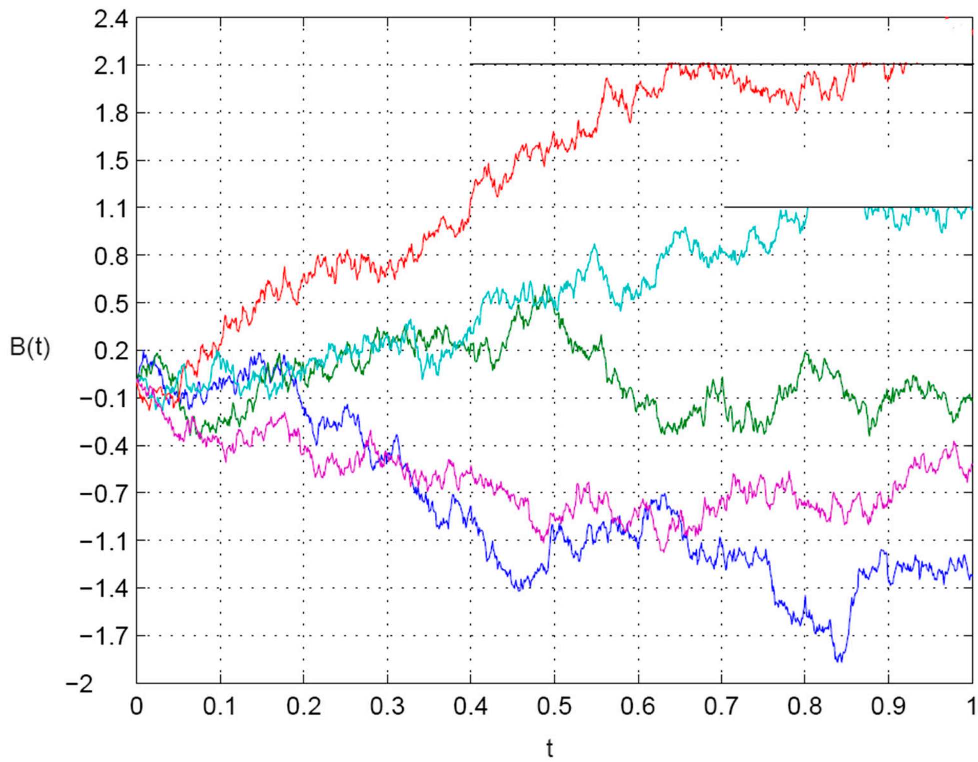

Figure 1. Brownian motion paths.

A continuous time stochastic process $\{W_t\}_{t\geq 0}$ is called Brownian motion with drift if it meets the requirements for Brownian motion; W_t has mean μt rather than 0, for some $\mu > 0$. We use this definition to investigate the profit process $\{Z_t\}_{t\geq 0}$, using a compound Poisson risk process meeting the requirements of $\{W_t\}_{t\geq 0}$, as also in this case, we consider the number of jumps which increases with time whilst at the same time their size decreases. For more details, see, for example, [12,27].

Here, we assume that the amounts of the individual claims $\{X_1, X_2, \ldots\}$ are independent, positive-valued random variables. The surplus process, therefore, increases in continuous time, as premiums are paid in each unit time period and has consecutive decrements due to the individual claims $\{X_1, X_2, \ldots\}$ at the random times $\{T_1, T_2, \ldots\}$.

Let

$$Z_t = U_t - U_0 = c \cdot t - S_t,\ t \geq 0 \tag{9}$$

then $Z_0 = 0$. As S_t has a compound distribution the process, Z_t has mean and variance according to the basic relationships valid in the collective risk model for long time periods. We express the mean and variance of Z_1 as follows:

$$E(Z_1) = \mu,\ Var(Z_1) = \sigma^2 \tag{10}$$

$\{S_t\}_{t\geq 0}$ is a continuous time process with stationary independent increments and this also applies to the processes $\{U_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$. Given that the number of increments increases, the process followed cannot be differentiated but the paths of the Brownian motion are continuous functions of the variable t. We can justify continuity, as the increments are small, and if we denote the total distance of a path by the random variable D in the interval (0; t) of the process U_t, then the following is true:

$$D = ct + S_t = c \cdot t + X_1 + \ldots + X_{N_t} \tag{11}$$

and the mean value of D is unbounded.

In order to use this fact together with the process Z_t to justify replacing the process by Brownian motion, we let, in accordance with [12], the random variable $X = \alpha Y$, with fixed mean and variance of the random variable Y. Then, D is a function of Y and α, and we have the following:

$$\lambda = \frac{\sigma^2}{E(Y^2)} \cdot \frac{1}{\alpha^2} \tag{12}$$

and the following:

$$c = \mu + \sigma^2 \frac{E(Y)}{E(Y^2)} \cdot \frac{1}{\alpha} \tag{13}$$

In order for $\lambda \to \infty$ we need to have from (12) that $\alpha \to 0$. The expected value of the random variable D, given (12) and (13), is as follows:

$$E(D) = t \cdot (c + \lambda E(X)) = t \cdot \left(\mu + 2\sigma^2 \frac{E(Y)}{E(Y^2)} \cdot \frac{1}{\alpha} \right) \tag{14}$$

and therefore, the following is true:

$$\lim_{\alpha \to 0} E(D) = \infty \tag{15}$$

This means that the expected value of the distance D in a finite time interval is infinitely large. Via the moment generating function Z_t we get the following:

$$\lim_{\alpha \to 0} m_{Z_t}(z) = e^{(z \cdot \mu \cdot t + \frac{z}{2}\sigma^2 \cdot t)}. \tag{16}$$

For every time $t > 0$, Z_t has, therefore, in the limit a normal distribution with mean $\mu \cdot t$ and variance $\sigma^2 \cdot t$. Since $Z_t = U_t - U_0$, given the equality of the means, we can approximate the process $\{Z_t\}_{t \geq 0}$ by a Brownian motion with drift. The approximation becomes more precise as the expected number of claims increases and the size of the jumps decreases. If $\{W_t\}_{t \geq 0}$ is a Brownian motion with drift with mean $m > t$ and variance $s^2 > t$, and $U_t = U_0 + W_t$ is a Brownian motion with drift, then the ruin probability in a finite time interval $(0;\ \tau)$ can be expressed as follows:

$$\Psi(U_0;\ \tau) = P(T < \tau) = P\left(\min_{0<t<\tau} U_t < 0 \right) \approx P\left(\min_{0<t<\tau} W_t < -U_0 \right) \tag{17}$$

This approximates the ruin probability for the surplus process since it includes all the cash flows, i.e., the resulting positive as well as negative values of the surplus. The ruin probability in some time interval (0, t) is the sum of the probabilities of all such possibilities. Therefore, the final expression for the ruin probability immediately before time τ, given the characteristics of the random variable Z_1 and Equations (4), (6) and (8) is the following:

$$P(Z_\tau < -U_0) + P(Z_\tau > U_0)e^{-R \cdot U_0} = \Phi\left(\frac{-U_0 - \mu\tau}{\sqrt{\sigma^2 \tau}} \right) + 1 - \Phi\left(\frac{U_0 - \mu\tau}{\sqrt{\sigma^2 \tau}} \right) \cdot e^{-\frac{2\theta\mu\lambda}{\sigma^2 \lambda} U_0}.$$

Hence, the following is true:

$$\Psi(U_0;\tau) = \Phi\left(-\frac{U_0+\mu\tau}{\sqrt{\sigma^2\tau}}\right) + e^{-\frac{2\mu}{\sigma^2}U_0}\cdot\Phi\left(-\frac{U-\mu\tau}{\sqrt{\sigma^2\tau}}\right) \tag{18}$$

where $\Phi(.)$ is the distribution function of the normalised normal distribution.

