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First Solution of Fractional Bioconvection with Power Law Kernel for a Vertical Surface

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Abstract: The present study provides the heat transfer analysis of a viscous fluid in the presence of bioconvection with a Caputo fractional derivative. The unsteady governing equations are solved by Laplace after using a dimensional analysis approach subject to the given constraints on the boundary. The impact of physical parameters can be seen through a graphical illustration. It is observed that the maximum decline in bioconvection and velocity can be attained for smaller values of the fractional parameter. The fractional approach can be very helpful in controlling the boundary layers of the fluid properties for different values of time. Additionally, it is observed that the model obtained with generalized constitutive laws predicts better memory than the model obtained with artificial replacement. Further, these results are compared with the existing literature to verify the validity of the present results.

Keywords: bioconvection; Caputo fractional; heat transfer; vertical surface

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1. Introduction

The term bioconvection is characterized by hydrodynamic instability and patterns in suspensions of biased swimming microorganisms. The velocity and spatial range of fluid motions are typically considerably higher than those associated with the velocity and size of individual cells, resulting in rapid cell transfer and the formation of specialized cell concentration visualization patterns. Biotechnology has evolved to integrate new and diverse sciences in the early 21st century, as introduced by Platt [1]. A variety of researchers dedicate their resources to revealing bioconvection characteristics, like Khan et al. [2], who conducted their research on the dynamics of the Cattaneo–Christov theory of heat and mass flow with bioconvection Oldroyd-B nanofluid. Assessment of bioconvection in Maxwell nanofluid equipped with nonlinear radiation using a Riga surface was carried out by Ramesh et al. [3]. Tlili et al. [4] previewed the analysis of micropolar nanofluid flow with partial slip and double stratification in MHD gyrotactic microorganisms. Heat production due to gyrotactic microorganisms with the bioconvection of a magnetohydrodynamic flow of nanofluid was examined by Khotha et al. [5]. Khan et al. [6] studied the entropy generation of bioconvection nanofluid streamlines between two spinning extendable discs. Shah et al. [7] investigated the microstructure and inertial properties of Maxwell base fluid stream-dependent magnetohydrodynamic suspended SWCNTs and MWCNTs with bioconvection and a vertical permeable cone. Research was conducted on the implications of activation energy, MHD, and bioconvection in the flow to the extended surface of third grade (non-Newtonian) fluid by Chu et al. [8]. Bhatti et al. [9] investigated the swimming of motile gyrotactic microorganisms and nanoparticles in blood flow with anisotropically tapered arteries. In that work, they discussed a theoretical study on the swimming of migratory gyrotactic microorganisms in a non-Newtonian blood-based nanofluid through an anisotropically narrowed artery.

Abbasi et al. [10] investigated the convective stream across a convective spinning stretching disc of viscoelastic nanofluid. With Maxwell, MHD, activation power, heat transfer, and Oldroyd-B on an extendable sheet, and also with a vertical rotating cylinder, the significance of bioconvection was studied by Waqas et al. [11,12]. The significance of various slips with gyrotactic motile microorganisms in the bioconvection stream of cross nanofluid over a wedge was studied. The value of viscoelastic nanofluid through a magnetic dipole and several slips on a cross nanofluid bioconvection stream past a wedge through gyrotactic motile microorganisms has been numerically examined by Alshomrani et al. [13,14]. A bioconvection study of third-grade nanofluid in von-Kármán flow with motile microorganisms was conducted by Ullah et al. [15]. The effectiveness of magnetohydrodynamic span and Carreau nanofluid has been established in entropy production details and gyrotactic motile microorganisms with transformations of von Neumann similarity by Naz et al. [16,17]. Alqarni et al. [18] investigated the transferred heat of micropolar nanofluid bioconvection flow through motile microorganisms and velocity slip conditions. Bioconvection with activation energy and Wu's slip of magnetized couple stress nanofluid was studied by Khan et al. [19]. Sajid et al. [20] considered the effects of double-diffusive convection and motile microorganisms on tangent hyperbolic base fluid bioconvection MHD. A simulation of finite elements of bioconvection and superior-Fick diffusion effects over a vertically extending layer on micropolar-based nanofluid flow was discussed by Ali et al. [21]. The Newtonian heating effect on a time mixed convection nanofluid microorganism stream through the stagnation realm of an impulsively spinning sphere was examined by Mahdy et al. [22]. The effects of energy on Eyring's motile microorganism-containing nanofluid on MHD flow were studied by Sharif et al. [23]. A computational simulation by coupled quasiliberalization magnetohydrodynamic bioconvective Casson nanofluid flow was carried out by Ansari et al. [24].

Fractional calculus is a field of mathematics used for the integration and differentiation of real or complex numbers. Even although calculus is historic, it has still gained attention in recent decades. Recently, mathematical simulations of heat transfer fluids, which play an important role in manufacturing, have also been studied by researchers, with significant applications. Usually, these simulations are expressed in standard integer-order partial differential equations. Note that the conventional PDEs cannot decipher the dynamical behavior of physical flow parameters and retention effects. To remove these defects, researchers focused on the fractional dynamic systems of heat transfer in simple and complex fluid models. Recently, many researchers have worked on the application of fractional calculus but they have not considered the unsteady effect of bioconvection in those heat transfer models. The wide range of fractional models, and their applications in applied sciences, can be seen in the following references [25–44].

