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Controllability of Impulsive ψ-Caputo Fractional Evolution Equations with Nonlocal Conditions

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Abstract: This paper is mainly concerned with the exact controllability for a class of impulsive ψ-Caputo fractional evolution equations with nonlocal conditions. First, by generalized Laplace transforms, a mild solution for considered problems is introduced. Next, by the Mönch fixed point theorem, the exact controllability result for the considered systems is obtained under some suitable assumptions. Finally, an example is given to support the validity of the main results.

1. Introduction

Fractional systems have gained considerable popularity and importance due to their wide range of applications in many mathematical, physical, and engineering disciplines such as the chaotic synchronization system [1], solutions of differential systems [2–4], impulsive problems [5,6], quantum theory [7], diffusion phenomena [8–10], delay problems [11,12], systems of thermoelasticity [13,14], etc. It turns out that fractional calculus can provide a more vivid and accurate description of many practical problems than integral ones. Increasingly more recent achievements in various aspects of science and technology have proved that fractional differential systems [15–20] have naturally replaced integer-order differential systems. What makes fractional calculus special is the fact that there exist various kinds of fractional operators which can be chosen to provide a more accurate modeling of real-world phenomena. In order to improve the precision of the objective modeling, in 2017, Almeida [21] introduced the new definition of fractional derivative by considering the Caputo fractional derivative with another function ψ, that is, the ψ-Caputo fractional derivative. In 2020, Jarad and Abdeljawad [22] introduced the generalized Laplace transforms and the inverse version about ψ-Caputo fractional derivative. As we all know, Laplace transforms can be used to solve the mild solutions of some fractional differential equations. Consequently, the application of fractional differential equations opens a new window in the framework of the ψ-Caputo fractional derivative.

On the other hand, impulsive differential systems are powerful tools to describe systems with short-term perturbations, which are observed in optimal control, biology, stability analysis, medicine, biotechnology, and electronics, please see in [23–26] and the references therein. Moreover, Byszewski [27] recently introduced another kind of Cauchy condition, which is called the nonlocal Cauchy condition, that plays a more important role on the above two systems. Since then, increasingly more researchers have paid their attention to kinds of differential equations with nonlocal initial conditions. For more details, see in [28,29] and the references therein.

As we all know, controllability is an important component of control theory and engineering. As one fundamental concept in mathematical control theory, controllability of the above two systems has increasingly received interest in recent years, and many controllability problems for integer-order and fraction-order evolution equations have been discussed in many papers such as in [29–33] and the references therein. In 2009,

Hernández and O'Regan [34] pointed out that under the compactness semigroup and some suitable assumptions, controllability results for some abstract control systems are only applicable in finite-dimensional space. Since then, many advancements have been made for various kinds of nonlinear evolution equations with a non-compact semigroup in infinite dimensional spaces. However, note that there still exist some unsolved controllable problems in the framework of infinite dimensional spaces, such as the exact and regional of controllability about ψ-Caputo fractional evolution equations.

Motivated by the above-mentioned discussions, we consider the controllability for the following impulsive ψ-Caputo fractional evolution equations with nonlocal conditions:

$$\begin{cases} {}^{c}_{0}\mathcal{D}^{\alpha}_{\psi}x(t) = \mathcal{A}x(t) + f(t, x(t), x(t)) + \mathcal{B}u(t), & a.e.\ t \in J := [0, \ell]; \\ \Delta x\mid_{t=t_i} = I_i(x(t_i)), & i = 1, 2, \ldots, k; \\ x(0) + g(x) = x_0, \end{cases} \tag{1}$$

where $\Delta x\mid_{t=t_i} = x(t_i + 0) - x(t_i - 0), 0 < \alpha < 1, \ell < +\infty$. $\mathcal{A}$ is an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X. Further, the control function u is given in $L^2[J, V]$. Here, V is a Banach space. $\mathcal{B}$ is a linear bounded operator from V to X. The function $g : PC[J, X] \to X$. $0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = \ell, I_i : X \to X$ ($i = 1, 2, \ldots, k$) are impulsive functions. Here, the nonlinear function f will be specified later. The Volterra integral operator $x(t) := \int_0^t \mathbb{K}(t, s)x(s)ds$ is equipped with integral kernel $\mathbb{K} \in C[\Omega, \mathbb{R}^+]$, $\Omega := \{(t, s) : 0 \leq t \leq s \leq \ell\}$.

As far as we know, there are few papers that have studied the exact controllability in the framework of ψ-Caputo fractional derivative and there is no paper considering such a problem. The aim of this paper is to fill this gap. What is more, for the sake of investigating the exact controllability of (1), the framework of a ψ-Caputo fractional derivative is constructed and a new concept of mild solutions is introduced (Section 3, Definition 3) for system (1).

The rest of this paper is organized as follows. Some background materials and preliminaries are introduced in Section 2. Section 3 is reserved for discussion about the concept of mild solution. In Section 4, some sufficient conditions for exact controllability are obtained. Finally, in Section 4, an example is given to support the validity of the main results.

2. Preliminaries

In this section, we will introduce some definitions and results that are used in this paper. In the following, $\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}d\alpha, \ \alpha > 0$ represents the Gamma function. ψ' denotes the first derivative of ψ. For convenience, we set $J_0 = [0, t_1]$ and $J_i = (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$. Let X be a Banach space with the norm $\|\cdot\|$, $PC[J, X] := \{x : x$ is a map from J into X such that $x(t)$ is continuous at $t \neq t_i$, and left continuous at $t = t_i$, and the right limit $x(t_i^+)$ exists for $i = 1, 2, \ldots, k\}$.