We can replace the process $\{Z_t\}_{t\geq 0}$ by the process $\{W_t\}_{t\geq 0}$, where $E(Z_t) = \mu t$ and $Var(Z_t) = \sigma^2 t$. Ruin may occur in a short period of time, which we express in the first term on the right-hand side of Equation (18). Alternatively, it can occur ultimately, which we express as a product in the second term on the right-hand side of Equation (18). The probability of ultimate ruin is expressed by means of Lundberg's inequality and the correction coefficient derived with the help of a Taylor series.

Equation (17) does not, however, represent the probability law of the time to ruin. We obtain this distribution by making an adjustment with respect to the ultimate ruin probability. This means that the distribution of the time until ruin, given that ruin occurs $P(T<\tau|T<\infty)$, can be expressed by the following equation which is important in what follows:

$$\frac{\Psi(U_0;\tau)}{\Psi(U_0)} = e^{\frac{2\cdot\mu}{\sigma^2}U_0}\Phi\left(-\frac{U_0+\mu\tau}{\sqrt{\sigma^2\tau}}\right) + \Phi\left(-\frac{U_0-\mu\tau}{\sqrt{\sigma^2\tau}}\right),\ \tau>0. \tag{19}$$

We also make use of this in the numerical example in the second part of this paper in connection with the inverse Gaussian distribution to estimate the ruin probability in the case of a particular reinsurance cover. Equation (19) is connected closely with the distribution function of the inverse Gaussian distribution. Random variables with this distribution belong to the reproductive exponential dispersion family, which allows us in a simple way to express the *CVaR* of the inverse Gaussian distribution [28]. Since we use this connection in the application part, we now look briefly at the distribution laws of a random variable with this distribution and show the relationship with the derivation of the risk measures.

2.3. Inverse Gaussian Distribution

For the distribution function of the inverse Gaussian distribution, $X \sim IG(m;\delta)$ with probability density function in the following form:

$$f_X(x) = \begin{cases} \sqrt{\frac{\delta}{2\pi x^3}}\cdot e^{-\frac{\delta(x-m)^2}{2m^2x}} & x>0 \\ 0 & x\leq 0 \end{cases} \tag{20}$$

We have the following:

$$F_X(x) = \begin{cases} 0 & x\leq 0 \\ \int\limits_0^x \sqrt{\frac{\delta}{2\pi z^3}}\cdot e^{-\frac{\delta(z-m)^2}{2m^2z}}\,\mathrm{d}z & x>0 \end{cases} \tag{21}$$

Differentiating Equation (19), with respect to τ, we get the density function of the inverse Gaussian distribution $T \sim IG\left(\frac{U_0}{\mu};\frac{U_0{}^2}{\sigma^2}\right)$ in the following form:

$$f_T(\tau) = \frac{U_0}{\sqrt{2\pi\sigma^2}}\tau^{-\frac{3}{2}}\cdot e^{\left(-\frac{(U_0-\mu\tau)^2}{2\sigma^2\tau}\right)},\ \tau>0 \tag{22}$$

The random variable T expresses the time to ruin in the case where this occurs with the following mean:

$$E(T) = \frac{U_0}{\mu} = \frac{U_0}{\lambda\theta\cdot E(X)} \tag{23}$$

For any process based on the compound Poisson process, we can easily get a simple numerical approximation. The accuracy of this approximation for the expected time to ruin is dependent on the relative sizes of the appropriate quantities, i.e., of the parameters of the individual distributions.

Since we also use in our analysis the conditional value at risk, we give, for the value $x_p = VaR_p(X)$ of the random variable with inverse Gaussian distribution $X \sim IG(m; \delta)$ defined by Equation (20), an explicit expression for the value $CVaR_p(X)$. We obtain this by expressing this probability law as a two-parameter Tweedy distribution:

$$f_{X_{REDF}}(x; \theta; \lambda) = e^{\lambda(\theta x + \sqrt{-2\theta})} \cdot q(x; \lambda) \tag{24}$$

The Tweedy distribution is a special case of the exponential dispersion models, which, on the one hand, generalise the normal distribution and make use of some of its important properties whilst also having the properties of a right-skewed distribution. The density of the inverse Gaussian distribution as expressed in Equation (20) can also be expressed in the form shown in Equation (24), which conforms to a distribution belonging to the reproductive exponential dispersion family. Given this form of the density, we can express relatively simply the *CVaR* also with the help of the original parameters of the inverse Gaussian distribution:

$$X \sim IG(m; \delta)$$

By means of this, we can express the conditional value at risk, thus the following is true:

$$CVaR_p(X) = m + \frac{m/\delta}{\overline{F}_X(x_p, m, \delta)} \cdot$$

$$\cdot \left\{ \sqrt{\delta \cdot x_p} \cdot \varphi\left(\frac{1}{m}\sqrt{\delta \cdot x_p} - \sqrt{\frac{\delta}{x_p}}\right) + e^{\frac{2\delta}{m}}\left[2\delta \cdot \Phi\left(-\frac{1}{m}\sqrt{\delta \cdot x_p} - \sqrt{\frac{\delta}{x_p}}\right) - \sqrt{\delta x_p} \cdot \varphi\left(-\frac{1}{m}\sqrt{\delta \cdot x_p} - \sqrt{\frac{\delta}{x_p}}\right)\right]\right\}. \tag{25}$$

This is covered in more detail in, for example, [29].

2.4. Optimal Reinsurance

The main reasons why an insurer buys reinsurance as a means of risk transfer are the following:

- To stabilise its financial results.
- To reduce its required capital.
- To increase its underwriting capacity.
- To gain access to the advantages from bigger funds.
- To diversify and reduce the probability of making a loss with which an insurer can, only with difficulty, come to terms.