The most interesting feature of the fractional operators is that there are several such operators. This helps researchers to select the most appropriate operator to characterize the implications of real global crises. Jarad et al. [45] modified fractional derivatives and Laplace transformed them. Vieru et al. [46] applied a Caputo fractional derivative to heat and mass transfer flow over a flat plate. They used the Laplace transform method to find exact solutions. They did not consider the bioconvection effect in their model. Abro et al. [47] investigated the thermal stability of Maxwell nanofluids by fractional derivatives with a singular kernel. Zhang et al. [48] analyzed a new mathematical model of COVID-19 through fractional derivatives of discrete and non-singular kernels. Rayal et al. [49] proposed a numerical analysis of a damped differential equation of an extended form associated with fractal–fractional derivatives through the use of the Lagrange wavelet fractional order. Singh et al. [50] presented a review of the chemical kinetics method of a fractional derivative of the Mittag-Leffler kernel form. Ghanbari et al. [51] studied new edge detection strategies focused on fractional derivatives of non-local and semi kernels. Son et al. [52] researched the liberalization regulatory problem with a granular neutrosophic fractional differential equation. Gambo et al. [53] discussed the unique nature of fractional differential equation solutions within the context of simplified Caputo fractional derivatives. Akgul et al. [54] analyzed the formulation of linear and nonlinear differential equations through simplified fractional derivatives. El-Nabulsi et al. [55] investigated results that were used to explore some novel aspects of fractional quantum field theory and many interesting consequences were revealed, in particular the complex quantum field theory, Dirac operators and the novel notion of mass without mass. El-Nabulsi et al. [56] studied the fractional field theory with Saxena–Kumbhat fractional integrals, fractional derivatives of order α, β and dynamical fractional integral exponents. Moshrefi et al. [57] analyzed the physical and geometrical interpretation of fractional operators. Cloot et al. [58] investigated the generalized groundwater flow equation using the concept of non-integer order derivatives. Lima et al. [59] studied the experimental signal analysis of robot impacts from a fractional calculus perspective. Bhalekar et al. [60] investigated the fractional Bloch equation with delay. In that work, they described the fractional Bloch equation with both fading memory and delay included. Odzijewicz et al. [61] invented variations of fractional calculus in the expression of a generalized fractional integral with physics applications. In that work, they investigated speculated fractional integrals with Lagrangian depending on classical derivatives, generalized fractional derivatives and integrals. Malinowska [62] studied the fractional variational calculus for non differentiable functions. Kulish et al. [63] investigated the application of fractional calculus to fluid mechanics. In that work, they examined the application of fractional calculus and of arbitrary differentiation to the solution of time-dependent, viscous diffusion fluid mechanics problems. Together with the Laplace transform method, the application of fractional calculus to the classical transient viscous diffusion equation in a semi-infinite space was shown to yield explicit analytical fractional solutions for the shear stress and fluid speed anywhere in the domain. El-Nabulsi et al. [64] investigated the path integral formulation of fractionally perturbed Lagrangian oscillators on a fractal. Meerschaert et al. [65] studied multidimensional advection and fractional dispersion. In that work, they found that the extension of the fractional diffusion equation to two or three dimensions is not as simple as an extension of the second order equation.

Vieru el al.'s [46] heat and mass transfer model is used in the absence of the bioconvection effect. After that, Shah et al. [25] studied analytical solutions for time-fractional boundary layer flow of viscous fluid over a vertical heat transfer surface including Caputo and Caputo–Fabrizio derivatives. Exact solutions were obtained through the Laplace transform method without the bioconvection effect.

The studies above were carried out with or without fractional derivatives in the absence of fractional bioconvection. On the other hand, the main task is to combine these two attractive topics, bioconvection and fractional derivatives. In the abovementioned literature, there is no single study of bioconvection with a Caputo fractional derivative. The theoretical model of heat transfer of fluid flow in the presence of bioconvection is solved with the Laplace transform method. A graphical discussion of flow parameters is presented through graphics.

2. Mathematical Formulation

Let us consider an unsteady heat transfer flow of a viscous fluid over a flat surface in an xy-coordinate system situated at $y = 0$. In the beginning, at $t = 0$, the plate and the fluid are at rest with reference surface temperature T_∞ and reference concentration of microorganisms N_∞. After some time, the plate begins to move at a constant velocity and the surface temperature T_w and the concentration of microorganisms of the plate N_w increase. Since the plate has infinite length, every physical quantity is the function of y and t only and shown in Figure 1.

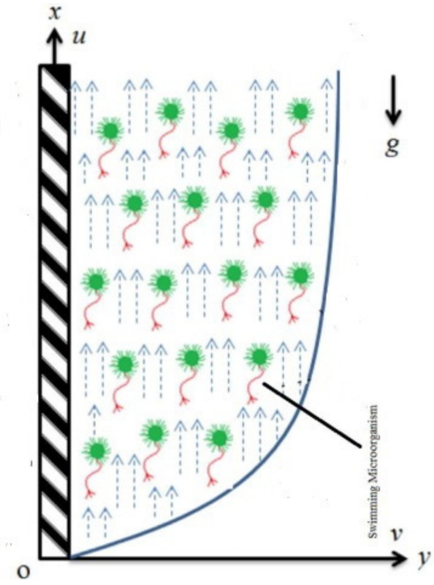

Figure 1. Geometry of the problem.

The momentum and bioconvection equations can be seen in limiting form in [66–69]

$$\rho u_t(y,t) = \mu u_{yy}(y,t) + g[\rho\beta_T(T - T_\infty) - \gamma(\rho_m - \rho)(N - N_\infty)], \tag{1}$$

while the heat equation is

$$\rho C_p T_t(y,t) = k T_{yy}(y,t), \tag{2}$$

and the unsteady bioconvection equation can be found in [66–69],

$$N_t(y,t) = D_n N_{yy}(y,t), \tag{3}$$

subject to initial and boundary conditions

$$u(y,0) = 0,\ T(y,0) = T_\infty,\ N(y,0) = N_\infty,\ \text{for all } y \geq 0, \tag{4}$$

$$u(0,t) = u_0 H(t),\ T(0,t) = T_w,\ N(0,t) = N_w,\ t > 0, \tag{5}$$

$$u(y,t) \longrightarrow 0,\ T(y,t) \longrightarrow T_\infty,\ N(y,t) \longrightarrow N_\infty,\ y \longrightarrow \infty,\ t > 0. \tag{6}$$

