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Generalized Strongly Increasing Semigroups

E. R. García Barroso [1], J. I. García-García [2,*] and A. Vigneron-Tenorio [3]

[1] Departamento de Matemáticas, Estadística e I.O. Sección de Matemáticas, Universidad de La Laguna, Apartado de Correos 456, 38200 La Laguna, Spain; ergarcia@ull.es
[2] Department of Mathematics/INDESS (Instituto Universitario para el Desarrollo Social Sostenible), University of Cadiz, 11510 Puerto Real, Spain
[3] Department of Mathematics/INDESS (Instituto Universitario para el Desarrollo Social Sostenible), University of Cadiz, 11406 Jerez de la Frontera, Spain; alberto.vigneron@uca.es
* Correspondence: ignacio.garcia@uca.es

Abstract: In this work, we present a new class of numerical semigroups called GSI-semigroups. We see the relations between them and other families of semigroups and we explicitly give their set of gaps. Moreover, an algorithm to obtain all the GSI-semigroups up to a given Frobenius number is provided and the realization of positive integers as Frobenius numbers of GSI-semigroups is studied.

Keywords: generalized strongly increasing semigroup; strongly increasing semigroup; Frobenius number; Apéry set; singular analytic plane curve

MSC: 20M14; 14H20

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Citation: García Barroso, E.R.; García-García, J.I.; Vigneron-Tenorio, A. Generalized Strongly Increasing Semigroups. *Mathematics* **2021**, *9*, 1370. https://doi.org/10.3390/math9121370

Academic Editors: Alexei Kanel-Belov and Alexei Semenov

Received: 3 May 2021
Accepted: 11 June 2021
Published: 13 June 2021

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, ...\}$ be the set of non-negative integers. A numerical semigroup is a subset S of $\mathbb{N}$ closed under addition, $0 \in S$ and $\mathbb{N} \backslash S$, its gapset, is finite. The least non-zero element in S is called the multiplicity of S, we denote it by $m(S)$. Given a non-empty subset $A = \{a_1, ..., a_n\}$ of $\mathbb{N}$, we denote by $\langle A \rangle$ the smallest submonoid of $(\mathbb{N}, +)$ containing A; the submonoid $\langle A \rangle$ is equal to the set $\mathbb{N}a_1 + \cdots + \mathbb{N}a_n$. The minimal system of generators of S is the smallest subset of S generating it, and its cardinality, denoted by $\mathrm{e}(S)$, is known as the embedding dimension of S. It is well known (see Lemma 2.1 from [1]) that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. The cardinality of $\mathbb{N} \setminus S$ is called the genus of S (denoted by $g(S)$) and its maximum is known as the Frobenius number of S (denoted by $\mathrm{F}(S)$).

Numerical semigroups appear in several areas of mathematics and their theory is connected with Algebraic Geometry and Commutative Algebra (see [2,3]) as well as with Integer Optimization (see [4]) and Number Theory (see [5]). There are several studies on different families of numerical semigroups, for instance symmetric semigroups, irreducible semigroups and strongly increasing semigroups (see [6,7]) or on the study or characterization of invariants, for instance the Frobenius number, the Apéry set, the set of gaps, the genus, etc. (see [1,7–10]). It is an open problem to find a formula that gives the Frobenius number and algorithms to calculate it. Let S be a numerical semigroup, admitting as minimal system of generators the coprime positive integers $a_1, ..., a_n$. It is well-known after [11] that the Frobenius number $\mathrm{F}(S) = a_1a_2 - a_1 - a_2$ for $n = 2$. The computation of a similar formula when $n \geq 3$ is more difficult and it remains open, in fact it is a hard problem from the computational point of view (see [12]). On the other hand, given $m \in S \setminus \{0\}$, the Apéry set of m in S is $Ap(S, m) = \{s \in S \ : \ s - m \notin S\}$. Clearly, $Ap(S, m) = \{w(0) = 0, w(1), ..., w(m-1)\}$, where $w(i)$ is the least element of S congruent to i modulo m, for all $i \in \{0, ..., m-1\}$. Moreover, the set of gaps of S is the non-negative elements of $Ap(S, m) - m$, hence $\mathrm{F}(S) = \max(Ap(S, m)) - m$. As happens for the Frobenius number, given a numerical semigroup S, it is an open problem to determine explicit formulas of the Apéry set of an element $m \in S$, which can be expressed in terms of the minimal system of generators of S and of m itself. In fact, if there were, the problem of Frobenius would be solved in particular. It is still an open problem to give an explicit formula for the particular case of numerical semigroups, which are minimally generated by three elements. In this sense, we can find a positive response to some special families, as in [13] [Theorem 2.8] , where the authors determined the explicit formula for the Apéry set of the multiplicity of numerical semigroups whose minimal generators are three pairwise relatively prime numbers.

Inspired in [6,8], and with the aim to study the sets of gaps of strongly increasing semigroups (shorted by SI-semigroups), we introduce the concept of generalized strongly increasing semigroup (shorted by GSI-semigroups). These numerical semigroups $\bar{S} = \langle v_0, \ldots, v_h, \gamma\rangle$ are the gluing of a semigroup S with $\mathbb{N}$, we denote them by $S \oplus_{d,\gamma} \mathbb{N}$, where $S = \langle v_0/d, \ldots, v_h/d\rangle$, $d = \gcd(v_0, \ldots, v_h) > 1$ and $\gamma \in \mathbb{N}$ such that $\gamma > \max\{d\mathrm{F}(S), v_h\}$.

In [14], [Theorem 12] and [15] [Theorem 9.2, p. 125], we found the description of the Apéry set of an element in the gluing of two semigroups. Given the relationship between the Apéry set of an element of a numerical semigroup and the set of gaps of such a semigroup, the mentioned results in [14,15] could be used to describe the set of gaps of a gluing semigroup, but that description would not be explicit, since the description of the Apéry set in [14,15] are not, in the sense that they do not provide a list of the elements of the Apéry set and/or an explicit formula based on the minimal system of generators of the gluing semigroup, as it is done for example in [13] [Theorem 2.8].

Our main result is Theorem 2, where we describe the set of gaps of GSI-semigroups listing all its elements as explicit formulae based on the minimal system of generators of the semigroup. As a consequence, we determine the explicit list of elements of the Apéry set of the multiplicity of GSI-semigroups (see Corollary 4), following the spirit of [13] [Theorem 2.8] rather than that of [14] [Theorem 12] or [15] [Theorem 9.2, p. 125]. Since every SI-semigroup is a GSI-semigroup, our description of the gaps is also valid for SI-semigroups. Semigroups of values associated with plane branches are always SI-semigroups, and their sets of gaps describe topological invariants of the curves (see [6]), which are used to classify singular analytic plane curves. Due to the fact that the condition for being a GSI-semigroup is straightforward to check from a given system of generators, it is easy to construct subfamilies of GSI-semigroups and thus of SI-semigroups.

In this work we also compare the class of GSI-semigroups with other families of numerical semigroups obtained as gluing of numerical semigroups. These are the classes of telescopic, free and complete intersection numerical semigroups. In [8], the set of complete intersection numerical semigroups with given Frobenius number is constructed, and some special subfamilies as free and telescopic numerical semigroups, and numerical semigroups associated with irreducible singular plane curves are studied. As we pointed above, we prove that SI-semigroups are always GSI-semigroups. We also prove that GSI-semigroups are not included in the other three above-mentioned families.