Hence, the final tool which we use in our analysis of a particular risk is a mathematical description of a reinsurance chain and the criteria for the optimal setting of the relevant parameters of the reinsurance protection. By the reinsurance chain, we mean, in general, a reinsurance cover made up of two or more types of reinsurance, for example, quota-surplus, surplus-quota, quota-excess of loss and quota-excess of loss with a reinsurer's limit. The composition of the chain needs to allow for the requirements of the particular proportional and non-proportional reinsurance and the order in which they are to be applied. At the end of the day, the criteria, making use of the relevant input data, are mutually substitutable. The choice of criteria depends on what we want to achieve. In general, we have at our disposal these four basic criteria.

1. Minimisation of the *VaR* or *CVaR*.
2. Maximisation of the total surplus, given a constant variance of the surplus.
3. Minimisation of the variance of the surplus, given a constant surplus.
4. Minimisation of the ruin probability, given a constant surplus.

Given the mathematical framework we have presented, we can use all these optimisation criteria and calculate the premium using various principles. In order to show a particular approach to reducing the risk, we use a premium calculated by using the mean value principle, RP_θ, and the variances, RP_η, expressed by the following equations:

$$RP_\theta = (1+\theta) \cdot E\left(S^{col}\right), \theta > 0 \tag{26}$$

$$RP_\eta = E\left(S^{col}\right) + \overline{\theta} \cdot Var\left(S^{col}\right), \ \overline{\theta} > 0 \tag{27}$$

where $E\left(S^{col}\right) = E(S_1)$ is the mean of the total claim amounts and $Var\left(S^{col}\right) = Var(S_1)$ is their variance. The approach to calculate the premium, which we introduce in the application part of this paper, can also be applied if we calculate the premium, using, for example, the Esscher principle or the exponential zero utility principle.

For example, the third criterion focuses on ensuring a specified surplus. Given the required surplus, it allows the insurer to set optimal protection by seeking to minimize the variance of the total claim amounts. This criterion provides the possibility of protecting the portfolio for a level of surplus, which the insurer would achieve without the costs of reinsuring, by means of an optimal distribution of the risk between the insurer and the reinsurer. This means that for the same ruin probability, one can, by means of an optimal choice of reinsurance parameters, reduce the required initial reserves.

The optimisation process, using one of the above criteria, assumes that we know the nature of the original risk and of the risk after reinsurance is allowed for.

An optimal structure of chains of reinsurance protection can help the insurer avoid taking on excessive risk, whilst keeping for itself an adequate retention, which should have a positive effect on the price of its offered products. Such an optimal structure for a given portfolio of insurance contracts can be set mathematically, using the general properties of the mean value and the variance, the derived properties of the compound distribution and the proposed split of the risk between the cedant and the reinsurer.

To use fully our knowledge of reinsurance also implies using a method for determining an optimal chain of reinsurance protection. We derive this in the conditions of the collective model, which are modified to allow for the requirements of the individual reinsurance protections and their composition. For both quota share and surplus reinsurance, the insured amount S, the claim amount X and the premium P are split between the insurer and the reinsurer in an agreed way. For quota share, there is an agreed proportion q, $0 < q < 1$, where q is the insurer's retention.

In the case of surplus reinsurance, the reinsurer takes, in respect of each insured sum, the amount which is in excess of an agreed sum known as the retention, which then represents the amount of risk retained by the insurer $\alpha > 0$. In this case, for an insured amount S, premium P and claim amount X, we have the following, from the point of view of the reinsurer:

$$Z_{S_\alpha} = \begin{cases} 0 & S \leq \alpha \\ (S-\alpha) & S > \alpha \end{cases} \tag{28}$$

The effect of this from the point of view of the insurer and the reinsurer is illustrated in Figure 2.

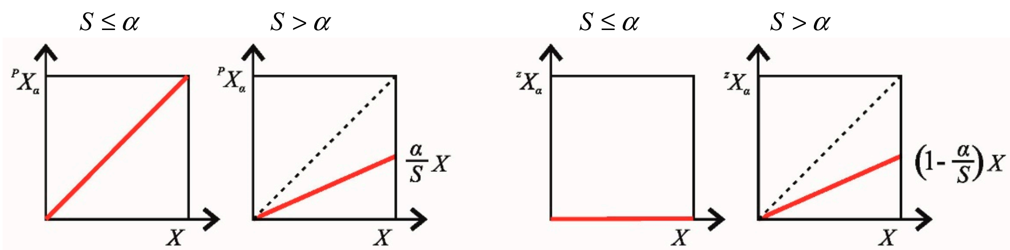

Figure 2. Amount of an individual claim borne by the insurer and the reinsurer in the case of surplus reinsurance.

In practice, surplus reinsurance is often used, as it leaves the insurer with greater freedom to choose their own reinsurance protection strategy. So, in the case that the insured sum is greater than the insurer's retention, the same equations, with regard to characteristics and distribution laws, apply regarding quota share but with the difference that here, instead of a quota, we consider the ratio of the retention to the total insured sum, namely $q = \frac{\alpha}{S}$.

The process of optimisation is based on knowledge of the characteristics of the risk. For example, for a proportional split of the risk, we have, for the characteristics of the total claim amount of the insurer and the reinsurer in the case of quota share reinsurance followed by surplus reinsurance with $S > \alpha$, the following:

$$E\left({}^{P}S_{q,\alpha}^{col}\right) = E(N) \cdot E\left({}^{P}X_{q,\alpha}\right) = q \cdot \frac{\alpha}{S} \cdot E\left(S^{col}\right) \tag{29}$$

$$E\left({}^{Z_1}S_{q}^{col}\right) = E(N) \cdot E\left({}^{Z_1}X\right) = (1-q) \cdot E\left(S^{col}\right) \tag{30}$$

$$E\left({}^{Z_2}S_{q,\alpha}^{col}\right) = E(N) \cdot E\left({}^{Z_2}X_{q,\alpha}\right) = q \cdot \left(1 - \frac{\alpha}{S}\right) \cdot E\left(S^{col}\right) \tag{31}$$

Similarly, for the variances, we have the following:

$$Var\left({}^{P}S_{q,\alpha}\right) = Var\left(q \cdot \frac{\alpha}{S} \cdot S^{col}\right) = q^2 \cdot \left(\frac{\alpha}{S}\right)^2 \cdot Var\left(S^{col}\right) \tag{32}$$

$$Var\left({}^{Z_1}S_{q}^{col}\right) = Var\left((1-q) \cdot S^{col}\right) = (1-q)^2 \cdot Var\left(S^{col}\right) \tag{33}$$

$$Var\left({}^{Z_2}S_{q,\alpha}^{col}\right) = Var\left(q \cdot \left(1 - \frac{\alpha}{S}\right) \cdot S^{col}\right) = q^2 \cdot \left(1 - \frac{\alpha}{S}\right)^2 \cdot Var\left(S^{col}\right) \tag{34}$$

where $Var\left(S^{col}\right) = E(N) \cdot Var(X) + E^2(X) \cdot Var(N)$.