It is clear that $PC[J, X]$ is a Banach space with the following norm

$$\|x\|_{PC} = \sup_{t \in J}\{\|x(t)\|\}, \quad \forall x \in PC[J, X].$$

Let $L^p[J, X]$ ($p \in [1, +\infty]$) denote the Banach space of all strongly measurable functions $x : J \to X$ with the norm

$$\|x\|_p = \begin{cases} (\int_J \|x(t)\|^p)^{\frac{1}{p}}, 1 \leq p < +\infty; \\ ess\sup_{t \in J}\|x(t)\| = \inf\{a \geq 0 : \|x(t)\| \leq a, \ a.e.\ t \in J\}, p = +\infty. \end{cases}$$

First, we recall some basic definitions and fundamental results about fractional calculus.

Definition 1 ([21]). *Let $\alpha > 0$, f be an integrable function defined on $[a, b]$ and $\psi \in C^1[a, b]$ be an increasing function with $\psi(t) \neq 0$ for all $t \in [a, b]$. The ψ-Riemann–Liouville fractional integral operator of a function f is defined by*

$$({}_a\mathcal{I}_\psi^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(s))^{\alpha-1} f(s) \psi'(s) ds. \tag{2}$$

Clearly, (2) is the classical Riemann–Liouville fractional integral when $\psi(t) = t$.

Definition 2 ([21]). *Let $n - 1 < \alpha < n$, $f \in C^n[a, b]$ and $\psi \in C^n[a, b]$ be an increasing function with $\psi(t) \neq 0$ for all $t \in [a, b]$. The ψ-Caputo fractional derivative of a function f is defined by*

$$({}_a^C\mathcal{D}_\psi^\alpha f)(t) = ({}_a\mathcal{I}_\psi^{n-\alpha} f^{[n]})(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} f^{[n]}(s) \psi'(s) ds,$$

where $n = [\alpha] + 1$ and $f^{[n]}(t) := (\frac{1}{\psi'(t)} \frac{d}{dt})^n f(t)$ on $[a, b]$.

Lemma 1 ([21]). *Let $f \in C^n[a, b]$ and $\alpha > 0$. Then, we have*

$$_a\mathcal{I}_\psi^\alpha \, {}_a^C\mathcal{D}_\psi^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{[n]}(a^+)}{k!} (\psi(t) - \psi(a))^k.$$

In particular, given $\alpha \in (0, 1)$, we have ${}_a^C\mathcal{D}_\psi^\alpha f(t) = f(t) - f(a)$.

Second, we list the Mönch fixed point theorem, which will be used in the proof of our main results. Here, the Hausdorff measure of non-compactness of a bounded set in X and $PC[J, X]$ are denoted by $\chi(\cdot)$ and $\chi_{PC}(\cdot)$, respectively.

Lemma 2 ([35]). *Suppose X is a Banach space. Let H be a countable set of strongly measurable function $x : J \to X$ such that there exists $y \in L[J, \mathbb{R}^+]$ with $\|x(t)\| \leq y(t)$, i.e., $t \in J$ for all $x \in H$. Then, $\chi(H(t)) \in L[J, \mathbb{R}^+]$ and*

$$\chi\left(\left\{ \int_J x(t) dt : x \in H \right\} \right) \leq 2 \int_J \chi(H(t)) dt,$$

where $\chi(\cdot)$ denotes the Hausdorff non-compactness measure.

Theorem 1 ([36] Mönch fixed point theorem). *Suppose X is a Banach space. Let D be a closed and convex subset of X and $u \in D$. Assume that the continuous operator $A : D \to D$ has the following property:*

$$C \subset D \text{ countable}, \ C \subset \overline{co}(\{u\} \cup A(C)) \text{ implies } C \text{ is relatively compact}.$$

Then, A has a fixed point in D.

3. The Concept of Mild Solution

In this section, based on the works in [22,37–39], the existence of a mild solution is obtained for our problems.

First, we introduce some facts about semigroups theory. For more details about it, please see in [40,41] and the references therein.

The infinitesimal generator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ of C_0-semigroup $\{T(t)\}_{t \geq 0}$ is defined by

$$\mathcal{A}x = \lim_{t \to 0^+} \frac{T(t)x - x}{x},$$

where $D(\mathcal{A}) := \{x \in X : \lim_{t \to 0^+} \dfrac{T(t)x - x}{x} \text{ exists}\}$.

A family of bounded and linear operators $\{T(t)\}_{t \geq 0}$ is called a C_0-semigroup if

(a) $T(0) = I$.
(b) $T(t_1 + t_2) = T(t_1)T(t_2)$ for all t_1, $t_2 \in [0, +\infty)$.
(c) For all $x \in X$ and $t \in [0, +\infty)$, $\lim_{t \to 0^+} T(t)x = x$.

Further, the C_0-semigroup $\{T(t)\}_{t \geq 0}$ is called an analytic semigroup if the map $t \mapsto T(t)$ is analytic in $[0, +\infty)$.

Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup of uniformly bounded linear operators $\{T(t)\}_{t \geq 0}$ on X. Thus, there exists $M \geq 1$ such that $M = \sup_{t \in [0, +\infty)} \|T(t)\|$.

Define the following two operators $S_\psi^\alpha(t, s)$ and $\mathcal{T}_\psi^\alpha(t, s)$ on X by

$$S_\psi^\alpha(t, s)x = \int_0^\infty \phi_\alpha(\theta) T((\psi(t) - \psi(s))^\alpha \theta) x \, d\theta, \quad \forall x \in X,$$

and

$$\mathcal{T}_\psi^\alpha(t, s)x = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T((\psi(t) - \psi(s))^\alpha \theta) x \, d\theta, \quad \forall x \in X, \tag{3}$$

for $0 \leq s \leq t \leq \ell$, where

$$\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1 - \frac{1}{\alpha}} \rho_\alpha(\theta^{-\frac{1}{\alpha}}),$$

and $\rho_\alpha(\theta)$ is defined by

$$\rho_\alpha(\theta) = \frac{1}{\pi} \sum_{k=1}^\infty (-1)^{k-1} \theta^{\alpha k - 1} \frac{\Gamma(\alpha k + 1)}{k!} \sin(k\pi\alpha),$$

where ϕ_α is a probability density function on $(0, +\infty)$. Here, ϕ_α has the following properties: $\phi_\alpha(\theta) \geq 0$ for all $\theta \in (0, +\infty)$ and $\int_0^\infty \phi_\alpha(\theta) \theta^r d\theta = \dfrac{\Gamma(1 + r)}{\Gamma(1 + \alpha r)}$ for $r > -1$.