Some algorithms for computing GSI-semigroups are provided in this work. One of them computes the set of GSI-semigroups up to a fixed Frobenius number. We prove that for any odd number f, there is at least one GSI-semigroup with Frobenius f, while this is not always the case for an even number f. Thus, GSI-semigroups with even Frobenius numbers are also studied, and we present an algorithm to check for a GSI-semigroup with a given even Frobenius number.

This work is organized as follows. In Section 2, we introduce the GSI-semigroups and some of their properties. We prove that SI-semigroups are GSI-semigroups (see Corollary 1). We also compare GSI-semigroups with other families such as free, telescopic and complete intersection numerical semigroups. In Section 3, the main result of this paper is presented (Theorem 2). This theorem gives us an explicit formula for the set of gaps of GSI-semigroups.

In Section 4, we give an algorithm for computing the set of GSI-semigroups up to a fixed Frobenius number, and we show some properties of Frobenius numbers of GSI-semigroups. We also provide an algorithm to test whether there is a GSI-semigroup with given even Frobenius number. In the last section, Section 5, some conclusions and possible future research are set out. Appendix A presents the code of the functions in GAP using for computing the examples in this work.

2. Generalized Strongly Increasing Semigroups

In this section, we introduce the family of GSI-semigroups and give some of their properties. We also compare this new family with other families of numerical semigroups already known.

The gluing of $S = \langle v_0, ..., v_h\rangle$ and $\mathbb{N}$ with respect to d and γ, with $\gcd(d, \gamma) = 1$ and $d, \gamma > 1$ (see [15] [Chapter 8]), is the numerical semigroup $\mathbb{N}dv_0 + \cdots + \mathbb{N}dv_h + \mathbb{N}\gamma$. We denote it by $S \oplus_{d,\gamma} \mathbb{N}$. In general, given S_1 and S_2, two numerical semigroups minimally generated by $\{v_0, ..., v_{r-1}\}$ and $\{v_r, ..., v_t\}$, respectively, $f \in S_1 \setminus \{v_0, ..., v_{r-1}\}$ and $d \in S_2 \setminus \{v_r, ..., v_t\}$ such that $\gcd(f, d) = 1$, the numerical semigroup minimally generated by $\{dv_0, ..., dv_{r-1}, fv_r, ..., fv_t\}$ is called a gluing of S_1 and S_2.

Definition 1. *A numerical semigroup $\bar{S}$ is a generalized strongly increasing semigroup whenever $\bar{S}$ is the gluing of a numerical semigroup $S = \langle v_0, ..., v_h\rangle$ with respect to d and γ (that is, $\bar{S} = S \oplus_{d,\gamma} \mathbb{N}$), where $d \in \mathbb{N} \setminus \{0, 1\}$ and $\gamma \in \mathbb{N}$ with $\gamma > \max\{d\mathrm{F}(S), d\max\{v_0, ..., v_h\}\}$ (note that d and γ are coprimes).*

Since the gluing does not depend on the order of the generators of the semigroup, the definition of GSI-semigroup is also independent of this order.

The first example of GSI-semigroups are numerical semigroups generated by two positive integers $\bar{S} = \langle a, b\rangle$ with $a < b$. For these semigroups, set $S = \mathbb{N} = \langle 1\rangle$, $d = a$, and $\gamma = b$. Since $\mathrm{F}(S) = -1$, $\gamma = b > \max\{d\mathrm{F}(S), d\} = \{a \cdot (-1), a\} = a$. By a result of Sylvester (see [11]), we know that the Frobenius numbers of these semigroups are given by the formula $a \cdot b - a - b$. Hence, every odd natural number is realizable as a Frobenius number of a GSI-semigroup.

Our definition of GSI-semigroup is motivated by the already established notion of a SI-semigroup. We remind you how they are defined (see [16] for further details).

A sequence of positive integers $(v_0, ..., v_h)$ is called a characteristic sequence if it satisfies the following two properties:

(CS1) Let $e_k = \gcd(v_0, ..., v_k)$ for $0 \leq k \leq h$. Then $e_k < e_{k-1}$ for $1 \leq k \leq h$ and $e_h = 1$.
(CS2) $e_{k-1}v_k < e_k v_{k+1}$ for $1 \leq k \leq h - 1$.

We put $n_k = \frac{e_{k-1}}{e_k}$ for $1 \leq k \leq h$. Therefore, $n_k > 1$ for $1 \leq k \leq h$ and $n_h = e_{h-1}$. If $h = 0$, the only characteristic sequence is $(v_0) = (1)$. If $h = 1$, the sequence (v_0, v_1) is a characteristic sequence if and only if $\gcd(v_0, v_1) = 1$. Property (CS2) plays a role if and only if $h \geq 2$.

Lemma 1. *(Ref. [16] [Lemma 1.1]) Let $(v_0, ..., v_h)$, $h \geq 2$ be a characteristic sequence. Then,*

(i) *$v_1 < \cdots < v_h$ and $v_0 < v_2$.*
(ii) *Let $v_1 < v_0$. If $v_0 \not\equiv 0 \pmod{v_1}$ then $(v_1, v_0, v_2, ..., v_h)$ is a characteristic sequence. If $v_0 \equiv 0 \pmod{v_1}$ then $(v_1, v_2, ..., v_h)$ is a characteristic sequence.*

We denote by $\langle v_0, ..., v_h\rangle$ the semigroup generated by the characteristic sequence $(v_0, ..., v_h)$. Observe that $\langle v_0, ..., v_h\rangle$ is a numerical semigroup. A semigroup $S \subseteq \mathbb{N}$ is Strongly Increasing (SI-semigroup) if $S \neq \{0\}$ and it is generated by a characteristic sequence. Note that by Lemma 1, we can assume that $v_0 < \cdots < v_h$.

Theorem 1. *Let $\bar{S}$ be a numerical semigroup with $\mathrm{e}(\bar{S}) = h + 1$. Then, $\bar{S}$ is strongly increasing if and only if one of the two next conditions holds:*

1. *$h = 1$, $\bar{S} = \mathbb{N} \oplus_{d,\gamma} \mathbb{N} = \langle d, \gamma \rangle$, where d and γ are two coprime integers.*
2. *$h > 1$, $\bar{S} = S \oplus_{d,\gamma} \mathbb{N}$, where $S = \langle v_0, ..., v_{h-1} \rangle$ is a strongly increasing semigroup with embedding dimension h and $\gamma, d > 1$ are two coprime integer numbers such that $\gamma > d \gcd(v_0, ..., v_{h-2})v_{h-1}$.*

Proof. The case $h = 1$ is trivial by definition of characteristic sequences.