3. The Surplus Process for a Given Portfolio of Non-Life Insurance Contracts—A Numerical Example

To illustrate the analysis of the risk taken on by an insurer, based on actual claim numbers and claim amounts over a particular period, it is necessary first to introduce briefly the required mathematical tools and their mutual relationships. Based on the mathematical framework presented, we set out some of the possible results that can be obtained with an appropriate commentary on the approach to their determination, as well as an in-depth interpretation and mutual comparison.

We assume, given the data we have for the portfolio, that the number of claims has a Poisson distribution with parameter $\lambda = 77$ and the amount of an individual claim can be described by the lognormal distribution $X \sim \mathrm{LN}\left(6; 0.9^2\right)$. For given risk adjustments, as well as optimally set up reinsurance protections, we determine the necessary reserves to cover these liabilities in continuous time. We apply the proposed approach to the injection of capital at suitable times so as to save on capital resources whilst maintaining a given level of security. To get to the final results, we make use of the following partial results, without reinsurance and with consideration of proportional reinsurance cover. These are as follows:

- Setting the ruin probability at a particular time t—setting the level of initial reserves such that there is a 0.99 probability of ensuring solvency.
- Setting the ruin probability by approximating the Poisson process using Brownian motion with a drift to time τ—setting the level of initial reserves such that there is a 0.99 probability of ensuring solvency.
- Estimation of the value $CVaR_{0.99}(T)$ of the distribution of the random variable of the time to ruin T and use of the value for setting the time period in which fluctuations in the portfolio are followed.

- Setting up an algorithm for the injection of capital before time τ, while economising on resources.

3.1. Ruin Probability over Time

Given the distribution of the number of claims our starting point is the compound Poisson process $\{S_t\}_{t \geq 0}$, where according to (1), the following is true:

$$S_t = X_1 + X_2 + \ldots + X_{N_t} = \sum_{i=1}^{N_t} X_i$$

for given values $t > 0$. The random variable S_t has a compound distribution with Poisson parameter $\lambda \cdot t$, $S_t \sim CoPo\left(\lambda \cdot t;\ X \sim LN\left(6;\ 0.9^2\right)\right)$ with the distribution laws and characteristics expressed according to the collective risk model equations. We work with an expected total claim amount $E\left(S^{kol}\right) = 46574.36$ and variance $Var\left(S^{kol}\right) = 7957.76^2$. According to (2), we can express the ruin probability at a particular point of time t for the reserves needed to cover the claims up to time t. We express the risk premium according to the mean value principle with a risk addition $\theta = 0.16$ and according to the variance principle for calculating the risk premium with an addition $\overline{\theta} = 0.00012$. We can estimate the ruin probability in continuous time at time t using the following equation:

$$\Psi(U_0; t) = P(T \leq t) = P(U_t < 0),\ \text{for } 0 \leq t \leq s$$

where $\Psi(U_0; t)$ represents the probability that the insurer's surplus falls below zero in the finite time interval. Note that all the results shown in the following tables were obtained using numerical approximations based on Equations (17)–(19).

Column 1 of Table 1 shows the initial reserves U_0 needed so that at time t, they are adequate to ensure solvency with a 99% probability. The non-highlighted ruin probabilities for these initial reserves depend on six different random variables by which the variance and mean increase with the expected number of claims arising to time t. For example, for $S_6 \sim CoPo\left(6\lambda;\ X \sim LN\left(6;\ 0.9^2\right)\right)$, the initial reserves required to cover the claims with a 99% probability are lower than those for the total claims $S_1 \sim CoPo\left(\lambda;\ X \sim LN\left(6;\ 0.9^2\right)\right)$. Figure 3 shows why this is so. For this comparison, we also have that with increasing reserves the ruin probability at time t reduces.

Table 1. Probability of ruin at time t = 1, 2, 3, . . . , 6 for a given level of initial reserves.

U_0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6
11,060.62	0.01	0.010524	0.007666	0.006569	0.005117	0.002110
11,276.85	0.009298	0.01	0.007342	0.004919	0.003189	0.002037
9708.93	0.015523	0.014370	0.01	0.006516	0.004151	0.002620
7217.44	0.032635	0.024670	0.015953	0.01	0.006218	0.003860
4135.76	0.072676	0.045341	0.027303	0.016474	0.01	0.006106
634.84	0.154765	0.083682	0.047657	0.027890	0.016602	0.01

It is clear from the above that when considering the reserves over time, one must consider all cash flows together. To this end, a Brownian motion process serves as an approximation for the compound Poisson process.

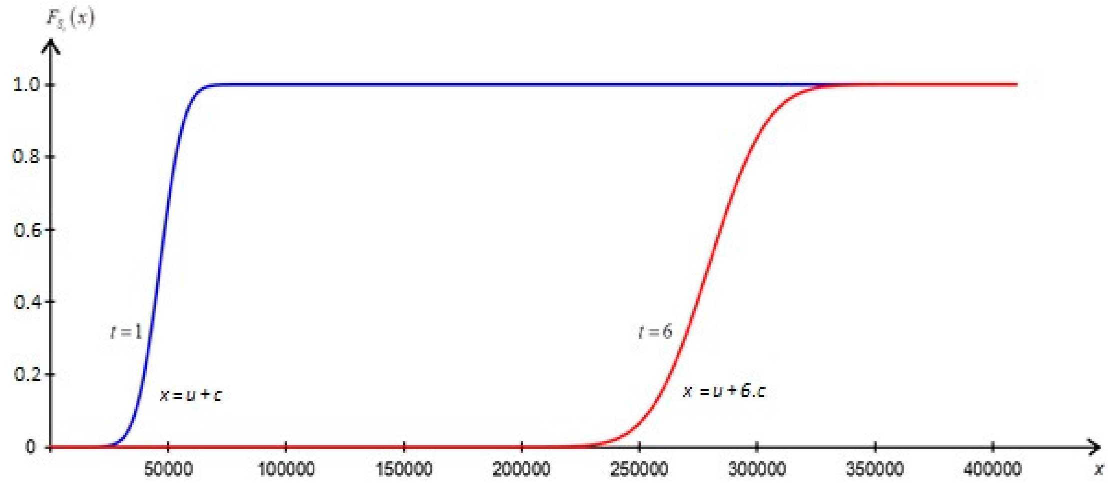

Figure 3. Relationship between the initial reserves *u* and a ruin probability of 0.01 at time *t* = 1 and *t* = 6 expressed by means of the distribution function.