We introduce the non-dimensional variables

$$y^* = \frac{u_0 y}{\nu},\ u^* = \frac{u}{u_0},\ t^* = \frac{t u_0^2}{\nu},\ \theta = \frac{T - T_\infty}{T_W - T_\infty},\ N^* = \frac{N - N_\infty}{N_W - N_\infty}, q^* = \frac{q}{q_o}, J^* = \frac{J}{J_o}, \tag{7}$$

into Equations (1)–(6) and ignore the $(*)$ notation

$$u_t(y,t) = u_{yy}(y,t) + \mathrm{Gr}[\theta(y,t) - \mathrm{Ra}N(y,t)], \tag{8}$$

$$\theta_t(y,t) = \frac{1}{\mathrm{Pr}}\theta_{yy}(y,t), \tag{9}$$

$$N_t(y,t) = \frac{1}{\mathrm{Lb}}N_{yy}(y,t), \tag{10}$$

with dimensionless conditions

$$u(y,0) = 0,\ \theta(y,0) = 0,\ N(y,0) = 0, \tag{11}$$

$$u(0,t) = H(t),\ \theta(0,t) = 1,\ N(0,t) = 1, \tag{12}$$

$$u(y,t) \longrightarrow 0,\ \theta(y,t) \longrightarrow 0,\ N(y,t) \longrightarrow 0, \text{as } y \longrightarrow \infty, \tag{13}$$

where $H(t)$ is the unit step function, Gr is the Grashof number, $\mathrm{Gr} = \frac{g\nu\beta_T(T_w - T_\infty)}{u_0^3}$, Pr is the Prandtl number, $\mathrm{Pr} = \frac{\rho C_p}{k}$, $\mathrm{Lb} = \frac{\nu}{D_N}$ is the bioconvection Lewis number and $\mathrm{Ra} = \frac{\gamma(\rho_m - \rho)(N_w - N_\infty)}{\beta_T(T_w - T_\infty)\rho}$ is the bioconvection Rayleigh number.

3. The Solution of the Problem with Classical Time Derivative

This section deals with the solution of the temperature and velocity field with and without a fractional derivative and the Laplace transform method.

3.1. The Solution of Bioconvection

In this section, the solution of bioconvection given in Equation (10) and subject to boundary conditions (11)–(13) by the assistance of the Laplace transform strategy can be obtained in the following way:

$$s\bar{N}(y,s) = \frac{1}{Lb}\bar{N}_{yy}(y,s), \tag{14}$$

while associated conditions in the transformed domain are:

$$\bar{N}(0,s) = \frac{1}{s},\ \bar{N}(y,s) \to 0,\ \text{as } y \to \infty. \tag{15}$$

The general solution of Equation (14) under condition (15):

$$\bar{N}(y,s) = \frac{1}{s}e^{-y\sqrt{\mathrm{Lb}s}}, \tag{16}$$

is found by applying the inverse Laplace to Equation (16)

$$N(y,t) = \mathrm{erfc}\left(\frac{-y\sqrt{\mathrm{Lb}}}{2\sqrt{t}}\right), \tag{17}$$

where erfc(.) is Gauss's complementary error function.

3.2. The Solution of Temperature Field

In this section, the solution of energy in Equation (9) is subject to the boundary conditions (11)–(13) with the assistance of the Laplace transform method and is given as

$$s\bar{\theta}(y,s) = \frac{1}{\mathrm{Pr}}\bar{\theta}_{yy}(y,s), \tag{18}$$

with associated conditions:

$$\bar{\theta}(0,s) = \frac{1}{s},\ \bar{\theta}(y,s),\ \to 0,\ \text{as } y \to \infty. \tag{19}$$

The solution of Equation (18) is subject to the conditions given in (19), so we have

$$\bar{\theta}(y,s) = \frac{1}{s}e^{-y\sqrt{\mathrm{Pr}s}}, \tag{20}$$

with its inverse Laplace transform form

$$\theta(y,t) = \mathrm{erfc}\left(\frac{-y\sqrt{\mathrm{Pr}}}{2\sqrt{t}}\right), \tag{21}$$

where erfc(.) is Gauss's complementary error function.

3.3. The Solution of the Velocity Field

This section deals with the solution of momentum in Equation (8) subject to the boundary conditions (11)–(13) through the assistance of the Laplace transform technique and the procedure is given below:

$$s\bar{u}(y,s) = \bar{u}_{yy}(y,s) + \mathrm{Gr}[\bar{\theta}(y,s) - \mathrm{Ra}\bar{N}(y,s)], \tag{22}$$

and relevant boundary conditions in transform domain are:

$$\bar{u}(y,s) = \frac{1}{s},\ \bar{u}(y,s) \to 0,\ \text{as } y \to \infty. \tag{23}$$

A general solution of Equation (22) subject to the constraints given in (23):

$$\bar{u}(y,s) = \frac{e^{-y\sqrt{s}}}{s} + \frac{\mathrm{Gr}}{\mathrm{Pr}-1}\left(\frac{e^{-y\sqrt{s}}}{s^2} - \frac{e^{-y\sqrt{\mathrm{Pr}s}}}{s^2}\right) - \frac{\mathrm{RaGr}}{\mathrm{Lb}-1}\left(\frac{e^{-y\sqrt{s}}}{s^2} - \frac{e^{-y\sqrt{\mathrm{Lb}s}}}{s^2}\right). \tag{24}$$

Taking the inverse Laplace of Equation (24) with the help of the following formula given in [25,46] gives:

$$L^{-1}\left\{\frac{1}{s^2}e^{-y\sqrt{as}}\right\} = \left(t + \frac{ay^2}{2}\right)erfc\left(\frac{y\sqrt{a}}{2\sqrt{t}}\right) - \frac{y\sqrt{at}}{\pi}e^{-\frac{-ay^2}{4t}}$$

$$\begin{aligned} u(y,t) = erfc\left(\tfrac{y}{2\sqrt{t}}\right) + \tfrac{\mathrm{Gr}}{\mathrm{Pr}-1}\left[\left(t + \tfrac{y^2}{2}\right)erfc\left(\tfrac{y}{2\sqrt{t}}\right) - \tfrac{y\sqrt{t}}{\pi}e^{-\frac{-y^2}{4t}}\right] - \\ \tfrac{\mathrm{Gr}}{\mathrm{Pr}-1}\left[\left(t + \tfrac{\mathrm{Pr}y^2}{2}\right)erfc\left(\tfrac{y\sqrt{\mathrm{Pr}}}{2\sqrt{t}}\right) - \tfrac{y\sqrt{\mathrm{Pr}t}}{\pi}e^{-\frac{-\mathrm{Pr}y^2}{4t}}\right] \\ -\tfrac{\mathrm{RaGr}}{\mathrm{Lb}-1}\left[\left(t + \tfrac{y^2}{2}\right)erfc\left(\tfrac{y}{2\sqrt{t}}\right) - \tfrac{y\sqrt{t}}{\pi}e^{-\frac{-y^2}{4t}}\right] \\ +\tfrac{\mathrm{RaGr}}{\mathrm{Lb}-1}\left[\left(t + \tfrac{\mathrm{Lb}y^2}{2}\right)erfc\left(\tfrac{y\sqrt{\mathrm{Lb}}}{2\sqrt{t}}\right) - \tfrac{y\sqrt{\mathrm{Lb}t}}{\pi}e^{-\frac{-\mathrm{Lb}y^2}{4t}}\right]. \end{aligned} \tag{25}$$

Equation (25) is valid only for $\mathrm{Pr} \neq 1$ and $\mathrm{Lb} \neq 1$.