Assume $h > 1$ and that $\bar{S} = \langle \bar{v}_0, ..., \bar{v}_h \rangle$ is a strongly increasing numerical semigroup with embedding dimension strictly greater than 2. Let $\bar{e}_i = \gcd(\bar{v}_0, ..., \bar{v}_i)$ for $0 \le i \le h$. Put $v_i = \frac{\bar{v}_i}{\bar{e}_{h-1}}$ for $0 \le i \le h-1$. Then, $(v_0, ..., v_{h-1})$ is a characteristic sequence. Let $S = \langle v_0, ..., v_{h-1} \rangle$. Since $\mathrm{e}(\bar{S}) = h + 1$, then $\mathrm{e}(S) = h$. Set $\gamma = \bar{v}_h$ and $d = \bar{e}_{h-1}$, we get $\bar{S} = S \oplus_{d,\gamma} \mathbb{N}$. We have that $\gamma = \bar{v}_h > \bar{v}_{h-1} = dv_{h-1}$, and since $\bar{S}$ is a SI-semigroup,

$$\begin{aligned}
\gamma = \bar{v}_h &> \frac{\bar{e}_{h-2}}{\bar{e}_{h-1}}\bar{v}_{h-1} = \frac{\gcd(\bar{v}_0, ..., \bar{v}_{h-2})}{\bar{e}_{h-1}}\bar{v}_{h-1} \\
&= \frac{\gcd(\bar{e}_{h-1}v_0, ..., \bar{e}_{h-1}v_{h-2})}{\bar{e}_{h-1}}\bar{e}_{h-1}v_{h-1} \\
&= \bar{e}_{h-1}\gcd(v_0, ..., v_{h-2})v_{h-1} \\
&= d\gcd(v_0, ..., v_{h-2})v_{h-1}.
\end{aligned}$$

Conversely, let $S = \langle v_0, ..., v_{h-1} \rangle$ be a strongly increasing semigroup with embedding dimension h, and $\gamma, d > 1$ be two coprime integer numbers such that $\gamma > d \gcd(v_0, ..., v_{h-2})v_{h-1}$. Denote $e_i = \gcd(v_0, ..., v_i)$ for $i = 0, ..., h-1$. Take $\bar{S} = \langle \bar{v}_0, ..., \bar{v}_h = \gamma \rangle$ the gluing semigroup $S \oplus_{d,\gamma} \mathbb{N}$ and define $\bar{e}_i = \gcd(\bar{v}_0, ..., \bar{v}_i)$ for $i = 0, ..., h$. We have that $\bar{e}_0 = de_0 > \cdots > \bar{e}_{h-1} = de_{h-1} = d > \bar{e}_h = \gcd(\gamma, d) = 1$. Since $e_{i-1}v_i < e_iv_{i+1}$, for $1 \le i \le h-1$ then $e_{i-1}v_id^2 < e_iv_{i+1}d^2$ and therefore $\bar{e}_{i-1}\bar{v}_i < \bar{e}_i\bar{v}_{i+1}$. By hypothesis $de_{h-2}v_{h-1} < \gamma$, hence $d^2e_{h-2}v_{h-1} < d\gamma$ and therefore $\bar{e}_{h-2}\bar{v}_{h-1} < \bar{e}_{h-1}\gamma$. We conclude that $\bar{S}$ is a SI-semigroup. □

The following result gives us a formula for the conductor (the Frobenius number plus 1) of a SI-semigroup.

Proposition 1. *(Ref. [6] [Proposition 2.3 (4)], [16] [Proposition 1.2]) Let $S = \langle v_0, ..., v_h \rangle$ be the semigroup generated by the characteristic sequence $(v_0, ..., v_h)$. The conductor of the semigroup S is*

$$c(S) = \sum_{i=1}^{h}(n_i - 1)v_i - v_0 + 1.$$

Moreover, the conductor of S is an even number and the genus of S is $g(S) = \frac{c(S)}{2}$.

By Proposition 1, we get

$$\begin{aligned}
\mathrm{F}(S) &= \sum_{i=1}^{h}(n_i - 1)v_i - v_0 = \sum_{i=1}^{h} n_iv_i - \sum_{i=1}^{h} v_i - v_0 \\
&= (n_1v_1 - v_2) + (n_2v_2 - v_3) + \cdots \\
&\quad +(n_{h-1}v_{h-1} - v_h) + n_hv_h - v_1 - v_0 \\
&\le -(h-1) + n_hv_h - v_1 - v_0 \\
&= e_{h-1}v_h - v_0 - v_1 - h + 1 < e_{h-1}v_h - v_0 - v_1.
\end{aligned}$$

Assume $\bar{S} = \langle \bar{v}_0, ..., \bar{v}_h \rangle$ is a SI-semigroup satisfying that $\bar{v}_0 < ... < \bar{v}_h = \gamma$. Set $d = \bar{e}_{h-1} = \gcd(\bar{v}_0, ..., \bar{v}_{h-1})$, $\gamma = \bar{v}_h$ and $S = \langle \bar{v}_0/d, ..., \bar{v}_{h-1}/d \rangle$. We have that $d\gamma = \bar{e}_{h-1}\bar{v}_h > \bar{e}_{h-1}\bar{v}_h - \bar{v}_0 - \bar{v}_1 > \mathrm{F}(\bar{S}) = \sum_{i=1}^{h}(\bar{n}_i - 1)\bar{v}_i - \bar{v}_0 = d\mathrm{F}(S) + (\bar{e}_{h-1} - 1)\bar{v}_h = d\mathrm{F}(S) + (d-1)\gamma$. Thus, $\gamma > d\mathrm{F}(S)$. Since $\bar{S}$ is a SI-semigroup, the property (CS2) is fulfilled. Using the ordered generators, we obtain $\bar{v}_h < \bar{e}_{h-1}\bar{v}_h = d\bar{v}_h < \gamma$.

So we can state the following result.

Corollary 1. *Every SI-semigroup is a GSI-semigroup.*

There are some semigroups with similar definitions to SI and GSI semigroups; for example, telescopic, free and complete intersection semigroups. All of them are defined using the gluings of different types of numerical semigroups.

Let S be a numerical semigroup and let $\{v_0, ..., v_h\}$ be its minimal set of generators. Put $e_0 = v_0$ and for $k \in \{1, ..., h\}$, set $e_k = \gcd(v_0, ..., v_k)$ and $n_k = \frac{e_{k-1}}{e_k}$. The semigroup S is free for the arrangement $(v_0, ..., v_h)$ if for all $k \in \{1, ..., h\}$, $n_k > 1$ and $n_k v_k \in \langle v_0, ..., v_{k-1} \rangle$ (see [5] [page 30]). In terms of gluings, a semigroup S is free whenever it is equal to $\mathbb{N}$ or it is the gluing of a free semigroup with $\mathbb{N}$ (see [15] [Chapter 8]). The semigroup S is telescopic if it is free for the rearrangement $v_0 < \cdots < v_h$. A semigroup S is complete intersection if its associated algebra is complete intersection (see [17]). In terms of gluings, a semigroup is complete intersection if it is equal to $\mathbb{N}$ or it is the gluing of two complete intersection numerical semigroups (see [18] [Proposition 9]).

It is well known that SI-semigroups are telescopic, telescopic are free semigroups and free semigroups are complete intersection (see [19] [Figure 1]). In general, GSI-semigroups are neither strongly increasing nor telescopic nor free nor complete intersection. Clearly, $\langle 6, 14, 22, 23 \rangle = \langle 3, 7, 11 \rangle \oplus_{2,23} \mathbb{N}$ and $23 > \max\{2F(\langle 3, 7, 11 \rangle), 2 \cdot 11\}$. Thus, this is a GSI-semigroup. We define the functions `IsSIncreasingNumericalSemigroup` and `IsGSI` to check if a numerical semigroup is a SI-semigroup and a GSI-semigroup, respectively, (the code of these functions is showed in Appendix A).