3.2. Calculation of the Increase in the Reserves at the End of the Followed Period by Approximating the Compound Poisson Process by a Brownian Motion

In connection with the compound Poisson process, we first express the corresponding surplus process $\{U_t\}_{t\geq 0}$ for which we have the following:

$$U_t = U_0 + c \cdot t - S_t,\ t \geq 0.$$

Using Equation (9), we can create the intermediate process $\{Z_t\}_{t\geq 0}$ and determine its characteristics, namely $E(Z_1) = \mu = 7451.9\ Var(Z_1) = \sigma^2 = 7957.76^2$. We can then use Equation (19) to express the ruin probability up to time τ. The results are shown in Table 2.

Table 2. Relationship between the initial reserves and the ruin probability expressed by means of Brownian motion.

$\Phi(U_0,\tau)/\Phi(U_0)$ = 0.01	τ = 1	τ = 2	τ = 3	τ = 4	τ = 5	τ = 6
27,356.28	0.01	0.188428	0.455344	0.663923	0.799253	0.881519
42,829.13	0.000007	0.01	0.095598	0.266530	0.457874	0.623028
56,384.02	0	0.000213	0.01	0.061678	0.169371	0.311966
68,954.28	0	0.000002	0.000623	0.01	0.048177	0.127610
80,899.04	0	0	0.000023	0.001087	0.01	0.040435
92,401.58	0	0	0	0.000081	0.001535	0.01

The initial reserves in Table 2 are chosen to cover the liabilities up to time $\tau = 1, 2, \ldots, 6$ with 99% probability. In other words, we set the initial reserves such that the ruin probability is kept at 1%, allowing for all cash flows. The remaining probabilities in the rows show how a particular level of the reserves affects the ruin probability at given times τ. For example, we can see from the figures in the first row of the table that reserves of an amount $U_0 = 27,356.28$ are enough to cover the first year with 99% probability. At the end of the sixth year, given this level of the reserves, there is an 88.15% probability that the surplus is negative. For the figures shown, we have that with time τ, the ruin probability increases and with increasing reserves, it decreases as is illustrated in Figure 4, where $u = U_0$.

In the cases in Table 2 where the ruin probability is zero, the initial reserves are unnecessarily high. We can avoid this by using the following approach, which we have divided into a number of steps. The first step is to consider (19) as the distribution function of the random variable $T \sim IG\left(\frac{U_0}{\mu};\ \frac{{U_0}^2}{\sigma^2}\right)$, which represents the time to ruin and has an inverse Gaussian distribution.

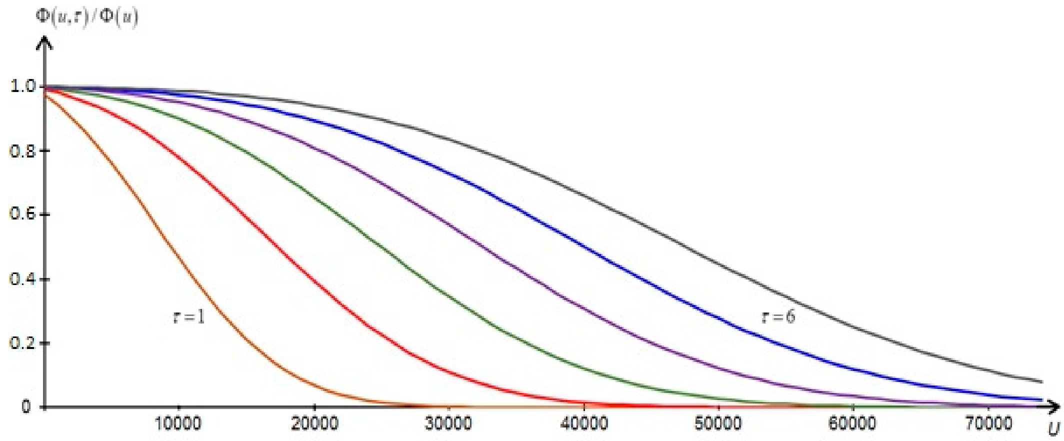

Figure 4. At times $\tau = 1, 2, \ldots, 6$ the ruin probability falls as the reserves increase and for a particular level of reserves the ruin probability increases with time.

In the second step, we set a maximum time when ruin can occur with a given probability. We use the value $CVaR_{0.99}(T)$ to improve on the value $VaR_{0.99}(T)$ in accordance with Equation (25). For the initial reserves shown in Table 2, we can then express the corresponding quantiles by means of the distribution function of the random variable with distribution $T \sim IG\left(\frac{U_0}{7451.9}; \frac{{U_0}^2}{957.76^2}\right)$. In what follows, we will also work with the value $CVaR_{0.99}(T)$, which we use to determine the maximum time until the surplus falls below zero with 0.99 probability. The results are shown in Table 3.

Table 3. Values of $VaR_{0.99}(T)$ and $CVaR_{0.99}(T)$ for the random variable of the time to ruin T.

U_0	$P(T<\tau) = 0.99$	$CVaR_{0.99}(T)$
27,356.28	10.58	12.52
42,829.13	13.86	15.98
56,384.02	16.79	18.99
68,954.28	19.22	21.53
80,899.04	21.47	23.75
92,401.58	23.59	26.11

The first row, for example, shows that for initial reserves of 27,356.28, needed with probability 0.99 to cover the claims in the first year, the value of $VaR_{0.99}(T)$ is 10.58. We can interpret this value as follows. With almost 100% certainty, ruin will occur approximately at the end of the 11th time period for the initial reserves so set, taking into account all cash flows in the surplus. We can improve on this further, using the $CVaR_{0.99}(T)$ value shown in the third column.

Based on this information, we calculate the values shown in Table 4, where we give the values for the first six years we have considered. We obtained these figures by carrying out the steps, described in the theory part of this paper, for ensuring the adequacy of the reserves in each year by the injection of additional capital at the appropriate time. That is to say that the insurer does not hold the required reserves at the start of the observed period but gradually adds further capital, thereby saving on resources in each year whilst keeping the ruin probability at the required 1% level and staying solvent.

For the reserve values shown in the first column, the ruin probability is 1% with the corresponding expected surplus shown in the third column. The values Δu_τ represent the amount of additional capital injected at the end of year t, t = 1, 2, . . . , 5, 11. In each year, we can consider the values in the last column of the table as the amount saved given initial reserves of 27,356.28. By analogy, we can continue up to t = 11, respectively t = 13, whereby the whole period would be covered in which insolvency could occur.

Table 4. Additional capital Δu_τ for $P(T<\tau|T<\infty)=0.01$.