Very similar to the argument in [42], we can obtain the following results.

Lemma 3. *The bounded linear operators S_ψ^α and $\mathcal{T}_\psi^\alpha$ have the following properties.*
(1) For $t \geq s \geq 0$ and $x \in X$,

$$\|S_\psi^\alpha(t, s)x\| \leq M\|x\| \text{ and } \|\mathcal{T}_\psi^\alpha(t, s)x\| \leq \frac{\alpha M}{\Gamma(1 + \alpha)} \|x\| = \frac{M}{\Gamma(\alpha)} \|x\|;$$

(2) For all $t \geq s \geq 0$, the operators S_ψ^α and $\mathcal{T}_\psi^\alpha$ are strongly continuous. That is, for every $x \in X$ and $0 \leq s \leq t_1 \leq t_2 \leq \ell$, we have

$$\|S_\psi^\alpha(t_2, s)x - S_\psi^\alpha(t_1, s)x\| \to 0 \text{ and } \|\mathcal{T}_\psi^\alpha(t_2, s)x - \mathcal{T}_\psi^\alpha(t_1, s)x\| \to 0, \text{ as } t_2 \to t_1.$$

Subsequently, for simplicity and convenience, set

$$\mathcal{L}_{t_1}^{t_2}\{x(t)\} := \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^{\alpha - 1} \mathcal{T}_\psi^\alpha(t_2, s)x(s)\psi'(s) ds, \tag{4}$$

where t_1, $t_2 \in J$, $x \in PC[J, X]$, and $\mathcal{T}_\psi^\alpha$ as defined in (3).

In order to obtain the mild solution for problem (1), we also need to introduce the following two lemmas.

Lemma 4. *If $0 < \alpha < 1$, then*

$${}_0^c\mathcal{D}_\psi^\alpha[\, S_\psi^\alpha(t, 0)x_0 \,] = \mathcal{A}[\, S_\psi^\alpha(t, 0)x_0 \,],$$

and

$$
{}^c_0\mathcal{D}^\alpha_\psi[\,\pounds^t_0\{h(t)\}\,] = \mathcal{A}[\,\pounds^t_0\{h(t)\}\,] + h(t),
$$

where S^α_ψ, $\mathcal{T}^\alpha_\psi$ *are defined in* (3), *and* $h \in PC[J, X]$.

Proof of Lemma 4. Similar to the argument of Lemma 3.3 in [38], applying the generalized Laplace transforms (Definition 3.1 in [22]) to (3), one can obtain that

$$
\begin{aligned}
L_\Psi\{\,S^\alpha_\psi(t,0)x_0\,\}(\lambda) &= L_\Psi\Big\{\,\int_0^\infty \phi_\alpha(\theta)T((\psi(t) - \psi(0))^\alpha\theta)x_0 d\theta\,\Big\}(\lambda) \\
&= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x_0.
\end{aligned}
\tag{5}
$$

Therefore, by (5) and Corollary 4 in [22],

$$
\begin{aligned}
L_\Psi\{\,{}^c_0\mathcal{D}^\alpha_\psi[\,S^\alpha_\psi(t,0)x_0\,]\,\}(\lambda) &= \lambda^\alpha \cdot (\,L_\Psi\{\,S^\alpha_\psi(t,0)x_0\,\}(\lambda) - \lambda^{-1}S^\alpha_\psi(t,0)x_0\,|_{t=0}\,) \\[2mm]
&= \lambda^\alpha \cdot (\,L_\Psi\{\,S^\alpha_\psi(t,0)x_0\,\}(\lambda) - \lambda^{-1}x_0\,) \\[2mm]
&= \lambda^\alpha(\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x_0) - \lambda^{\alpha-1}x_0 \\[2mm]
&= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} \cdot \mathcal{A}x_0 \\[2mm]
&= \mathcal{A}\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}x_0.
\end{aligned}
$$

This together with (5) guarantees that

$$
{}^c_0\mathcal{D}^\alpha_\psi[\,S^\alpha_\psi(t,0)x_0\,] = \mathcal{A}[\,S^\alpha_\psi(t,0)x_0\,].
$$

Using a similar process of the proof of Lemma 3.1 in [37], one can get that

$$
L_\Psi\{\,\pounds^t_0\{h(t)\}\,\}(\lambda) = (\lambda I - \mathcal{A})^{-1} \cdot L_\Psi\{\,h(t)\,\}(\lambda).
\tag{6}
$$

Moreover,

$$
\begin{aligned}
L_\Psi\{\,{}^c_0\mathcal{D}^\alpha_\psi(\,\pounds^t_0\{h(t)\}\,)\,\}(\lambda) &= \lambda^\alpha \cdot (\,L_\Psi\{\,\pounds^t_0\{h(t)\}\,\}(\lambda) - \lambda^{-1}(\pounds^t_0\{h(t)\})\,|_{t=0}\,) \\[2mm]
&= \lambda^\alpha \cdot (\,\lambda^\alpha I - \mathcal{A}\,)^{-1} \cdot L_\Psi\{h(t)\}.
\end{aligned}
\tag{7}
$$

Based on (6) and (7), we have

$$
{}^c_0\mathcal{D}^\alpha_\psi[\,\pounds^t_0\{h(t)\}\,] = \mathcal{A}[\,\pounds^t_0\{h(t)\}\,] + h(t).
$$

As a result, the conclusion of this lemma follows. □

Lemma 5. *The function*

$$
x(t) = S^\alpha_\psi(t,0)x_0 + \pounds^t_0\{h(t)\} + \sum_{0<t_i<t} S^\alpha_\psi(t,t_i)I_i(x(t_i))
\tag{8}
$$

defined on J is a mild solution of the following nonhomogeneous impulsive linear fractional equation:

$$
\begin{cases}
{}_0^c\mathcal{D}_\psi^\alpha x(t) = \mathcal{A}x(t) + h(t), & \text{a.e. } t \in J := [0, \ell], \ t \neq t_i; \\[2mm]
\Delta x \mid_{t=t_i} = I_i(x(t_i)), & i = 1, 2, \ldots, k; \\[2mm]
x(0) = x_0,
\end{cases}
\tag{9}
$$

where $h \in PC[J, X]$.