Applying our functions and the functions `IsFreeNumericalSemigroup`, `IsTelescopicNumericalSemigroup` and `IsCompleteIntersection` of [20] to the semigroup $\langle 6, 14, 22, 23 \rangle$, we obtain the following outputs:

```
gap> IsFreeNumericalSemigroup(
        NumericalSemigroup(6,14,22,23));
false
gap> IsTelescopicNumericalSemigroup(
        NumericalSemigroup(6,14,22,23));
false
gap> IsCompleteIntersection(
        NumericalSemigroup(6,14,22,23));
false
gap> IsSIncreasing(NumericalSemigroup(6,14,22,23));
false
gap> IsGSI(NumericalSemigroup(6,14,22,23));
true
```

From the results of the above computations, we conclude that the class of GSI-semigroups contains the class of SI-semigroups, but it is different to the classes of free, telescopic and complete intersection semigroups.

3. Set of Gaps of a GSI-Semigroup

In this section, we state and prove the main theorem of this article. We have seen that GSI-semigroups are easy to obtain from any numerical semigroup just by gluing it with $\mathbb{N}$ with appropriate elements d and γ. Hence, these semigroups form a large family within the set of numerical semigroups. Now, we deepen their study by explicitly determining their set of gaps.

Hereafter, the notation $[a \bmod n]$ for an integer a and a natural number n means the remainder of the division of a by n, and $[a]_n$ denotes the co-set of a modulo n. For any two real numbers $a \leq b$, we denote by $[a, b]_{\mathbb{N}}$ the set of natural numbers belonging to the real interval $[a, b]$. Let $\lfloor a \rfloor$ be the integral part of the real number a.

Theorem 2. *Let $S = \langle v_0, ..., v_h \rangle$ be a numerical semigroup with $v_0 < \cdots < v_h$, $d \geq 2$ and v_{h+1} as two natural co-prime numbers such that $v_{h+1} > \max\{d\mathrm{F}(S), dv_h\}$. Then, the gaps of the GSI-semigroup $\bar{S} = S \oplus_{d,v_{h+1}} \mathbb{N}$ are*

$$\mathbb{N} \setminus \bar{S} = \{1, ..., dv_0 - 1\} \cup \{x \in (dv_0, v_{h+1}) \cap \mathbb{N} : x \notin dS\} \cup \mathcal{A}_d \cup \bigcup_{\ell=1}^{d-2} \mathcal{B}_{d,\ell} \tag{1}$$

where

$$\mathcal{B}_{d,\ell} = \left\{ v_{h+1} + [\ell v_{h+1} \bmod d] + kd : 0 \leq k \leq \left\lfloor \frac{\ell v_{h+1}}{d} \right\rfloor - 1 \right\}$$

and

$$\mathcal{A}_d = \bigcup_{k=1}^{d-1} \left(d(\mathbb{N} \setminus S) + kv_{h+1} \right) \; (\mathcal{A}_d = \emptyset \text{ when } S = \mathbb{N}).$$

Moreover, the sets on the right-hand side of (1) are pairwise disjoint.

Proof. Let

$$\mathcal{H} := \{1, ..., dv_0 - 1\} \cup \{x \in (dv_0, v_{h+1}) \cap \mathbb{N} \mid x \notin dS\} \cup \mathcal{A}_d \cup \bigcup_{\ell=1}^{d-2} \mathcal{B}_{d,\ell}.$$

First, we will prove the inclusion $\mathcal{H} \subseteq \mathbb{N} \setminus \bar{S}$. It is clear that $\{1, ..., dv_0 - 1\}$ is included in $\mathbb{N} \setminus \bar{S}$.

Consider $x \in (dv_0, v_{h+1}) \cap \mathbb{N}$ such that $x \notin dS$, and suppose that $x \in \bar{S}$. Since $x < v_{h+1}$, then there are λ_i with $0 \leq i \leq h$ such that $x = \lambda_0 dv_0 + \cdots + \lambda_h dv_h = d(\lambda_0 v_0 + \cdots + \lambda_h v_h) \in dS$, which is a contradiction. Hence, we conclude that $\{x \in (dv_0, v_{h+1}) \cap \mathbb{N} : x \notin dS\} \subseteq \mathbb{N} \setminus \bar{S}$.

Suppose that $S \neq \mathbb{N}$. Fix $1 \leq k \leq d-1$ and let $x \in d(\mathbb{N} \setminus S) + kv_{h+1}$. We get $x = d\alpha + kv_{h+1}$, for some $\alpha \in \mathbb{N} \setminus S$. Suppose that $x \in \bar{S}$. So, there exist $\alpha_1, ..., \alpha_h, \beta \in \mathbb{N}$ such that $x = d\alpha + kv_{h+1} = d\alpha_1 v_1 + \cdots + d\alpha_h v_h + \beta v_{h+1}$ and then $(k - \beta)v_{h+1} = d(\alpha_1 v_1 + \cdots + \alpha_h v_h - \alpha)$. If $k = \beta$, the element α has to belong to S, which is not possible. Moreover, since d and v_{h+1} are co-prime, d divides $k - \beta$ when $k - \beta \neq 0$. If $\beta > k$, $d\alpha = d(\alpha_1 v_1 + \cdots + \alpha_h v_h) + (\beta - k)v_{h+1}$ with $v_{h+1} \in S$, that is, $\alpha \in S$. Again, it is not possible. We may therefore assume $k > \beta$, then $k - \beta \geq d$ and $k \geq d$. In any case, the set $d(\mathbb{N} \setminus S) + kv_{h+1}$ is included in $\mathbb{N} \setminus \bar{S}$ for any integer k in $[1, d-1]_{\mathbb{N}}$.

Let us prove now that $\mathcal{B}_{d,\ell} \subseteq \mathbb{N} \setminus \bar{S}$. Suppose that $x = v_{h+1} + [\ell v_{h+1} \bmod d] + kd \in \bar{S}$ for some $1 \leq \ell \leq d-2$ and $0 \leq k \leq \left\lfloor \frac{\ell v_{h+1}}{d} \right\rfloor - 1$. Let $\alpha_0, \alpha_1, ..., \alpha_{h+1} \in \mathbb{N}$ such that $x = v_{h+1} + [\ell v_{h+1} \bmod d] + kd = \alpha_0 dv_0 + \cdots + \alpha_h dv_h + \alpha_{h+1} v_{h+1}$. Hence, $(\alpha_{h+1} - 1)v_{h+1} - [\ell v_{h+1} \bmod d] = d(k - \alpha_0 v_0 - \cdots - \alpha_h v_h)$ and $[(\alpha_{h+1} - 1 - \ell)v_{h+1}]_d = [0]_d$. Since d and v_{h+1} are co-prime, then d divides $\alpha_{h+1} - 1 - \ell$. However, $\max \mathcal{B}_{d,\ell} = (\ell + 1)v_{h+1} - d$ so we have $\alpha_{h+1} \in \{0, 1, ..., \ell\}$, hence $-1 - \ell \leq \alpha_{h+1} - 1 - \ell \leq -1$ or equivalently $1 + \ell \geq -\alpha_{h+1} + 1 + \ell \geq 1$ and $-\alpha_{h+1} + 1 + \ell$ is a multiple of d, which is a contradiction since $\ell < d - 1$.