3.4. Fractional Modeling

In the literature presented in the Introduction, researchers addressed bioconvection with classical time derivatives or the steady case, either analytically or numerically, in the absence of a fractional derivative approach. Usually, these models are represented in terms of traditional integer-order partial differential equations (PDEs). However, traditional PDEs cannot decode the complex behavior of physical flow parameters and memory effects. The classical calculus measures the instant rate of change of the output when the input level changes. Therefore, it is not able to include the previous state of the system, called the memory effect. In fractional calculus (FC), the rate of change is affected by all points of the considered interval, so it is able to incorporate the previous history/memory effects of any system. The order of the fractional derivative is treated as an index of memory. Therefore, we considered the present fractional model. Further, in the fractional derivative approach in convective problems, scientists have ignored the bioconvection effect. Therefore, our main target was to combine these two branches in order to cover the gap which is still not reported in the existing literature. For this purpose, we generalized the ordinary model with fractional derivatives and found analytical solutions of fractional energy equations and fractional bioconvection. There are two method for fractional modeling in the existing literature and: (i) we develop the fractional model of governing equations by replacing the integer order with non-integer order derivative, (ii) we develop the fractional model for energy balance equations and bioconvection through generalized constitutive relations and use these solutions in a classical momentum equation to obtain an analytical solution for the velocity field.

3.5. Solution of the Fractional Model Using Generalized Constitutive Relations

Firstly, we develop the constitutive fractional model for energy balance equations and bioconvection equations and solve them with the Laplace transform method, followed by the velocity field. The momentum equation for viscous fluid containing the bioconvection term is given by

$$\rho \ \frac{\partial u(y,t)}{\partial t} = \frac{\partial \tau(y,t)}{\partial y} + g[\rho\beta_T(T - T_\infty) - \gamma(\rho_m - \rho)(N - N_\infty)]. \tag{26}$$

The constitutive equation is

$$\tau(y,t) = \mu \frac{\partial u(y,t)}{\partial y}. \tag{27}$$

The equation of thermal balance is

$$(\rho C_p) \ \frac{\partial T(y,t)}{\partial t} = -\frac{\partial q(y,t)}{\partial y}. \tag{28}$$

By Fourier's law, the thermal flux equation of heat conduction is

$$q(y,t) = -k\frac{\partial T(y,t)}{\partial t}. \tag{29}$$

The equation of diffusion balance is

$$\frac{\partial N(y,t)}{\partial t} = -\frac{\partial J(y,t)}{\partial y}. \tag{30}$$

The equation of bioconvection concentration is

$$J(y,t) = -D_n\frac{\partial N(y,t)}{\partial y}. \tag{31}$$

By introducing the dimensionless variables from Equation (7) into (26)–(31), we have the momentum equation containing the bioconvection term, given by

$$\frac{\partial u(y,t)}{\partial t} = c_o\frac{\partial \tau(y,t)}{\partial y} + \mathrm{Gr}[\theta(y,t) - \mathrm{Ra}N(y,t)]. \tag{32}$$

The constitutive equation is

$$\tau(y,t) = c_1\frac{\partial u(y,t)}{\partial y}. \tag{33}$$

The equation of thermal balance is

$$\frac{\partial \theta(y,t)}{\partial t} = -c_2\frac{\partial q(y,t)}{\partial y}. \tag{34}$$

By Fourier's law, the thermal flux equation of heat conduction is

$$q(y,t) = -c_3\frac{\partial \theta(y,t)}{\partial t}. \tag{35}$$

The equation of diffusion balance is

$$\frac{\partial N(y,t)}{\partial t} = -c_4\frac{\partial J(y,t)}{\partial y}. \tag{36}$$

The equation of bioconvection concentration is

$$J(y,t) = -c_5 \frac{\partial N(y,t)}{\partial y}. \tag{37}$$

By using Equation (33) in Equation (32), we have the momentum equation in dimensionless form:

$$\frac{\partial u(y,t)}{\partial t} = \frac{\partial u^2(y,t)}{\partial y^2} + \mathrm{Gr}[\theta(y,t) - \mathrm{Ra}N(y,t)]. \tag{38}$$

where $c_0 = \frac{\tau_o}{\rho u_o^2}$, $c_1 = \frac{\rho u_o^2}{\tau_o}$, $c_2 = \frac{q_o}{\rho C_p(T_w - T_\infty)}$, $c_3 = \frac{k u_o (T_w - T_\infty)}{q_o \nu}$, $c_4 = \frac{J_o}{u_o(N_w - N_\infty)}$, $c_5 = \frac{D_n u_o (N_w - N_\infty)}{J_o \nu}$.

Hristov [70] and Povstenko [71] found the constitutive thermal flux equation with the generalized Fourier's law:

$$Q(y,t) = -b_{1-\beta}\, {}^C D_t^{1-\beta}\left[\frac{\partial \theta(y,t)}{\partial y}\right], \quad 0 < \beta \leq 1, \tag{39}$$

and the constitutive equation for bioconvection balance equation:

$$J(y,t) = -c_{1-\gamma}\, {}^C D_t^{1-\gamma}\left[\frac{\partial N(y,t)}{\partial y}\right], \quad 0 < \gamma \leq 1. \tag{40}$$

In the equations above, ${}^C D_t^{\beta}$ or ${}^C D_t^{\gamma}$ is the Caputo time fractional operator defined in [46].