U_0	$P(T<\tau \mid T<\infty)=0.01$	$E(Z_\tau)$	Δu_τ, τ = 1, 2, ... 5, 11
27,356.28	$P(T<1|T<\infty)=0.01$	7451.89	$\Delta u_1 = 15,472.85$
42,829.13	$P(T<2|T<\infty)=0.01$	1490.79	$\Delta u_2 = 13,554.89$
56,384.02	$P(T<3|T<\infty)=0.01$	22,355.69	$\Delta u_3 = 12,570.26$
68,954.28	$P(T<4|T<\infty)=0.01$	29,807.59	$\Delta u_4 = 11,945.16$
80,899.04	$P(T<5|T<\infty)=0.01$	37,259.49	$\Delta u_5 = 11,502.14$
92,401.58	$P(T<6|T<\infty)=0.01$	44,711.38	Δu_6 =11,168.71
$\vdots$	$\vdots$	$\vdots$	$\vdots$
146,035.85	$P(T<11|T<\infty)=0.01$	81,970.87	$\Delta \overline{u} =$ 12,702

3.3. The Effect of Proportional Reinsurance on the Parameters of the Random Variable Z_1

In order to assess whether it is desirable to reinsure our portfolio of insurance contracts proportionally, we consider a chain made up of quota share and surplus reinsurance. By using the rules for setting reinsurance protection, see [5], we derive for the two principles for calculating the risk premium the parameters μ and σ. The means $\mu = E(Z_1)$ are specified in Table 5. The derivations apply for the considered primary and secondary distributions and the mean value principle with risk margins q, ς_q, ξ_α and the variance principle with risk margins $\overline{\theta}$, $\overline{\varsigma}_q$, $\overline{\xi}_\alpha$.

Table 5. The parameters of the insurer's random variable Z_1.

Risk Premium	Quota Share Reinsurance	Surplus Reinsurance	Quota-Surplus Chain	Surplus-Quota Chain
Mean value principle	$\mu = \mu_{\theta,q}$ $\sigma^2 = Var(Z_1)\cdot q^2$	$\mu = \mu_{\theta,\alpha}$ $\sigma^2 = Var(Z_1)\cdot\left(\frac{\alpha}{S}\right)^2$	$\mu = \mu_{\theta,q,\alpha}$ $\sigma^2 = Var(Z_1)\cdot q^2\cdot\left(\frac{\alpha}{S}\right)^2$	$\mu = \mu_{\theta,\alpha,q}$ $\sigma^2 = Var(Z_1)\cdot q^2\cdot\left(\frac{\alpha}{S}\right)^2$
Variance principle	$\mu = \mu_{\eta,q}$ $\sigma^2 = Var(Z_1)\cdot q^2$	$\mu = \mu_{\eta,\alpha}$ $\sigma^2 = Var(Z_1)\cdot\left(\frac{\alpha}{S}\right)^2$	$\mu = \mu_{\eta,q,\alpha}$ $\sigma^2 = Var(Z_1)\cdot q^2\cdot\left(\frac{\alpha}{S}\right)^2$	$\mu = \mu_{\eta,\alpha,q}$ $\sigma^2 = Var(Z_1)\cdot q^2\cdot\left(\frac{\alpha}{S}\right)^2$

The following Table 6 gives expressions for the parameter μ.

Table 6. Specification of the parameter μ.

$\mu = \mu_{\theta,q}$	$\mu = E\left(S^{col}\right)\cdot\left[\theta - \varsigma_q\cdot(1-q)\right]$
$\mu = \mu_{\theta,\alpha}$	$\mu = E\left(S^{col}\right)\cdot\left[\theta - \xi_\alpha\cdot\left(1-\frac{\alpha}{S}\right)\right]$
$\mu = \mu_{\theta,\alpha,q}$	$\mu = E\left(S^{col}\right)\cdot\left[\theta - \xi_\alpha\cdot\left(1-\frac{\alpha}{S}\right) - \frac{\alpha}{S}\cdot\varsigma_q\cdot(1-q)\right]$
$\mu = \mu_{\theta,q,\alpha}$	$\mu = E\left(S^{col}\right)\cdot\left[\theta - \varsigma_q\cdot(1-q) - q\cdot\xi_\alpha\cdot\left(1-\frac{\alpha}{S}\right)\right]$
$\mu = \mu_{\eta,q}$	$\mu = Var\left(S^{col}\right)\cdot\left[\overline{\theta} - \overline{\varsigma}_q\cdot(1-q)^2\right]$
$\mu = \mu_{\eta,\alpha}$	$\mu = Var\left(S^{col}\right)\cdot\left[\overline{\theta} - \overline{\xi}_\alpha\cdot\left(1-\frac{\alpha}{S}\right)^2\right]$
$\mu = \mu_{\eta,q,\alpha}$	$\mu = Var\left(S^{col}\right)\cdot\left[\overline{\theta} - \overline{\varsigma}_q\cdot(1-q)^2 - \overline{\xi}_\alpha\cdot q^2\cdot\left(1-\frac{\alpha}{S}\right)^2\right]$
$\mu = \mu_{\eta,\alpha,q}$	$\mu = Var\left(S^{col}\right)\cdot\left[\overline{\theta} - \overline{\xi}_\alpha\cdot\left(1-\frac{\alpha}{S}\right)^2 - \overline{\varsigma}_q\cdot\left(\frac{\alpha}{S}\right)^2\cdot(1-q)^2\right]$

In the case of proportional reinsurance, we can use the expression for the parameter μ to determine the insurer's expected surplus at time τ. In all cases, we have for the variance $\sigma^2 = q^2\cdot\left(\frac{\alpha}{S}\right)^2\cdot Var\left(S^{col}\right)$. If we consider only quota reinsurance, then we have, in accordance with (28), that $\alpha \geq S$, where S is the insured amount. In the case of surplus reinsurance, q = 1. If we then put the modified parameters μ and σ into (18), we can determine the ruin probability at time τ for values of the parameters q and α of the reinsurance chain. These can be chosen, for example, on the basis of experience, but we can set them by applying an optimisation criterion. In the next part, we show how these approaches affect the results.

3.4. Estimation, with 99% Probability, of the Expected Surplus Using Optimally Set Quota-Surplus Reinsurance Chains

We present the results obtained according to the parameters chosen for the quota-surplus chain. We compare these with the situation in which we set the quota and the own retention by applying an optimisation criterion. We look for values of q and α, which minimise the variance for a set surplus. In our case, we set the surplus to that which the insurer would make without reinsurance protection.

(a) Let us analyse the risk given the initial reserves needed by the insurer to cover its risk with a 99% probability and the achieved surplus in the case of quota reinsurance with $q = 0.3$. We use a risk premium, calculated using the mean value principle and a quota reinsurance margin $\zeta_q = 0.2$. These values are then inserted, together with the parameters $E(S^{col}) = 46,574.36$, $Var(S^{col}) = 7957.76^2$ and the risk margin $\theta = 0.16$, into the relevant equation for $\mu_{\theta,q}$, shown in Table 6. The result together with the variance $\sigma^2 = q^2 \cdot Var\left(S^{col}\right)$ is then inserted into Equation (19). We set the ruin probability to 1% and determine the reserves needed to cover the risk up to time τ. The resulting figures are summarised in Table 7.