Taking into account the reasoning done so far we have

$$\mathcal{H} = \{1, ..., dv_0 - 1\} \cup \{x \in (dv_0, v_{h+1}) \cap \mathbb{N} \mid x \notin dS\} \cup \mathcal{A}_d \cup \bigcup_{\ell=1}^{d-2} \mathcal{B}_{d,\ell} \subseteq \mathbb{N} \setminus \bar{S}.$$

Now, let us prove that the sets on the right-hand side of (1) are pairwise disjoint.

When $\mathcal{A}_d$ is a non-empty set, let $\mathcal{A}_{d,k} = d(\mathbb{N} \setminus S) + kv_{h+1}$ for any fixed $1 \leq k \leq d-1$. In this case, if $\mathcal{B}_{d,\ell}$ is non-empty we have

$$\max \mathcal{B}_{d,\ell} < \min \mathcal{A}_{d,\ell+1} \;\text{ for } 1 \leq \ell \leq d-2. \tag{2}$$

Observe that

$$[x]_d = [kv_{h+1}]_d \text{ for any } x \in \mathcal{A}_{d,k} \tag{3}$$

and

$$[y]_d = [(\ell+1)v_{h+1}]_d \text{ for any } y \in \mathcal{B}_{d,\ell}. \tag{4}$$

Since $1 \leq k, \ell < d$ we get that any two sets $\mathcal{A}_{d,k}$ and $\mathcal{A}_{d,k'}$ are disjoint for $k \neq k'$ and any two sets $\mathcal{B}_{d,\ell}$ and $\mathcal{B}_{d,\ell'}$ are also disjoint for $\ell \neq \ell'$. Moreover, $\mathcal{A}_d$ and $\mathcal{B}_{d,\ell}$ are also disjoint for any $1 \leq \ell \leq d-2$. Indeed, let $x \in \mathcal{A}_d \cap \mathcal{B}_{d,\ell}$ for some $1 \leq \ell \leq d-2$. Hence, there is $k \in \{1, ..., d-1\}$ such that $x \in \mathcal{A}_{d,k}$ and by (3) and (4), $[x]_d = [kv_{h+1}]_d = [(\ell+1)v_{h+1}]_d$. Given that d and v_{h+1} are co-prime and $1 \leq k, \ell < d$, we get $k = \ell + 1$. So, $x \in \mathcal{A}_{d,\ell+1} \cap \mathcal{B}_{d,\ell}$, which is a contradiction by inequality (2).

In order to finish the proof, we will show that there is no gap of $\bar{S}$ outside $\mathcal{H}$.

First at all, observe that if $x \in \mathbb{N} \setminus \bar{S}$ and $x < v_{h+1}$, $x \in \{1, ..., dv_0 - 1\} \bigcup \{x \in (dv_0, v_{h+1}) \cap \mathbb{N} \mid x \notin dS\}$.

Claim 1: if $x \in \mathbb{N} \setminus \bar{S}$ and $v_{h+1} < x$, then $[x]_d = [kv_{h+1}]_d$, for some $k \in \{1, ..., d-1\}$.

Indeed, if we suppose that $x = \lambda d$ for some $\lambda \in \mathbb{N}$, by hypothesis we get $d\mathrm{F}(S) < v_{h+1} < x = \lambda d$, in particular $\lambda > \mathrm{F}(S)$, so $x \in dS \subset \bar{S}$. Since $[x]_d \neq [0]_d$ and $\gcd(d, v_{h+1}) = 1$ we get $[x]_d \in \{[1]_d, ..., [d-1]_d\} = \{[kv_{h+1}]_d \ : \ 1 \leq k \leq d-1\}$, that is, any $x \in \mathbb{N} \setminus \bar{S}$ with $v_{h+1} < x$ is congruent with kv_{h+1} modulo d for some integer $k \in \{1, ..., d-1\}$.

We distinguish two cases, depending on $\mathcal{A}_d$. First, we suppose that $\mathcal{A}_d \neq \emptyset$.

Claim 2: The greatest gap of $\bar{S}$, which is congruent with kv_{h+1} modulo d is $\max \mathcal{A}_{d,k}$, for $1 \leq k \leq d-1$.

Let $x \in \mathbb{N} \setminus \bar{S}$ with $[x]_d = [kv_{h+1}]_d$ and $x > \max \mathcal{A}_{d,k}$, then $x = d\mathrm{F}(S) + kv_{h+1} + \lambda d$ for some non-zero natural number λ. So, $x = d(\mathrm{F}(S) + \lambda) + kv_{h+1} \in \bar{S}$ since $\mathrm{F}(S) + \lambda \in S$.

Claim 3: There are no gaps of $\bar{S}$ congruent with $(\ell+1)v_{h+1}$ modulo d between $\max \mathcal{B}_{d,\ell}$ and $\min A_{d,\ell+1}$.

Remember that $[\max \mathcal{B}_{d,\ell}]_d = [\min A_{d,\ell+1}]_d = [(\ell+1)v_{h+1}]_d$. Suppose that $x \in \mathbb{N} \setminus \bar{S}$ with $\max \mathcal{B}_{d,\ell} < x < \min A_{d,\ell+1}$ and $[x]_d = [(\ell+1)v_{h+1}]_d$. Since $\max \mathcal{B}_{d,\ell} = (\ell+1)v_{h+1} - d$ and $\min A_{d,\ell+1} = (\ell+1)v_{h+1} + d$, the only possibility for x is $(\ell+1)v_{h+1}$, which is an element of $\bar{S}$.

By Claims 1 and 2, we deduce that for any $x \in \mathbb{N} \setminus \bar{S}$ with $v_{h+1} < x$ there exists an integer $k_0 \in \{1, ..., d-1\}$ such that $[x]_d = [k_0 v_{h+1}]_d$ and $v_{h+1} < x \leq \max \mathcal{A}_{d,k_0}$. In particular, there is an integer λ such that $x = k_0 v_{h+1} + \lambda d$. Hence, if $x \in [\min \mathcal{A}_{d,k_0}, \max \mathcal{A}_{d,k_0}]_{\mathbb{N}}$ then $x \in \mathcal{A}_{d,k_0}$. Indeed, in this case $\lambda \in \mathbb{N}$ and $\lambda \notin S$, otherwise $x \in \bar{S}$.

By Claim 3, we can conclude that if $v_{h+1} < x < \min \mathcal{A}_{d,k_0}$ then $v_{h+1} < x \leq \max \mathcal{B}_{d,k_0-1} = k_0 v_{h+1} - d$.

Claim 4: The set of all the integers in $(v_{h+1}, \max \mathcal{B}_{d,k_0-1}]$ congruent with $k_0 v_{h+1}$ modulo d is $\mathcal{B}_{d,k_0-1}$.

By (4) we have $[\max \mathcal{B}_{d,k_0-1}]_d = [k_0 v_{h+1}]_d$. Moreover

$$\{\max \mathcal{B}_{d,k_0-1}, \max \mathcal{B}_{d,k_0-1} - d, \max \mathcal{B}_{d,k_0-1} - 2d, ..., \min \mathcal{B}_{d,k_0-1}\} = \mathcal{B}_{d,k_0-1}$$

and $\min \mathcal{B}_{d,k_0-1} - d = v_{h+1} + [(k_0-1)v_{h+1} \mod d] - d < v_{h+1}$.

Hence, x has to belong to $\mathcal{B}_{d,k_0-1}$ and we are done with the proof for the case $\mathcal{A}_d \neq \emptyset$.