By introducing Equations (39) and (40) into Equations (34) and (36), respectively, we have. In [72,73] the authors used the same kind of fractional modeling for some heat transfer models.

$$\frac{\partial \theta(y,t)}{\partial t} = -c_2 \frac{\partial\left[-b_{1-\beta}\, {}^C D_t^{1-\beta} \frac{\partial \theta(y,t)}{\partial y}\right]}{\partial y}, \tag{41}$$

$$\frac{\partial N(y,t)}{\partial t} = -c_4 \frac{\partial\left[-c_{1-\gamma}\, {}^C D_t^{1-\gamma} \frac{\partial N(y,t)}{\partial y}\right]}{\partial y}. \tag{42}$$

In order to get the equivalent forms of Equations (41) and (42), we apply left inverse operators $I_t^{1-\beta}(.)$ and $I_t^{1-\gamma}(.)$ and on both sides we have

$${}^C D_t^{\beta} \theta(y,t) = c_6 \frac{\partial^2 \theta(y,t)}{\partial y^2}, \tag{43}$$

$${}^C D_t^{\gamma} N(y,t) = c_7 \frac{\partial^2 N(y,t)}{\partial y^2}. \tag{44}$$

The solution of Equation (43) is subject to conditions given in Equation (19) by means of the Laplace transform method

$$\bar{\theta}(y,s) = \frac{1}{s} exp\left(-y\sqrt{\frac{s^{\beta}}{c_6}}\right), \tag{45}$$

written in a suitable form,

$$\theta(y,s) = \frac{1}{s} + \sum_{i=1}^{\infty} \frac{\left(\frac{-y}{\sqrt{c_6}}\right)^i}{i!\, s^{1-\frac{\beta i}{2}}}. \tag{46}$$

By taking the inverse Laplace transform of Equation (46), we have

$$\theta(y,t) = 1 + \sum_{i=1}^{\infty} \frac{(\frac{-y}{\sqrt{c_6}})^i t^{-\beta i/2}}{i!\Gamma(1-\beta i)}. \tag{47}$$

Therefore, by following a similar procedure, the solution of bioconvection can be obtained as follows:

$$N(y,t) = 1 + \sum_{j=1}^{\infty} \frac{(\frac{-y}{\sqrt{c_7}})^j t^{-\gamma j/2}}{j!\Gamma(1-\eta j)}. \tag{48}$$

In order to obtain the analytical solutions for momentum equation, we use the expressions of energy and bioconvection of fractional order in momentum Equation (38) after applying the Laplace transform subject to conditions (23), we have

$$u(y,s) = \frac{e^{-y\sqrt{s}}}{s} + \frac{\mathrm{Gr}}{(\frac{s^{\beta-1}}{c_6}) - 1}\left[\frac{e^{-y\sqrt{s}}}{s^2} - \frac{e^{-y\sqrt{\frac{s^\beta}{c_6}}}}{s^2}\right] - \frac{\mathrm{GrRa}}{(\frac{s^{\gamma-1}}{c_7}) - 1}\left[\frac{e^{-y\sqrt{s}}}{s^2} - \frac{e^{-y\sqrt{\frac{s^\gamma}{c_7}}}}{s^2}\right]. \tag{49}$$

In order to obtain the inverse Laplace transform analytically, Equation (49) can be written in its suitable form

$$\begin{aligned}\bar{u}(y,s) = \tfrac{1}{s} + \textstyle\sum_{i=1}^{\infty} \frac{(-y)^i}{i! s^{1-\frac{i}{2}}} + \mathrm{Gr}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c_6)^{j+1}(-y)^k}{k! s^{1+\beta j+\beta-j-\frac{k}{2}}} - \mathrm{Gr}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(c_6)^{j+1-\frac{l}{2}}(-y)^l}{l! s^{1+\beta j+\beta-j-\frac{\beta l}{2}}} \\ -\mathrm{GrRa}\textstyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c_7)^{m+1}(-y)^n}{n! s^{1+\beta m+\beta-m-\frac{n}{2}}} + \mathrm{GrRa}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(c_7)^{m+1-\frac{p}{2}}(-y)^p}{p! s^{1+\beta m+\beta-m-\frac{\beta p}{2}}}.\end{aligned} \tag{50}$$

Taking the inverse Laplace of Equation (50), we have

$$\begin{aligned}u(y,t) = 1 + \textstyle\sum_{i=1}^{\infty} \frac{(-y)^i t^{-\frac{i}{2}}}{i!\Gamma(1-\frac{i}{2})} + \mathrm{Gr}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(c_6)^{j+1}(-y)^k t^{\beta j-j-\frac{k}{2}}}{k!\Gamma(1+\beta j+\beta-j-\frac{k}{2})} - \mathrm{Gr}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(c_6)^{j+1-\frac{l}{2}}(-y)^l t^{\beta j+\beta-j-\frac{\beta l}{2}}}{l!\Gamma(1+\beta j+\beta-j-\frac{\beta l}{2})} \\ -\mathrm{GrRa}\textstyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c_7)^{m+1}(-y)^n t^{\beta m-m-\frac{n}{2}}}{n!\Gamma(1+\beta m+\beta-m-\frac{n}{2})} + \mathrm{GrRa}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \frac{(c_7)^{m+1-\frac{p}{2}}(-y)^p t^{\beta m+\beta-m-\frac{\beta p}{2}}}{p!\Gamma(1+\beta m+\beta-m-\frac{\beta p}{2})}.\end{aligned} \tag{51}$$

4. Results and Discussion

The unsteady bioconvection effect is studied in this investigation with a fractional derivative for a vertical surface. By the Laplace transform method, exact solutions are obtained and presented in the form of Wright's function. Some graphs are plotted to see the physical impact of flow parameters and are presented in this section. Figure 2 shows the comparison between the Caputo fractional and classical velocity field by keeping other parameters constant. It is clear from the figure that for $\alpha = 0.95$ the fluid velocity shows decay and for $\alpha = 0.6$ more decay is detected. Additionally, for $\alpha = 0.4$ and $\alpha = 0.2$, the velocity field reaches the maximum decline. This fact explains the memory effect of the present fractional operator for varying values of the fractional parameter. The Caputo fractional explains the memory of the function and further shows dual behavior for long and short times due to the power law kernel that appears in its definition. Such results are very important in some experimental data. Hence, by using a fractional approach, the velocity exhibits more memory than the classical approach.