Proof of Lemma 5. By Definition 3.3 in [37], $x(t) = S_\psi^\alpha(t, 0)x_0 + \mathcal{L}_0^t\{h(t)\}$ is the mild solution on J_0.

For $t \in J_i$ ($i = 1, 2, \ldots, k$), by (8) and Lemma 4, we have

$$
\begin{aligned}
{}_0^c\mathcal{D}_\psi^\alpha x(t) &= {}_0^c\mathcal{D}_\psi^\alpha [\, S_\psi^\alpha(t, 0)x_0 + \mathcal{L}_0^t\{h(t)\} + \sum_{j=0}^i S_\psi^\alpha(t, t_j)I_j(x(t_j)) \,] \\
&= \mathcal{A}S_\psi^\alpha(t, 0)x_0 + \mathcal{A}\mathcal{L}_0^t\{h(t)\} + h(t) + \mathcal{A}\sum_{j=0}^i S_\psi^\alpha(t, t_j)I_j(x(t_j)) \\
&= \mathcal{A}[\, S_\psi^\alpha(t, 0)x_0 + \mathcal{L}_0^t\{h(t)\} + \sum_{j=0}^i S_\psi^\alpha(t, t_j)I_j(x(t_j)) \,] + h(t) \\
&= \mathcal{A}x(t) + h(t).
\end{aligned}
$$

Moreover, one can easily obtain that $x(0) = x_0$ and $\Delta x \mid_{t=t_i} = I_i(x(t_i))$.
To sum up, Lemma 5 is proved. □

Based on Lemma 5, we introduce the definition of mild solution and exact controllability of system (1).

Definition 3. *A function* $x \in PC[J, X]$ *is called a mild solution of (1) if it satisfies*

$$
x(t) = S_\psi^\alpha(t, 0)(x_0 - g(x)) + \mathcal{L}_0^t\{\, f(t, x(t), x(t)) + \mathcal{B}u(t) \,\} + \sum_{0 < t_i < t} S_\psi^\alpha(t, t_i)I_i(x(t_i)), \ t \in J.
$$

It is obvious that the mild solution here is the same as in [39] when $\psi(t) = t$.

Definition 4. *The problem (1.1) is said to be exactly controllable on J if, for every* $x_0, x_1 \in X$, *there exists a control* $u \in L^2[J, V]$ *such that the mild solution of (1.1) in J satisfies* $x(\ell) + g(x) = x_1$.

4. Controllability Results

In this section, our aim is to obtain the exact controllability result for problem (1). For simplicity and convenience, set

$$
K^* := \sup_{t \in J} \int_0^t \mathbb{K}(t, s)\, ds.
$$

Furthermore, we give the following hypotheses:

Hypothesis 1 (H1). $\{T(t)\}_{t \geq 0}$ *is a* C_0-*semigroup and* $\lim\limits_{t \to 0^+} T(t) = I$ *(the identity operator)*.

Hypothesis 2 (H2). *The linear operator* $\mathbb{W} : L^2[J, V] \to X$ *defined by*

$$
\mathbb{W}u = \int_0^\ell \mathcal{T}_\psi^\alpha(\ell, s)\mathcal{B}u(s)\, ds
$$

satisfies the following:

(1) $\mathbb{W}$ *has an invertible operator* $\mathbb{W}^{-1}$, *which take values in* $L^2[J, V] \backslash \ker \mathbb{W}$, *and there exist two positive constants* L_B *and* L_w, *such that* $\|\mathcal{B}\| \leq L_B$ *and* $\|\mathbb{W}^{-1}\| \leq L_w$;

(2) there exists $\eta \in L^1[J, \mathbb{R}^+]$ *such that*

$$\chi(\mathbb{W}^{-1}(D)(t)) \leq \eta(t) \cdot \chi(D), \quad t \in J,$$

for any countable subset $D \subset X$.

Hypothesis 3 (H3). *The function* $f : J \times X \times X \to X$ *satisfies the following properties:*

(1) for a.e. $t \in J$, $f(t, \cdot, \cdot) : X \times X \to X$ *is continuous;*

(2) for each $(x, y) \in X \times X$, $f(\cdot, x, y) : J \to X$ *is strongly measurable;*

(3) for any $r > 0$, *there exists a function* $h_r \in L^\infty[J, E]$ *such that*

$$\sup\{\|f(t, x, y)\| : \|x\| \leq r, \|y\| \leq K^* r\} \leq h_r(t), \quad t \in J,$$

and

$$\varlimsup_{r \to \infty} \frac{\|h_r\|_\infty}{r} < L_h < \infty,$$

where $\varlimsup$ *means the upper limit of* $\dfrac{\|h_r\|_\infty}{r}$ *as* $r \to \infty$.

(4) There exists $\xi \in L^1[J, \mathbb{R}^+]$ *such that*

$$\chi(f(t, D_1, D_2)) \leq \xi(t)(\chi(D_1) + \chi(D_2)), \quad t \in J,$$

for any bounded countable subsets D_1, $D_2 \subset X$.

Hypothesis 4 (H4). $g : PC[J, X] \to X$ *is a continuous operator and there exist* a, b, $c \geq 0$ *such that*

$$\|g(y)\| \leq a\|y\|_{PC} + b, \quad \forall y \in PC[J, X];$$

$$\chi(g(D)) \leq c \cdot \chi_{PC}(D),$$

for any bounded countable $D \subset PC[J, X]$.