Suppose now that $\mathcal{A}_d = \emptyset$, that is $S = \mathbb{N}$ and $\bar{S}$ is generated by d and v_1 ($h = 0$).

Claim 5: If $\mathcal{A}_d = \emptyset$, then $\max \mathcal{B}_{d,\ell}$ is the greatest gap of $\bar{S}$, which is congruent with $(\ell+1)v_{h+1}$ modulo d, for $1 \leq \ell \leq d-2$.

Observe that $\max \mathcal{B}_{d,\ell} = (\ell+1)v_1 - d$. For any natural number $x > \max \mathcal{B}_{d,\ell}$ with $[x]_d = [(\ell+1)v_1]_d$, there is $\alpha \in \mathbb{N} \setminus \{0\}$ such that $x = (\ell+1)v_1 - d + \alpha d = (\ell+1)v_1 + (\alpha-1)d \in \bar{S}$. □

The above result provides us an explicit formula for the gaps, except the elements of $\mathcal{A}_d$. We now give some examples of GSI-semigroups where the set $\mathcal{A}_d$ is easily known.

Example 1. *Let $S = \langle 2,7 \rangle$. We have $\mathbb{N} \setminus S = \{1,3,5\}$ and $\mathrm{F}(S) = 5$. Take now $d = 2$ and $\gamma = 15$. Since $\gamma > \max\{2 \cdot 5, 2 \cdot 7\}$, the semigroup $S \oplus_{2,15} \mathbb{N}$ is a GSI-semigroup. The set $\mathcal{A}_2$ is equal to $2\{1,3,5\} + 1 \cdot 15 = \{17,21,25\}$ and therefore $\mathrm{F}(S \oplus_{2,15} \mathbb{N}) = 25$.*

Example 2. *Consider now the semigroup $S = \langle 5,6,7,8,9 \rangle$, $d = 3$ and $\gamma = 31$. We have $\mathbb{N} \setminus S = \{1,2,3,4\}$ and $\mathrm{F}(S) = 4$. Since $\gamma > \max\{3 \cdot 4, 3 \cdot 9\}$, the semigroup $S \oplus_{3,31} \mathbb{N}$ is a GSI-semigroup and $\mathcal{A}_3 = (3\{1,2,3,4\} + 1 \cdot 31) \cup (3\{1,2,3,4\} + 2 \cdot 31) = \{34,37,40,43,65,68,71,74\}$. Thus, $\mathrm{F}(S \oplus_{3,31} \mathbb{N}) = 74$.*

Corollary 2. *Let $S = \langle v_0, ..., v_h \rangle$ be a semigroup, and $d, v_{h+1} \in \mathbb{N}$ two natural numbers such that $\bar{S} = \langle dv_0, ..., dv_h, v_{h+1} \rangle$ is a GSI-semigroup. Then,*

$$\mathrm{F}(\bar{S}) = \begin{cases} \max \mathcal{A}_d & \text{if } \mathcal{A}_d \neq \emptyset \\ \max \mathcal{B}_{d,d-2} & \text{otherwise,} \end{cases}$$

where $\mathcal{A}_d$ and $\mathcal{B}_{d,d-2}$ are from (1).

Proof. If $\mathcal{A}_d \neq \emptyset$, then, by inequality (2) , $\mathrm{F}(\bar{S}) = \max \mathcal{A}_d = d\mathrm{F}(S) + (d-1)v_{h+1}$. Otherwise, $S = \mathbb{N}$ and $\bar{S}$ is generated by d and v_1 ($h = 0$). So, $\mathrm{F}(\bar{S}) = (d-1)(v_1 - 1) - 1 = \max \mathcal{B}_{d,d-2}$. □

Corollary 3. *The Frobenius number of a GSI-semigroup $S \oplus_{d,\gamma} \mathbb{N}$ is*

$$\mathrm{F}(S \oplus_{d,\gamma} \mathbb{N}) = d\mathrm{F}(S) + (d-1)\gamma. \tag{5}$$

Proof. It is a consequence of the proof of Theorem 2. □

From the proof of Theorem 2, we obtain the Frobenius number of a GSI-semigroup $S \oplus_{d,\gamma} \mathbb{N}$, which is equal to

$$\mathrm{F}(S \oplus_{d,\gamma} \mathbb{N}) = d\mathrm{F}(S) + (d-1)\gamma. \tag{6}$$

Corollary 4. *Let $S = \langle v_0, ..., v_h \rangle$ be a numerical semigroup with $v_0 < ... < v_h$ and $d \geq 2$ and v_{h+1} be two natural coprime numbers such that $v_{h+1} > \max\{d\mathrm{F}(S), dv_h\}$. Then the Apéry set of v_0 in the GSI-semigroup $\bar{S} = S \oplus_{d,v_{h+1}} \mathbb{N}$ is*

$$Ap(\bar{S}, v_0) = \Big((\mathcal{P} + v_0) \cup \{0\} \Big) \cap \bar{S},$$

where $\mathcal{P}$ is the partition (1).

Proof. It is a consequence of the equality

$$Ap(T, b) = T \cap \Big([0, b) \cup \big((\mathbb{N} \setminus T) + b \big) \Big),$$

for any numerical semigroup T and any $b \in T$. □

4. Algorithms for GSI-Semigroups

In this section, we propose some algorithms for computing GSI-semigroups. These algorithms focus on computing the GSI-semigroups up to a given Frobenius number, and on checking whether there is at least one GSI-semigroup with a given even Frobenius number. For any odd number, there is a GSI-semigroup with this number as its Frobenius number; however, this does not happen for a given even number. Thus, in this section we dedicate a special study to GSI-semigroups with even Frobenius number.

Algorithm 1 computes the set of GSI-semigroups with Frobenius number less than or equal to a fixed non-negative integer. Note that in step 5 of the algorithm, we use $\mathrm{F}(\bar{S}) = d\mathrm{F}(S) + (d-1)\gamma$ and $\gamma > d\mathrm{F}(S)$, which implies that $\mathrm{F}(\bar{S}) \geq d^2\mathrm{F}(S)$ where $\bar{S} = S \oplus_{d,\gamma} \mathbb{N}$.

Denote by $M(S)$ the largest element of the minimal system of generators of a numerical semigroup S.

Algorithm 1: Computation of the set of GSI-semigroups with Frobenius number less than or equal to f.

Data: $f \in \mathbb{N} \setminus \{0\}$, $C = \{S \mid S$ is a numerical semigroup, $F(S) \leq \lfloor f/4 \rfloor\}$
Result: The set $\{\bar{S} \mid \bar{S}$ is a GSI-semigroup with $\mathrm{F}(\bar{S}) \leq f\}$.

```
1 𝒜 = ∅;
2 forall k ∈ {−1} ∪ {1, 2, ..., ⌊f/4⌋} do
3     B = {S ∈ C | F(S) = k} ;
4     forall S ∈ B do
5         D_S = {d ∈ ℕ \ {0,1} | d²F(S) ≤ f} ;
6         G_{d,S} = {(d, γ) ∈ D_S × ℕ | gcd(γ, d) = 1, γ >
            max{dF(S), dM(S)}, dF(S) + (d − 1)γ ≤ f} ;
7     𝒜 = 𝒜 ∪ {S ⊕_{d,γ} ℕ | (d, γ) ∈ G_{d,S}};
8 return 𝒜 ;
```

Remark 1. *If A is a minimal system of generators of a numerical semigroup S and $d \in \mathbb{N} \setminus \{0,1\}$, then dA is a minimal system of generators of $dS = \{ds \mid s \in S\} \subset d\mathbb{N}$. Furthermore, if $\gamma \in \mathbb{N} \setminus \{1\}$ and $\gcd(d,\gamma) = 1$, then $\gamma \notin d\mathbb{N} \setminus \{0\}$. Thus, $\gamma \notin dS$ and $dA \cup \{\gamma\}$ is a minimal system of generators of $\langle dA \cup \{\gamma\} \rangle$.*

We give in Table 1 all the GSI-semigroups with Frobenius number less than or equal to 15.