Figure 3 presents the outcome of the bioconvection Rayleigh number Ra and the comparison between the Caputo fractional and classical approach to variable velocity field was carried out by keeping other parameters constant. It is clear from the figure that the fluid velocity falls when increasing the values of bioconvection Rayleigh number Ra. This is due to the fact that, for larger values of Ra, the buoyancy effect from the transportation of microorganisms decreases. Moreover, fractional velocity exhibits a greater decline than the classical approach.

In Figure 4, the same behavior Pr can be observed for the velocity with integer and non-integer orders. From the figure, it is clear that velocity is a decreasing function of Pr.

Physically, Pr is the dimensionless number that tests the relative breadth of a boundary layer of thermal conductivity and momentum. With higher values of Pr, thermal conductivity is reduced, the viscosity of the fluid is enhanced and, finally, a decline in velocity is observed. It is also observed that the boundary layer breadth declines.

Figure 5 presents increasing values of the time and shows that velocity can be enhanced for longer periods of time. Figure 6 shows a comparison between Caputo and classical Grashof number (Gr) in the field variable velocity field, by keeping other parameters constant, and an opposite trend of Pr can be seen. It is observed that field variable fluid velocity can be enhanced for larger values of the Grashof number (Gr). These phenomena occur with increases in the velocity due to the increased thermal buoyancy effect. Figure 7 shows bioconvection when keeping other parameters constant and varying the values of the Caputo fractional parameter α, and it is observed that bioconvection shows a decline for larger values of α and the maximum decline can be attained for smaller values of α near the plate. It is also found that the momentum boundary layer thickness decreases.

Figure 8 shows the evaluation among Caputo fractional and classical bioconvection fields by keeping other parameters constant, and it is shown that the bioconvection field can be decreased with higher values of Lb. This is because Lb is reciprocal to mass diffusivity, so its large values decrease the bioconvection profile. Figure 9 is plotted for the validation of the present results. In the absence of bioconvection, the present result was reduced to the result obtained in [25] and they are clearly in good agreement. Further, in order to justify the present fractional modeling with a classical time derivative, we created a fractional model with artificial replacement of the integer order derivative with a non-integer order and a fractional model obtained with generalized constitutive laws and found very interesting results, which are presented in Figures 10–12. In the literature, many results are obtained with artificial replacement after making the governing equation dimensionless. Here, we present a comparison between the fractional modeling obtained with artificial replacement and generalized constitutive laws for temperature, bioconvection and velocity field. It is clear from the figures that artificial replacement exhibits memory, but not accurately, as the correct fractional approach through constitutive laws does. In addition, a correct mathematical model obtained with generalized laws shows a rapid decline in the boundary layers of the fluid properties.

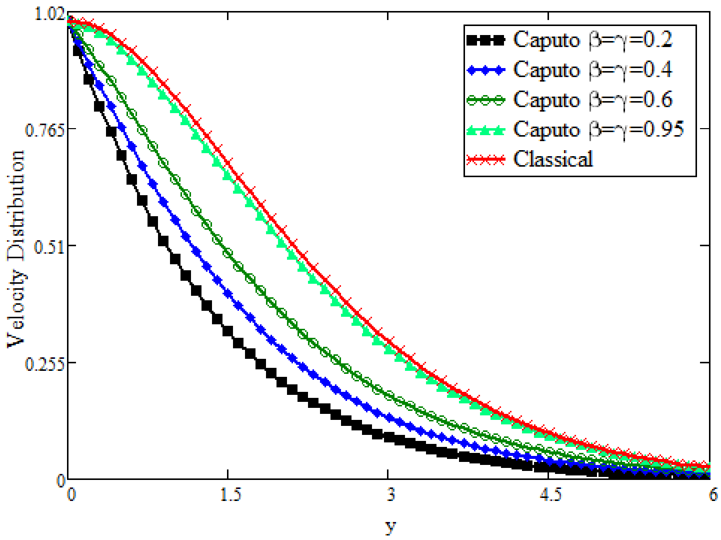

Figure 2. Comparative analysis of velocity distribution for different β and γ values.

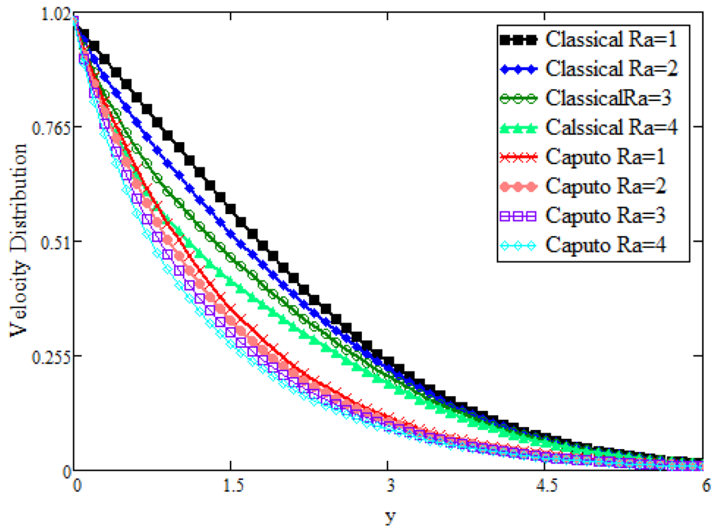

Figure 3. Comparative analysis of velocity distribution for different Ra values.

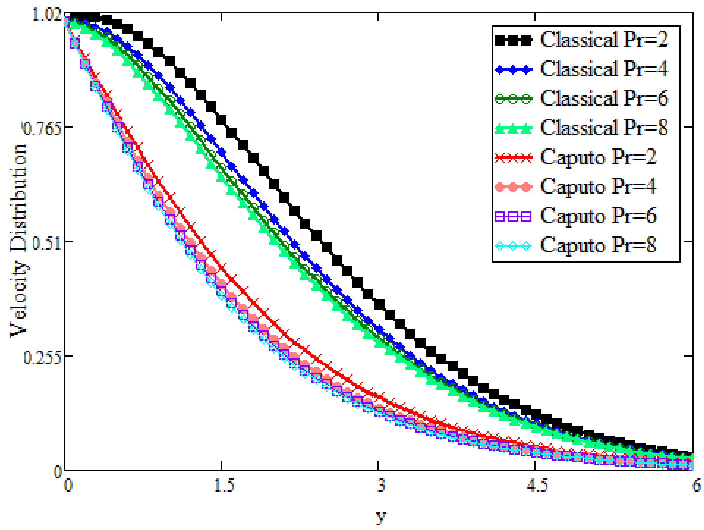

Figure 4. Comparative analysis of velocity distribution for different Pr values.