Hypothesis 5 (H5). $I_i : X \to X$ $(i = 1, 2, \ldots, k)$ *is a continuous operator and there exist non-negative numbers* a_i, b_i, c_i *such that*

$$\|I_i(x)\| \leq a_i\|x\| + b_i, \quad \forall x \in X;$$

$$\chi(I_i(D)) \leq c_i \cdot \chi(D),$$

for any bounded countable $D \subset PC[J, X]$, $i = 1, 2, \ldots, k$.

Now, we are in a position to prove the exact controllability result of (1). For convenience, let

$$Y_0 = \frac{M}{\Gamma(\alpha + 1)} \cdot (\psi(\ell) - \psi(0))^\alpha;$$

$$\Omega_r = \{x \in PC[J, X] : \|x\|_{PC} \leq r\} \ (\forall r > 0);$$

$$Y_* = (1 + 2Y_0 L_B \|\eta\|_1) \cdot (1 + K^*)\|\xi\|_1 + M L_B \|\eta\|_1 \cdot (1 + \sum_{i=1}^{k} c_i).$$

Theorem 2. *Assume that (H1)–(H5) hold. Then, the system (1) is exactly controllable on* J *provided that*

$$Ma + Y_0 L_h + Y_0^2 L_B L_w L_h < 1 \text{ and } M(c + \sum_{i=1}^{k} c_i) + 2Y_0 Y_* < 1.$$

Proof of Theorem 2. In order to obtain the result, we need to define a control:

$$u_x(t) = \mathbb{W}^{-1}[\, x_1 - g(x) - S_\psi^\alpha(\ell, 0)x_0 - \pounds_0^\ell\{f(s, x(s), x(s))\} - \sum_{i=1}^k S_\psi^\alpha(\ell, t_i)I_i(x(t_i))\,](t),$$

for $x \in PC[J, X]$, $x_1 \in X$, and $t \in J$.

Moreover, define an operator H on $PC[J, X]$ by

$$Hx(t) = S_\psi^\alpha(t, 0)(\, x_0 - g(x)\,) + \pounds_0^t\{f(s, x(s), x(s)) + \mathcal{B}u_x(s)\} + \sum_{0 < t_i < t} S_\psi^\alpha(\ell, t_i)I_i(x(t_i)),$$

for $x \in PC[J, X]$ and $t \in J$. It is obvious that $Hx \in PC[J, X]$.

It is obvious that if x is a fixed point of H, then it is a mild solution of (1) that satisfies $x(\ell) + g(x) = x_1$, which implies that the system (1) is exactly controllable. Therefore, we need only to find a fixed point of H in the following work. For this sake, we divide the proof of Theorem 2 into three steps:

Step 1. Claim that $H : PC[J, X] \to PC[J, X]$ is continuous.

To do this, suppose $\{x_n\}_{n=1}^\infty \subset PC[J, X]$ such that

$$x_n \to x^*, \quad \text{as } n \to +\infty.$$

Then, there exists $r > 0$ such that $\{x_n\}_{n=1}^\infty \subset \Omega_r$. By (H4) and (H5), we have

$$\|Hx_n - Hx^*\|_{PC}$$

$$\leq \quad M\|g(x_n) - g(x^*)\| + \frac{M}{\Gamma(\alpha)} \int_0^\ell (\psi(t) - \psi(s))^{\alpha-1}\|f(s, x_n, x_n) - f(s, x^*, x^*)\| \cdot \psi'(s)ds \qquad (10)$$

$$+ \quad \frac{M}{\Gamma(\alpha)} \int_0^\ell (\psi(t) - \psi(s))^{\alpha-1}\|\mathcal{B}u_{x_n} - \mathcal{B}u_{x^*}\| \cdot \psi'(s)ds + M \cdot \sum_{i=1}^k \|I_i(x_n) - I_i(x^*)\|.$$

Notice that

$$\|\mathcal{B}u_{x_n} - \mathcal{B}u_{x^*}\| \quad \leq L_B L_w \cdot (\, \| g(x_n) - g(x^*) \| + Y_0 \cdot \|f(t, x_n, x_n) - f(t, x^*, x^*)\|$$

$$(11)$$

$$+ M \cdot \sum_{i=1}^k \|I_i(x_n) - I_i(x^*)\| \,).$$

Therefore, by (10) and (11), (H2)–(H5), and the Lebesgue dominated convergence theorem, we obtain $\|Hx_n - Hx^*\|_{PC} \to 0$, as $n \to \infty$, namely, that $H : PC[J, X] \to PC[J, X]$ is continuous.

Step 2. Claim that there exists $r > 0$ such that $H(\Omega_r) \subset \Omega_r$.

Suppose on the contrary, for each $r > 0$, there exists $x \in \Omega_r$ such that $\|Hx\|_{PC} > r$. By Lemma 3 and (H3)–(H5), one can get that

$$r < \|Hx\|_{PC} \quad \leq M\|x_0 - g(x)\| + Y_0 \cdot \|h_r\|_\infty + Y_0 \cdot \|\mathcal{B}u_x\| + M \cdot \sum_{i=1}^k \|I_i(x)\|$$

$$\leq M \cdot (\|x_0\| + a \cdot r + b) + Y_0 \cdot \|h_r\|_\infty + Y_0 \cdot \|\mathcal{B}u_x\| \qquad (12)$$

$$+ M \cdot \sum_{i=1}^k (a_i\|x\| + b_i).$$

Notice that by (H2)–(H5), one can see that

$$\|\mathcal{B}u_x\| \quad \leq L_B L_w \cdot [\ \|x_1\| + (a\|x\|_{PC}) + b) + M\|x_0\| + \Upsilon_0\|h_r\|_\infty + M\sum_{i=1}^{k}(a_i\|x\| + b_i)\ ]. \quad (13)$$

Then, it follows from (12) and (13) and our assumptions that

$$1 \leq Ma + \Upsilon_0 L_h + \Upsilon_0^2 L_B L_w L_h < 1.$$

This is a contradiction, which means that there exists $r > 0$ such that $H(\Omega_r) \subset \Omega_r$.