Table 1. Sets of GSI-semigroups with Frobenius number up to 15.

Frobenius Number	Set of GSI-Semigroups
1	$\{\langle 2,3\rangle\}$
2	$\emptyset$
3	$\{\langle 2,5\rangle\}$
4	$\emptyset$
5	$\{\langle 2,7\rangle, \langle 3,4\rangle\}$
6	$\emptyset$
7	$\{\langle 2,9\rangle, \langle 3,5\rangle\}$
8	$\emptyset$
9	$\{\langle 2,11\rangle, \langle 4,6,7\rangle\}$
10	$\emptyset$
11	$\{\langle 2,13\rangle, \langle 3,7\rangle, \langle 4,5\rangle, \langle 4,6,9\rangle\}$
12	$\emptyset$
13	$\{\langle 2,15\rangle, \langle 3,8\rangle, \langle 4,6,11\rangle\}$
14	$\emptyset$
15	$\{\langle 2,17\rangle, \langle 4,6,13\rangle, \langle 6,8,10,11\rangle\}$

Remember that every numerical semigroup generated by two elements is a GSI-semigroup. Hence, for any odd natural number there exists at least one GSI-semigroup with such Frobenius number.

From Table 1, one might think that there are no GSI-semigroups with even Frobenius number. This is not so, and we can check that $\langle 9,12,15,16\rangle = \langle 3,4,5\rangle \oplus_{3,16} \mathbb{N}$ is a GSI-semigroup and its Frobenius number is 38,

```
gap> FrobeniusNumber(NumericalSemigroup(9,12,15,16));
38
gap> IsGSI(NumericalSemigroup(9,12,15,16));
true
```

This is the first even integer that is realizable as the Frobenius number of a GSI-semigroup. We explain this fact: we want to obtain an even number f from the Formula (6), $f = \mathrm{F}(S \oplus_{d,\gamma} \mathbb{N}) = d\mathrm{F}(S) + (d-1)\gamma$. Since $\gcd(d,\gamma) = 1$, then d has to be odd and $\mathrm{F}(S)$ even. Thus, the lowest number f is obtained for the numerical semigroup S with the smallest even Frobenius number, the smallest odd number $d \geq 3$ and the smaller feasible integer γ, that is, $S = \langle 3,4,5\rangle$, $d = 3$ and $\gamma = 16$. Thus, the GSI-semigroup with the minimum even Frobenius number is $\langle 3,4,5\rangle \oplus_{3,16} \mathbb{N}$.

Note that not every even number is obtained as the Frobenius number of a GSI-semigroup $\langle 3,4,5\rangle \oplus_{3,\gamma} \mathbb{N}$ for some $\gamma \geq 16$ with $\gcd(d,\gamma) = 1$. In this way, we only obtain the values of the form $36 + 2k$ with $k \in \mathbb{N}$ and $k \not\equiv 0 \mod 3$ (if $\gamma = 16 + k$ for $k \in \mathbb{N}$, $\mathrm{F}(\langle 3,4,5\rangle \oplus_{3,\gamma} \mathbb{N}) = 38 + 2k$). The numbers of the form $42 + 6k$, with $k \in \mathbb{N}$, are not obtained (see Table 2).

Table 2. Values of γ such that $\gcd(3,\gamma) \neq 1$ are marked with *.

γ	16	17	18	19	20	21	22	...
$\mathrm{F}(\langle 3,4,5\rangle \oplus_{3,\gamma} \mathbb{N})$	38	40	*	44	46	*	50	...

We now look for GSI-semigroups with Frobenius number of the form $42 + 6k$. Reasoning as above, we use again the semigroup $\langle 3,4,5\rangle$ and set now $d = 5$. In this case, the smallest Frobenius number is 114, and it is given by the semigroup $\langle 3,4,5\rangle \oplus_{5,26} \mathbb{N}$. In general, for the semigroups $\langle 3,4,5\rangle \oplus_{5,\gamma} \mathbb{N}$, the formula of their Frobenius numbers is $5 \cdot 2 + 4\gamma$ with $\gamma \geq 26$ and $\gamma \not\equiv 0 \mod 5$ (see Table 3). For $S = \langle 3,4,5\rangle$, we fill all the even Frobenius number $f \geq 114$, excepting if f is of the form $f = 10 + 4k$ with $k = 15k'$. That is, f cannot be a number of the form $f = 10 + 60k'$ with $k' \in \mathbb{N} \setminus \{0,1\}$, for instance 130 and 190.

Table 3. Values γ such that $\gcd(5,\gamma) \neq 1$ are marked with *.

γ	26	27	28	29	30	31	32	...
$\mathrm{F}(\langle 3,4,5\rangle \oplus_{5,\gamma} \mathbb{N})$	114	118	122	126	*	134	138	...

The above procedures are useful to construct GSI-semigroups with even Frobenius numbers, but with them we cannot determine if a given even positive integer is realizable as the Frobenius number of a GSI-semigroup.

Given an even number f, we are interested in providing an algorithm to check if there exists at least one GSI-semigroup $S \oplus_{d,\gamma} \mathbb{N}$ such that $\mathrm{F}(S \oplus_{d,\gamma} \mathbb{N}) = f$.

As γ has to be greater than or equal to $d\mathrm{F}(S) + 1 \geq 3\mathrm{F}(S) + 1$ (recall that $\gamma > \max\{d\mathrm{F}(S), dM(S)\}$ and $d \geq 3$) and formula (6), we obtain that if $\mathrm{F}(S \oplus_{d,\gamma} \mathbb{N}) = f$, then $2 \leq \mathrm{F}(S) \leq \lfloor \frac{f-2}{9} \rfloor$.

Let $t \in \left[2, \frac{f-2}{9}\right]$ be the Frobenius number of S. Hence, $f = dt + (d-1)\gamma \geq d^2 t + d - 1$ and $d \in \left[3, \left\lfloor \frac{-1+\sqrt{4ft+4t+1}}{2t} \right\rfloor\right]$.

The next lemma follows from the previous considerations.

Lemma 2. *Given an even number f, $S \oplus_{d,\gamma} \mathbb{N}$ is a GSI-semigroup with Frobenius number f if and only if* $\mathrm{F}(S)$ *is an even number belonging to* $\left[2, \frac{f-2}{9}\right]$, *$d$ is an odd number verifying*

$$d \in \left[3, \left\lfloor \frac{-1+\sqrt{4f\mathrm{F}(S)+4\mathrm{F}(S)+1}}{2\mathrm{F}(S)} \right\rfloor\right],$$

and $\gamma = \frac{f-d\mathrm{F}(S)}{d-1}$ *is an integer number such that* $\gcd(\gamma, d) = 1$ *and* $\gamma > \max\{d\mathrm{F}(S), dM(S)\}$.