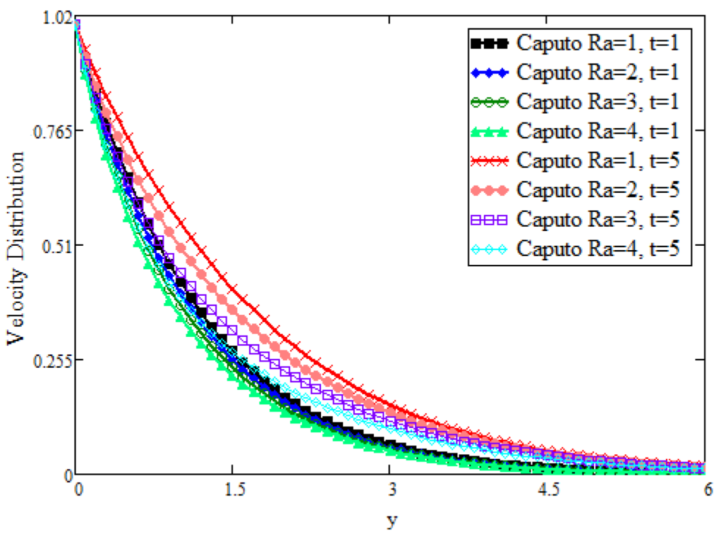

Figure 5. Comparative analysis of velocity distribution for different time values.

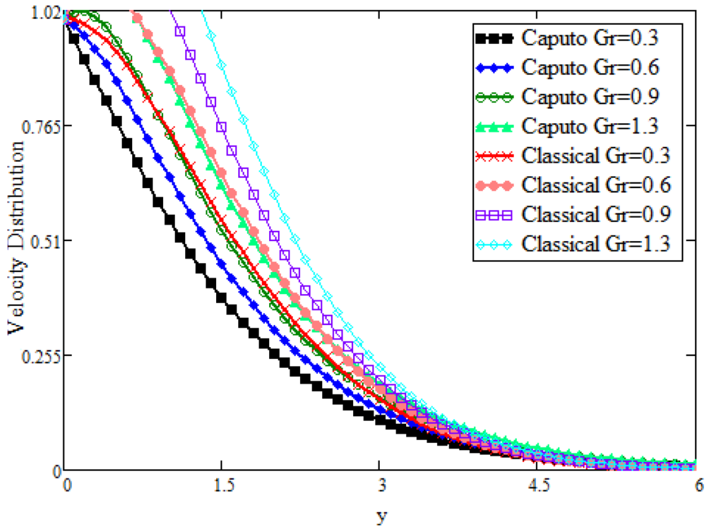

Figure 6. Comparative analysis of velocity distribution for different Gr values.

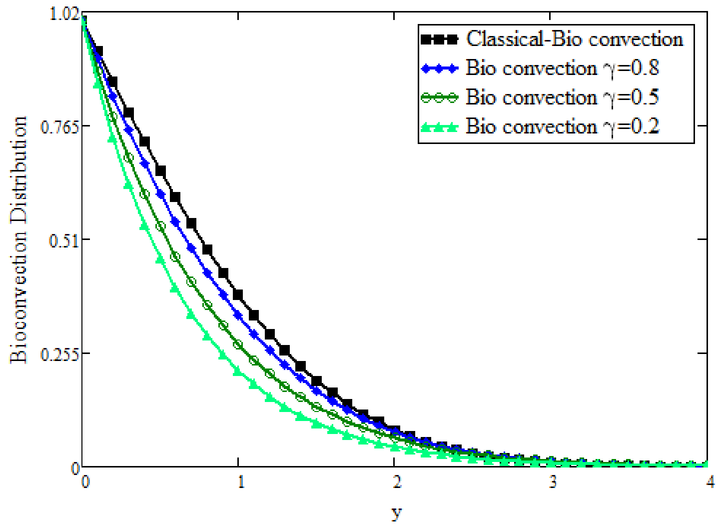

Figure 7. Comparative analysis of bioconvection distribution for different γ values.

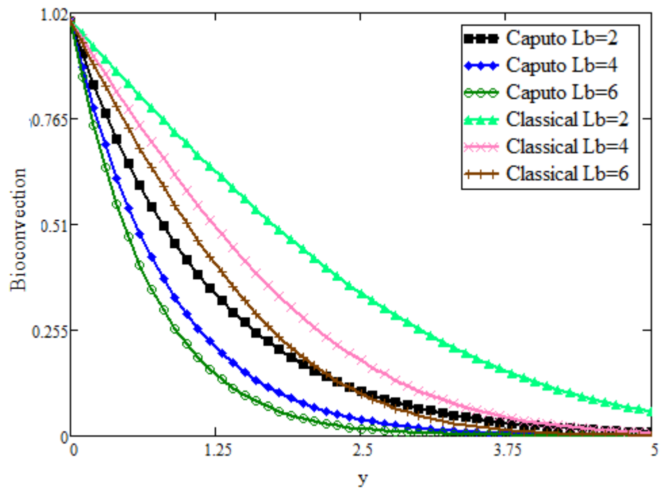

Figure 8. Comparative analysis of bioconvection distribution for different Lb values.