Table 7. Initial reserves for a quota $q = 0.3$ with a ruin probability $\Phi(u,\tau)/\Phi(u) = 0.01$ and the resulting surplus at time τ.

	τ = 1	τ = 2	τ = 3	τ = 4	τ = 5	τ = 6
Initial reserve	6992.46	10,392.09	13,204.27	15,713.49	18,029.69	20,209.16
Surplus	931.49	862.98	2794.46	3725.95	4657.44	5588.96

By comparing the results in Tables 2 and 7, we note that in order to ensure a 1% ruin probability without reinsurance, we need initial reserves at time τ = 1 of $u = 27,356.28$ with an achieved surplus of 7451.89.

With quota reinsurance with q = 0.3 over the same time period, the initial reserves fall to $U_0 = 6992.46$, but the surplus also reduces to only 931.49. This shows that as the own retention reduces, not only does the risk, but also the surplus. In our case, the reduction in risk manifests itself as a reduction in the initial reserves required. On the other hand, if we had used with the quota reinsurance the reserves needed to cover the risk without reinsurance, we would have discovered that these reserves are unnecessarily high.

(b) If we now repeat this process but with a quota-surplus reinsurance chain, there are changes in the choice of the value of $\mu = \mu_{\theta,q,\alpha}$ in the expression for the variance for which we now have $\sigma^2 = q^2 \cdot \left(\frac{\alpha}{S}\right)^2 \cdot Var\left(S^{col}\right)$ and in the level of expenses for the surplus reinsurance via the premium addition $\xi_\alpha = 0.22$. The results from the reinsurance with fixed parameters q = 0.3 and α/S = 0.4 are shown in Table 8.

Table 8. Initial reserves for a quota q = 0.3, α/S = 0.4 and ruin probability $\Phi(u,\tau)/\Phi(u) = 0.01$ and the resulting surplus at time τ.

	τ = 1	τ = 2	τ = 3	τ = 4	τ = 5	τ = 6
Initial reserve	5203.65	7921.91	9632.86	11,117.10	12,427.91	13,613.18
Surplus	316.71	633.42	950.12	1266.82	1583.53	1900.23

By comparing the figures shown in Table 8 with those in Table 2, we deduce that retaining all the risk, i.e., without reinsuring, is more advantageous than if we use the proposed reinsurance. Given the requirement of a 1% ruin probability, diversification is pointless. The product is not viable.

We will decide if it is possible in this case to set effectively the parameters for this form of reinsurance after we apply an optimisation criterion.

So instead of choosing fixed values for the reinsurance parameters, we search for optimal values that meet our requirements, using the following constrained minimising function:

$$L(q, \alpha, \gamma) = q^2 \left(\frac{\alpha}{S}\right)^2 Var\left(S^{col}\right)\tau + \gamma\left(\mu_{q,\alpha}\tau - k\right)$$

where the surplus is constrained to a value k and γ is the Lagrange multiplier. The value k is set equal to the values of the surplus shown in the first and sixth rows of Table 4, i.e., the surplus that is obtained without reinsurance at times $\tau = 1$ and $\tau = 6$.

For a surplus $k = 7500$, we get the optimal parameters $q = 0.5815$ and $\alpha/S = 0.3472$ and for a surplus $k = 45{,}000$, they are $q = 0.7939$ and $\alpha/S = 0.7646$. The expression for $\mu_{q,\alpha}$ and the margins remain unchanged. By repeating the preceding procedure but using these optimal reinsurance parameters, we get the results shown in Tables 9 and 10. We can see the effect of using reinsurance by comparing these figures with those ignoring reinsurance shown in Table 2. The first column in Tables 9 and 10 shows the amount of the initial reserves.

Table 9. Optimal reinsurance parameters $q = 0.5815$, $\alpha/S = 0.3472$ and an ensured surplus of 7500.

$\frac{\Phi(u,\tau)}{\Phi(u)} = 0.01$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$	$\tau = 6$
5284.42	0.01	0.161378	0.388256	0.579263	0.716534	0.810130
8163.48	0.000018	0.01	0.081637	0.219382	0.378882	0.526575
10,652.97	0	0.000274	0.01	0.056060	0.148620	0.271689
12,942.29	0	0	0.000756	0.01	0.044017	0.111911
15,104.39	0	0	0	0.001277	0.01	0.037120
17,176.54	0	0	0	0.000120	0.001766	0.01

Table 10. Optimal reinsurance parameters $q = 0.7939$, $\alpha/S = 0.7646$ and an ensured surplus of 45,000.

$\frac{\Phi(u,\tau)}{\Phi(u)} = 0.01$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$	$\tau = 6$
19,441.25	0.01	0.319648	0.714426	0.904179	0.973174	0.9914956
1736.12	0	0.01	0.167591	0.488742	0.755928	0.900910
42,888.67	0	0.000035	0.01	0.112494	0.355492	0.6227574
53,455.41	0	0	0.000155	0.01	0.085559	0.274105
63,650.04	0	0	0	0.000343	0.01	0.069934
72,531.24	0	0	0	0	0.000794	0.01

A comparison of the figures in Tables 4 and 10 shows that without reinsurance, we need initial reserves of amount 92,401.58 to ensure, immediately before time $\tau = 6$, a ruin probability of 1% whilst attaining a surplus of 44,711.38. With the optimal reinsurance protection in the form of quota share followed by surplus reinsurance, it is enough to have initial reserves of amount 72,531.24.

A further way to save capital is for the insurer to inject additional capital at the end of a particular time period in combination with an optimal reinsurance programme. Given the initial reserves shown in the first columns of Tables 9 and 10, which we determined so that in each time period there was a 99% probability that the insurer's surplus would not fall below zero, we can proceed to determine the amount of the capital injections.

The figures shown in Tables 11 and 12 were determined as before on the assumption of a 1% probability of ruin. They allow us to follow in column 3 the amounts of capital that need to be injected at the end of each year, where the last figure, $\Delta\overline{u}$, is the average yearly capital injection. Column 2 of the two tables shows the corresponding expected profit.

Table 11. $P(T < \tau | T < \infty) = 0.01$, surplus 7500.

Initial Reserves	$E(Z_\tau)$	Capital Injection $\Delta u_\tau, \tau = 1, 2, \ldots, 6$
5284.42 for $\tau = 1$	1250	$\Delta u_1 = 2879.06$
8163.48 for $\tau = 2$	2500	$\Delta u_2 = 2489.49$
10,652.97 for $\tau = 3$	3750	$\Delta u_3 = 2289.32$
12,942.29 for $\tau = 4$	5000	$\Delta u_4 = 2162.11$
15,104.39 for $\tau = 5$	6250	$\Delta u_5 = 2072.14$
17,176.54 for $\tau = 6$	7500	$\Delta \overline{u} = 2378$

Table 12. $P(T < \tau | T < \infty) = 0.01$, surplus 45,000.