Step 3. Claim that if $D \subset \Omega_r$ is countable and there exists $u_0 \in \Omega_r$ such that

$$D \subset \overline{co}(\ \{u_0\} \cup H(D)\ ), \quad (14)$$

then D is relatively compact.

Suppose that $D := \{x_n\}_{n=1}^{\infty} \subset \Omega_r$. First, we show that $\{Hx_n\}_{n=1}^{\infty}$ is equicontinuous on each J_i ($i = 1, 2, \ldots, k$). For this sake, we need only to claim that $H(\Omega_r)$ is equicontinuous on each J_i. If this is true, then $\overline{co}(\ \{u_0\} \cup H(D)\ )$ is also equicontinuous on each J_i.

To do this, setting $\mathcal{F}(t, x) := f(t, x, x) + \mathcal{B}u_x$ and $(t, s) := (\psi(t) - \psi(s))^{\alpha-1}\mathcal{T}_\psi^\alpha(t, s)$, then for any $x \in D$ and $t_1 < t_2 \in J_i$, we have

$$\|Hx(t_1) - Hx(t_2)\|$$

$$\leq \|[\ S_\psi^\alpha(t_2, 0) - S_\psi^\alpha(t_1, 0)\ ](x_0 - g(x))\| + \|\sum_{j=1}^{i}(\ S_\psi^\alpha(t_2, t_j) - S_\psi^\alpha(t_1, t_j)\ )I_j(x(t_j))\|$$

$$+ \|\int_0^{t_2}(t_2, s)\mathcal{F}(s, x(s))\psi'(s)ds - \int_0^{t_1}(t_1, s)\mathcal{F}(s, x(s))\psi'(s)ds\|$$

$$\leq \|[\ S_\psi^\alpha(t_2, 0) - S_\psi^\alpha(t_1, 0)\ ](x_0 - g(x))\| + \sum_{j=1}^{i}\|\ S_\psi^\alpha(t_2, t_j) - S_\psi^\alpha(t_1, t_j)\| \cdot \|I_j(x(t_j))\|$$

$$+ \|\int_0^{t_1}((t_2, s) - (t_1, s)) \cdot \mathcal{F}(s, x(s))\psi'(s)ds\|$$

$$+ \|\int_{t_1}^{t_2}(t_2, s) \cdot \mathcal{F}(s, x(s))\psi'(s)ds\|$$

$$\leq \|[\ S_\psi^\alpha(t_2, 0) - S_\psi^\alpha(t_1, 0)\ ](x_0 - g(x))\| + \sum_{j=1}^{i}\|\ S_\psi^\alpha(t_2, t_j) - S_\psi^\alpha(t_1, t_j)\| \cdot \|I_j(x(t_j))\|$$

$$+ \|\int_0^{t_1}[\ (\psi(t_2) - \psi(s))^{\alpha-1} - (\psi(t_1) - \psi(s))^{\alpha-1}\ ] \cdot \mathcal{T}_\psi^\alpha(t_2, s)\mathcal{F}(s, x(s))\psi'(s)ds\|$$

$$+ \|\int_0^{t_1}(\psi(t_1) - \psi(s))^{\alpha-1} \cdot [\ \mathcal{T}_\psi^\alpha(t_2, s) - \mathcal{T}_\psi^\alpha(t_1, s)\ ]\mathcal{F}(s, x(s))\psi'(s)ds\|$$

$$+ \|\int_{t_1}^{t_2}(t_2, s) \cdot \mathcal{F}(s, x(s))\psi'(s)ds\|$$

$$= \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5.$$

By Lemma 3, it is obvious that $\Lambda_1 \to 0$ and $\Lambda_2 \to 0$ as $t_2 \to t_1$. Moreover, one can get that

$$\Lambda_3 \leq \frac{M\|\mathcal{F}\|}{\Gamma(\alpha+1)}[\ (\psi(t_2 - \psi(t_1)))^\alpha + (\psi(t_1 - \psi(0)))^\alpha - (\psi(t_2 - \psi(0)))^\alpha\ ].$$

Therefore, $\Lambda_3 \to 0$ as $t_2 \to t_1$. Let ε be small enough, we have

$$
\begin{aligned}
\Lambda_4 &\leq \int_0^{t_1-\varepsilon} (\psi(t_1) - \psi(s))^{\alpha-1} \cdot \|\mathcal{T}_\psi^\alpha(t_2, s) - \mathcal{T}_\psi^\alpha(t_1, s)\| \cdot \|\mathcal{F}\| \psi'(s) ds \\
&+ \int_{t_1-\varepsilon}^{t_1} (\psi(t_1) - \psi(s))^{\alpha-1} \cdot \|\mathcal{T}_\psi^\alpha(t_2, s) - \mathcal{T}_\psi^\alpha(t_1, s)\| \cdot \|\mathcal{F}\| \psi'(s) ds \\
&\leq \|\mathcal{F}\| \cdot \int_0^{t_1-\varepsilon} (\psi(t_1) - \psi(s))^{\alpha-1} \psi'(s) ds \cdot \sup_{s \in [0, t_1-\varepsilon]} \|\mathcal{T}_\psi^\alpha(t_2, s) - \mathcal{T}_\psi^\alpha(t_1, s)\| \\
&+ \frac{2M\|\mathcal{F}\|}{\Gamma(\alpha)} \int_{t_1-\varepsilon}^{t_1} (\psi(t_1) - \psi(s))^{\alpha-1} \psi'(s) ds.
\end{aligned}
$$

Using a similar process of the proof of Lemma 2.9 in [43] and the absolute continuity of the Lebesgue integral, one can get that $\Lambda_4 \to 0$ as $t_2 \to t_1$ and $\varepsilon \to 0$. $\Lambda_5 \to 0$ as $t_2 \to t_1$. Therefore,

$$
\|Hx(t_2) - Hx(t_1)\| \to 0 \text{ as } |t_2 - t_1| \to 0 \text{ with respect to } x \in D.
$$

Namely, $H(D)$ is equicontinuous on every J_i.