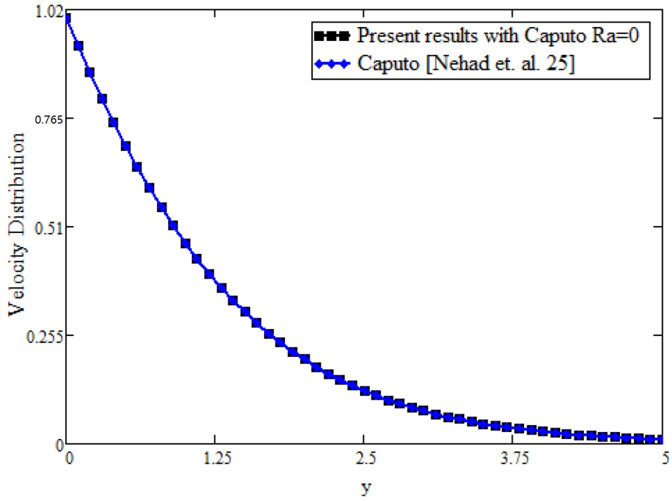

Figure 9. Comparison between present Results in the absence of bioconvection and Nehad el al. [25].

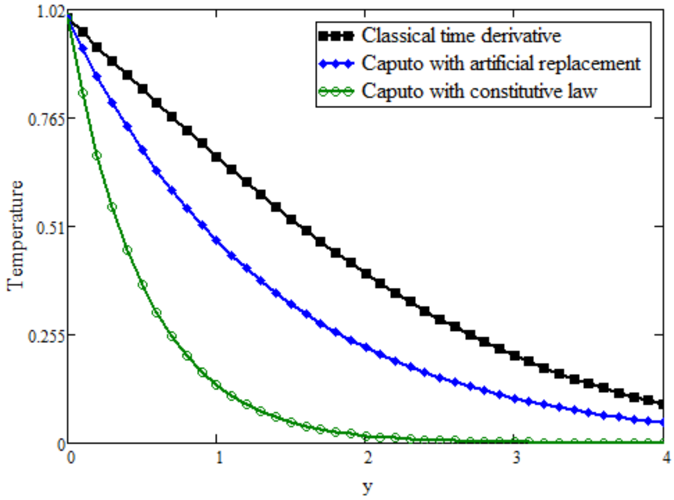

Figure 10. Comparison of temperature between classical time derivative, Caputo fractional model with artificial replacement and Caputo model with generalized constitutive law.

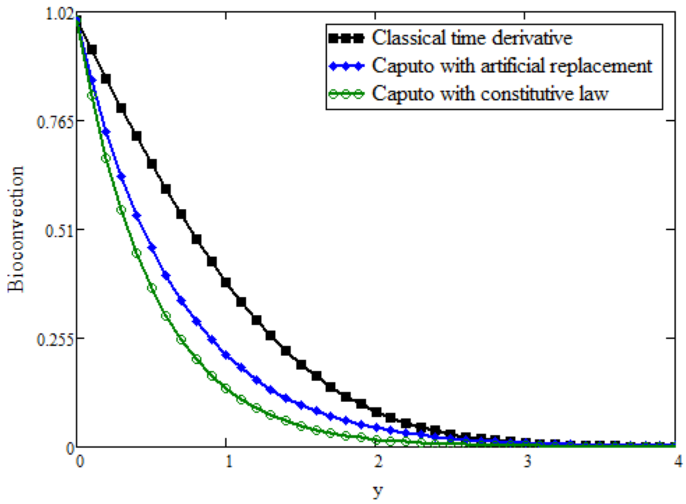

Figure 11. Comparison of bioconvection between classical time derivative, Caputo fractional model with artificial replacement and Caputo model with generalized constitutive law.

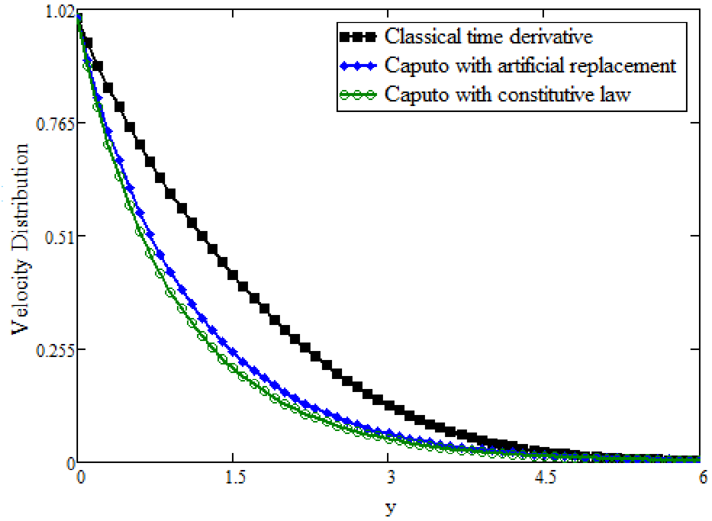

Figure 12. Comparison of velocity between classical time derivative, Caputo fractional model with artificial replacement and Caputo model with generalized constitutive law.

5. Conclusions

The present study deals with bioconvection with a heat transfer model with a Caputo fractional model. The exact solutions are obtained for the dimensionless governing equations with Laplace transform methods. Some physical impacts of flow parameters have been discussed through graphical illustrations. The major outcomes are the following:

- The fractional parameter can be used to control the boundary layers of the fluid properties like bioconvection, temperature and velocity.
- The fractional approach can be the best fit in some experimental work, where the needed and desired results can be achieved for different values of fractional parameters and time.
- The fractional model obtained with generalized constitutive laws gives better and more accurate results in terms of memory than the fractional approach with artificial replacement.
- The present results are compared with the existing literature in the absence of bioconvection and they are in good agreement.

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Abbreviations

The following abbreviations are used in this manuscript:

Symbol	Name
ρ	Fluid density
s	Laplace transform
μ	Viscosity
Pr	Prandtl number
g	Gravitational acceleration
Lb	Bioconvection Lewis number
T	Fluid temperature
Gr	Grashof number
T_w	Temperature at wall
Ra	Bioconvection Rayleigh number
T_∞	Ambient temperature of the fluid
erfc(.)	Gauss's error function of complimentary
β_T	Volumetric coefficient of thermal expansion
ϕ	The dimensionless nanoparticle volume fraction
ρ_m	Mass density
t	Time (s)
ρ_∞	Fluid density
N^*	Concentration of microorganisms
N	The dimensionless concentration of microorganisms
N_∞	The density of motile microorganisms
C_p	Specific heat at constant pressure
k	Thermal conductivity
D_N	Diffusivity of microorganisms
θ	Dimensionless temperature
u	Velocity

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