Initial Reserves	$E(Z_\tau)$	Capital Injection $\Delta u_\tau, \tau = 1, 2, \ldots, 6$
19,441.25 for $\tau = 1$	7500	Δu_1 = 12,294.87
31,736.12 for $\tau = 2$	15,000	Δu_2 = 11,152.55
42,888.67 for $\tau = 3$	22,500	Δu_3 = 10,566.74
53,455.41 for $\tau = 4$	30,000	Δu_4 = 10,194.63
63,650.04 for $\tau = 5$	37,500	Δu_5 = 8881.20
72,531.24 for $\tau = 6$	45,000	$\Delta \overline{u}$ = 10,618.00

Table 13 allows us to see the effect of using optimised reinsurance protection. It compares the results we obtained using the same approach but without reinsurance with those from using optimally determined parameters for a quota-surplus reinsurance chain. Figures for the initial reserves U_0 are shown for two time periods τ = 1 and τ = 6 for the same level of risk and surplus.

Table 13. Example savings in the initial reserves without and with reinsurance whilst maintaining solvency.

	τ = 1	τ = 6
Without reinsurance	$U_0 = 26,917.75$ $\Delta u_1 = 15,023.49$ $E(Z_{\tau=1}) = 7451.89$	$U_0 = 92,401.58$ $\Delta u_5 = 11,502.14$ $E(Z_{\tau=6}) = 44,711.38$
Surplus k = 7500 Optimal parameters quota-surplus α/S = 0.3472, q = 0.5815	$U_0 = 5125.25$ $\Delta u_1 = 2879.06$ $E(Z_{\tau=1}) = 1250$	$U_0 = 17,176.54$ $\Delta u_5 = 2072.14$ $E(Z_{\tau=6}) = 7500$
Surplus k = 45000 Optimal parameters quota-surplus α/S = 0.7647, q = 0.7940	$U_0 = 19,441.25$ $\Delta u_1 = 12,294.87$ $E(Z_{\tau=1}) = 7500$	$U_0 = 72,531.24$ $\Delta u_5 = 8881.2$ $E(Z_{\tau=6}) = 45,000$

4. Results

In the paper, we followed the progress of an insurer's surplus in continuous time for a given initial distribution of the number of claims and individual claim amounts. The approach would be the same in the case of another primary and secondary distribution. The expression of the maximal total claim amount with a specific probability and the setting of the conditional value at risk, *CVaR*, may be different, as these values depend on the form of the distribution function. If the distribution function of the total claim amount cannot be stated explicitly, it is possible either to approximate it or to use a simulation. The other steps leading to a reduction in the required initial reserves, ignoring reinsurance, can be carried out for a freely chosen compound distribution for which the necessary moments exist.

The theoretical apparatus is conceived such that, by its use, it is possible to analyse an insurance portfolio and develop various scenarios for the development of the risk in continuous time. The starting point for the proposed approach to setting the initial reserves in connection with the ruin probability is approximating the compound Poisson process by Brownian motion. The expected time of ruin depends on four key quantities, which describe the surplus process. By its use, it is possible to set all the parameters of the process such that the resulting ruin probability meets the required level. In our case, the task was to set up reserves in continuous time, which would ensure a 99% probability of solvency. In combination with the maximum time of ruin, the additional amount of capital can be estimated that is needed to avoid the surplus becoming negative.

When considering an optimal insurance programme, on the basis of the outlined strategy for dividing the risk between the insurer and the reinsurer, we can choose one of a number of optimisation criteria. For example, if the insurer sets importance on security, it constrains the variance of the surplus to a required value and uses the following criterion: maximisation of surplus for a constant variance. On the other hand, by constraining the insurer's expected surplus to a particular level, we can look for the values of the parameters of the reinsurance protection, which, given the stated constraint, minimises the relevant variance.

Based on data for the number of claims and individual claim amounts, Table 1 shows the initial reserves required for a chosen realisation of the Poisson process, which are adequate to ensure solvency with a 99% probability up to the time shown. Table 2 shows the same information but after evaluating all the surplus paths, namely all cash-flows up to time τ, together with the effect on the required initial reserves. Given that the time to ruin is a random variable T with an inverse Gaussian distribution, we can, by setting the value $VaR_{0.99}(T)$, obtain the maximum time up to which ruin may arise with 99% probability, or refine this value by using $CVaR_{0.99}(T)$. During the period up to this time, we implement an algorithm for the potential increase in the reserves needed for the next year. For example, the values in the first row of Table 4 show the reserves needed in the first year and their increase at the end of this year such that the probability that they will be adequate in the second year is 99%.

It was possible to reduce the reserves further, whilst maintaining the required probability of ruin, by using a specified reinsurance programme. The order in which different reinsurance covers are applied affects the resulting values of the initial reserves and thus also the ruin probabilities at each time. It is, therefore, useful to consider meaningful scenarios for possible reinsurance chains and study and compare the results. It is necessary to point out that the number of independent random variables, for which we seek an unbounded or bounded extreme, is conditional on the number of types of cover included in the resulting reinsurance chain.

In our paper, we worked with quota reinsurance and surplus reinsurance and variations of these. Tables 5 and 6 show the derived characteristics for approximating the Poisson process by Brownian motion for two methods for calculating the premium. The parameters of the reinsurance protection were set, using optimisation criteria. Using these parameters, the insurer's total claim amount was reduced and the earlier approach without reinsurance was repeated, incorporating the appropriate reinsurance protection. The paper compares and discusses the results obtained, and the final Table 13 summarises the estimates for the initial reserves U_0.

5. Conclusions

The results we give show that with the presented approach, it is possible to develop a number of scenarios from which the insurer can choose those which fit its strategic aims. These allow the insurer to see the differences in the results obtained with a particular type of reinsurance but also the differences in those obtained using a random choice of parameters for a particular type of reinsurance and when they are set optimally. We have shown that an arbitrary set of reinsurance protection parameters does not necessarily lead to a good result. On the other hand, through optimisation, we can set not only suitable reinsurance protection for the investigated risk, but also optimal parameters. An analysis based on these leads to a safe level for initial reserves, whilst maintaining a suitable chosen level of risk. The possibility of choosing optimisation criteria allows the insurer to prefer security also in the case where surplus is set at an expected level.

The results from the scenarios obtained using the above proposed approach can contribute to the appropriate and responsible decisions of the insurer. Further, the injection of additional capital at a particular time contributes to a saving in the capital that the insurer must have at their disposal to cover the risks they take on.

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