Next, notice that

$$
\|x_n(s) - x_m(s)\| \leq \|x_n - x_m\|_{PC}, \quad s \in J,
$$

which implies

$$
\chi(\ \{x_n(s)\}_{n=1}^\infty\ ) \leq \chi_{PC}(\ \{x_n\}_{n=1}^\infty\ ), \quad s \in J.
$$

Then, for each $t \in J$, we have

$$
\begin{aligned}
&\chi(\ \{(Hx_n)(t)\}_{n=1}^\infty\ ) \\
&\leq \chi(\ \{\ S_\psi^\alpha(t, 0)(x_0 - g(x_n)\ \}_{n=1}^\infty\ ) + \chi(\ \{\int_0^t (t, s)\mathcal{F}(s, x_n(s))\psi'(s)ds\}\ ) \\
&+ \chi(\ \{\ \sum_{0 < t_i < t} S_\psi^\alpha(t, t_i) I_i(x_n(t_i))\ \}_{n=1}^\infty\ ) \\
&\leq Mc \cdot \chi_{PC}(\ \{x_n\}_{n=1}^\infty\ ) + 2M \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \cdot \chi(\ \{\ \mathcal{F}(s, x_n(s))\ \}_{n=1}^\infty\ )\psi'(s)ds \\
&+ M \cdot \sum_{i=1}^k c_i \cdot \chi(\ \{x_n(t_i)\}_{n=1}^\infty\ ).
\end{aligned}
\tag{15}
$$

From (H2) and (H3), one can see that

$$
\chi(\{\mathcal{F}(s, x_n(s))\}_{n=1}^\infty) \leq \chi(\ \{f(s, x_n(s), x_n(s))\}_{n=1}^\infty\ ) + \chi(\ \{\mathcal{B}x_n(s)\}_{n=1}^\infty\ )
$$

$$
\leq \chi(\ \{f(s, x_n(s), x_n(s))\}_{n=1}^\infty\ ) + L_B \cdot \eta(s) \cdot [\ M \cdot \chi(\ \{g(x_n\}_{n=1}^\infty\ )
$$

$$
+ 2Y_0 \cdot \chi(\ \{\ f(s, x_n(s), x_n(s))\ \}_{n=1}^\infty\ ) + M \cdot \sum_{i=1}^k c_i \cdot \chi(\ \{x_n(t_i)\}_{n=1}^\infty\ )\ ]
$$

$$
\leq (1 + 2Y_0 L_B \|\eta\|_1) \cdot (1 + K^*)\|\xi\|_1 \cdot \chi(\ \{x_n(s)\}_{n=1}^\infty\ ) + M L_B \|\eta\|_1 \cdot (1 + \sum_{i=1}^k c_i) \cdot \chi_{PC}(\ \{x_n\}_{n=1}^\infty\ )
$$

$$
\leq [\ (1 + 2Y_0 L_B \|\eta\|_1) \cdot (1 + K^*)\|\xi\|_1 + M L_B \|\eta\|_1 \cdot (1 + \sum_{i=1}^k c_i)\ ] \cdot \chi_{PC}(\ \{x_n\}_{n=1}^\infty\ )
$$

$$
= Y_* \cdot \chi_{PC}(\ \{x_n\}_{n=1}^\infty\ ).
$$

This together with (15) implies that

$$
\begin{aligned}
\chi(\,\{(Hx_n)(t)\}_{n=1}^{\infty}\,) \quad &\leq M\cdot(c+\sum_{i=1}^{k}c_i)\cdot\chi_{PC}(\,\{x_n\}_{n=0}^{\infty}\,)+2Y_0Y_*\cdot\chi_{PC}(\,\{x_n\}_{n=1}^{\infty}\,) \\
&= [\,M\cdot(c+\sum_{i=1}^{k}c_i)+2Y_0Y_*\,]\cdot\chi_{PC}(\,\{x_n\}_{n=1}^{\infty}\,).
\end{aligned}
\tag{16}
$$

As $\{Hx_n\}_{n=1}^{\infty}$ is equicontinuous on each J_i, one can get that

$$
\chi_{PC}(\,\{Hx_n\}_{n=1}^{\infty}\,) = \sup_{0\leq i\leq k}\sup_{t\in J_i}\chi(\,\{Hx_n(t)\}_{n=1}^{\infty}\,).
$$

This together with (14) and (16) guarantees that

$$
\chi_{PC}(\,\{x_n\}_{n=1}^{\infty}\,)\leq \chi_{PC}(\,\{Hx_n\}_{n=1}^{\infty}\,)\leq [\,M\cdot(c+\sum_{i=1}^{k}c_i)+2Y_0Y_*\,]\cdot\chi_{PC}(\,\{x_n\}_{n=1}^{\infty}\,).
$$

From our assumptions, we know $\chi_{PC}(\,\{x_n\}_{n=1}^{\infty}\,)=0$, which implies that $D=\{x_n\}_{n=1}^{\infty}$ is relatively compact. Thus, H has a fixed point in Ω_r by Theorem 1. To sum up, system (1) is exactly controllable on J. $\square$

5. An Example

In this section, an illustrative example is worked out to show the effectiveness of the obtained result.

Example 1. *Let $X=L^2[0,\pi]$ be equipped with the norm and inner product defined by*

$$
\|x\|_2 = (\int_0^{\pi}|x(y)|^2dy)^{\frac{1}{2}} \text{ and } <x,z> = (\int_0^{\pi}x(y)\overline{z(y)}dy)^{\frac{1}{2}},\ \forall x,z\in X.
$$

Consider the following nonlinear partial integro-differential system:

$$
\begin{cases}
\dfrac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}x(t,y) = \dfrac{\partial^2}{\partial y^2}x(t,y)+\dfrac{\sin(t)}{1+e^t}[\,\varepsilon_1\cdot(1-\Gamma(\tfrac{3}{2}))\cdot x(t,y) \\
\qquad\qquad +\varepsilon_1\cdot\Gamma(\tfrac{3}{2})\cdot\displaystyle\int_0^t(\tfrac{3t}{2}-s)^2x(s,y)ds\,]+\varepsilon_2 u(t,y); \\
x(t,0)=x(t,\pi)=0; \\
x(0,y)+\dfrac{1}{10}\cdot\displaystyle\int_0^1 x(t,y)dt=0; \\
\Delta x\,|_{t=\frac{1}{2}}=x(\tfrac{1}{2},y),
\end{cases}
\tag{17}
$$

where $t\in J:=[0,1]$, $y\in[0,\pi]$ and $\varepsilon_i\ (i=1,2)$ are positive numbers will be specified later.

Conclusion: System (17) is exactly controllable on J as $\varepsilon_i\ (i=1,2)$ are sufficiently small.

Proof of Conclusion. System (17) can be regarded as the form of system (1), where

$$
\alpha=\frac{1}{2},\ \ell=1,\ g(x)=\frac{1}{10}\cdot\int_0^1 x(t,y)dt,\ I_1(x(t_1,y))=x(\tfrac{1}{2},y),\ \psi(t)=t;
$$

$$
f(t,x(t,y),x)=\frac{\sin(t)}{1+e^t}[\,\varepsilon_1\cdot(1-\Gamma(\tfrac{3}{2}))\cdot x(t,y)+\varepsilon_1\cdot\Gamma(\tfrac{3}{2})\cdot\int_0^t(\tfrac{3t}{2}-s)^2x(s,y)ds\,].
$$

Consider the operator $\mathcal{A}:D(\mathcal{A})\subset X\to X$ defined by

$$
\mathcal{A}x=\frac{\partial^2}{\partial y^2}x.
$$

As is well known, $\mathcal{A}$ has a discrete spectrum. The eigenvalues are $\{-n^2 : n \in \mathbb{N}\}$ with the corresponding normalized eigenvectors $e_n(y) = \sqrt{\dfrac{2}{\pi}} \sin(ny)$. Then,

$$\mathcal{A}x = -\sum_{n=1}^{\infty} n^2 < x, e_n > e_n, \ x \in D(\mathcal{A}).$$

Moreover, $\mathcal{A}$ is the infinitesimal generator of a uniformly bounded analytic semigroup $\{T(t)\}_{t \geq 0}$, where

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n, \ x \in X.$$

Obviously, $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$. Thus, one can choose $M := \sup_{t \in [0,\infty)} \|T(t)\| = 1$. In addition, the operator $\mathcal{B} = \varepsilon_2 I$ (the identity operator). For $y \in [0, \pi]$, the linear operator $\mathbb{W}$ is defined by

$$\mathbb{W}u = \varepsilon_2 \int_0^1 \mathcal{T}_t^{\frac{1}{2}}(1,s) Iu(s,y) ds.$$

It is easy to see that

$$\|\mathbb{W}u\| \leq \|\varepsilon_2 \int_0^1 \mathcal{T}_t^{\frac{1}{2}}(1,s) Iu(s) ds\| \leq \varepsilon_2 \|u\|,$$

which means $\|\mathbb{W}\| \leq \varepsilon_2$. Thus, (H2) holds by choosing $L_B = \varepsilon_2$ and a suitable $L_w > 0$.

By careful calculation, one can obtain that for all $x \in \Omega_r$,

$$\|f(t,x(t),x(t))\| \leq \varepsilon_1 r \cdot \left(1 - \Gamma(\frac{3}{2}) + \Gamma(\frac{3}{2}) \cdot \int_0^t (\frac{3t}{2} - s)^2 ds \right) = \varepsilon_1 r \cdot \left(1 - \Gamma(\frac{3}{2}) + \Gamma(\frac{3}{2}) t^2 \right).$$

Thus, (H3) holds by choosing $h_r(t) = \varepsilon_1 r \cdot \left(1 - \Gamma(\frac{3}{2}) + \Gamma(\frac{3}{2}) t^2 \right)$ and $L_h = \Gamma(\frac{3}{2}) \cdot \varepsilon_1$.

Moreover, one can easily verify that (H4) and (H5) hold by choosing $a = \dfrac{1}{10}$, $a_1 = 1$, and $b = b_1 = 1$. Suppose ε_1 and ε_2 are sufficiently small such that

$$\varepsilon_1 < \min\{\frac{1}{3}, \Gamma(\frac{3}{2})\} \ \text{ and } \ \varepsilon_2 < \frac{\Gamma(\frac{3}{2})}{3L_w \varepsilon_1},$$

$$Ma + Y_0 L_h + Y_0^2 L_B L_w L_h = \frac{1}{10} + \varepsilon_1 + \frac{L_w \varepsilon_1 \varepsilon_2}{\Gamma(\frac{3}{2})} < \frac{1}{10} + \frac{1}{3} + \frac{1}{3} < 1.$$

Consequently, system (17) is exactly controllable on J by Theorem 2. $\square$

6. Discussion

In this paper, first, based on generalized Laplace transforms and semigroup theory, the concept of a mild solution is obtained for a class of impulsive fractional evolution equations with nonlocal conditions in the framework of ψ-Caputo fractional derivatives. As far as we know, there is no definition of the mild solution available for the considered systems (1). Second, by Mönch fixed point theorem, the exact controllability result is investigated with a non-compact semigroup. The exact controllability obtained in the present paper can be applied in the broadest context such as many fractional evolution system with various boundary conditions involving the classical Caputo or Hadamard fractional derivatives.

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