We present a family formed by semigroups S of even Frobenius number with $\mathrm{F}(S) \geq 10$ and such that $M(S) \leq \mathrm{F}(S)$.

Proposition 2. *For every even number $f \geq 10$, the numerical semigroup S_f minimally generated by $A = \{f/2-1, f-1\} \cup ([f/2+2, f-3] \cap \mathbb{N})$ has a Frobenius number that equals f.*

Proof. Since $2(f/2-1) = f-2 \notin A$, $(f/2-1)+(f/2+2) = f+1$ and $\gcd(A) = 1$, the set A is a minimal system of generators of S_f and $f \notin S_f$.

The elements $f+1 = (f/2-1)+(f/2+2)$, $f+2 = (f/2-1)+(f/2+3)$, ..., $f+(f/2-4) = (f/2-1)+(f-3)$, $f+(f/2-3) = (f/2-1)+(f/2-1)+(f/2-1)$, $f+(f/2-2) = (f/2-1)+(f-1)$, $f+(f/2-1) = (f/2+2)+(f-3)$ are $f/2-1$ consecutive elements in S_f. Hence, $\mathrm{F}(S_f) = f$. $\square$

The numerical semigroups with Frobenius numbers 2, 4, 6 and 8 are the following:

$$\{\langle 3,4,5\rangle\}, \tag{7}$$

$$\{\langle 3,5,7\rangle, \langle 5,6,7,8,9\rangle\}, \tag{8}$$

$$\{\langle 4,5,7\rangle, \langle 4,7,9,10\rangle, \langle 5,7,8,9,11\rangle, \langle 7,8,9,10,11,12,13\rangle\}, \tag{9}$$

and

$$\begin{gathered}\{\langle 3,7,11\rangle, \langle 3,10,11\rangle, \langle 5,6,7,9\rangle, \langle 5,6,9,13\rangle, \langle 5,7,9,11,13\rangle, \\ \langle 5,9,11,12,13\rangle, \langle 6,7,9,10,11\rangle, \langle 6,9,10,11,13,14\rangle, \\ \langle 7,9,10,11,12,13,15\rangle, \langle 9,\ldots,17\rangle\},\end{gathered} \tag{10}$$

respectively.

The semigroups of the sets (7)–(10) and the families of Proposition 2 are the seeds to determine the even natural numbers that are realizable as Frobenius numbers of GSI-semigroups. More precisely, fixed with an even natural number f, these seeds allow us to check if there exist GSI-semigroups with Frobenius number f, and in this case to construct one of them. This is done with Algorithm 2.

Note that several steps of Algorithm 2 can be computed in a parallel way. We now illustrate it with a couple of examples.

Example 3. *Let $f = 42$, since $\lfloor \frac{42-2}{9} \rfloor = 4$, by Algorithm 2, only the numerical semigroups with Frobenius number 2 and 4 must be considered.*

If $\mathrm{F}(S) = 2$, then $d \in \{3,5\}$, since the odd numbers of the set $\left[3, \lfloor \frac{-1+\sqrt{505}}{4} \rfloor\right]_{\mathbb{N}}$ are 3 and 5. For $d = 3$, we have that $\gamma = \frac{42-3\cdot 2}{3-1} = 18$, but $\gcd(d,\gamma) = 1$ so we do not obtain any GSI-semigroup with Frobenius number 42 from S with $\mathrm{F}(S) = 2$ and $d = 3$. For $d = 5$, $\gamma = \frac{42-5\cdot 2}{5-1} = 8 \not> 8 = d\mathrm{F}(S) = 5\cdot 2 = 10$, obtaining again no GSI-semigroups.

If $\mathrm{F}(S) = 4$, then $d = 3$, since $[3,3] = \{3\}$, which is odd. We obtain that $\gamma = \frac{42-3\cdot 4}{3-1} = 15$. By (8), for $\mathrm{F}(S) = 4$, we have $M(S) \geq 7$. In this case, $15 \not> \max\{d\mathrm{F}(S), dM(S)\} = \max\{3\cdot 4, 3\cdot 7\} = 21$.

Hence, there are no GSI-semigroups with Frobenius number 42.

Example 4. *Consider $f = 4620$. Using the code in Appendix A, we check that there are no GSI-semigroups of the form $S \oplus_{d,\gamma} \mathbb{N}$, with $\mathrm{F}(S) \in \{2,4,6,8\}$. Nevertheless, the number 4620 is* *realizable as the Frobenius number of a GSI-semigroup: the Frobenius number of* $S_{12} \oplus_{13,372} \mathbb{N}$*,* $S_{12} \oplus_{17,276} \mathbb{N}$ *and* $S_{12} \oplus_{19,244} \mathbb{N}$ *is* 4620*.*

With the code below, we also obtain other examples of Frobenius numbers of GSI-semigroups that cannot be constructed from semigroups S with $\mathrm{F}(S) \in \{2,4,6,8\}$*.*

```
gap> t:=30000; # Bound of the Frobenius numbers.
gap> s1:=Difference([2..(t-2)],
              Union(ListOfFrobeniusD(2,t/2,t),
                  Union(ListOfFrobeniusD(4,t/2,t),
                      Union(ListOfFrobeniusD(6,t/2,t),
                          ListOfFrobeniusD(8,t/2,t))))
              );
gap> s2:=ListOfFrobeniusD(12,t/2,t);
gap> Print(Intersection(s1,s2));
[ 4620, 7980, 26460 ]
```

The new Frobenius numbers are 7980 *and* 26460*. Some GSI-semigroups with these Frobenius numbers are:* $S_{12} \oplus_{13,652} \mathbb{N}$ *and* $S_{12} \oplus_{17,486} \mathbb{N}$ *for* 7980*, and* $S_{12} \oplus_{13,2192} \mathbb{N}$ *and* $S_{12} \oplus_{17,1641} \mathbb{N}$ *for* 26460*.*

Algorithm 2: Computation of a GSI-semigroup with even Frobenius number f (if possible).

Data: f an even number.
Result: If there exists, a GSI-semigroup with Frobenius number f.

if $f < 38$ **then**
  return $\emptyset$.
$\mathcal{S} = \{S \text{ numerical semigroup} \mid \mathrm{F}(S) \in 2\mathbb{N} \cap [2, \min\{8, \lfloor \frac{f-2}{9} \rfloor\}]\}$;
forall $S \in \mathcal{S}, d \in \left[3, \left\lfloor \frac{-1+\sqrt{4f\mathrm{F}(S)+4\mathrm{F}(S)+1}}{2\mathrm{F}(S)} \right\rfloor\right]$ *odd, and* $\gamma = \frac{f-d\mathrm{F}(S)}{d-1} \in \mathbb{N}$ **do**
  if $\big((\gamma > \max\{d\mathrm{F}(S), dM(S)\}) \wedge (\gcd(d,\gamma) = 1)\big)$ **then**
    return $S \oplus_{d,\gamma} \mathbb{N}$;
$\mathcal{A} = \left\{ t \in [10, \lfloor \frac{f-2}{9} \rfloor] \mid t \text{ even} \right\}$;
while $\mathcal{A} \neq \emptyset$ **do**
  $t = \mathrm{First}(\mathcal{A})$;
  $\mathcal{A} = \mathcal{A} \setminus \{t\}$;
  $\mathcal{B} = \{d \in \left[3, \left\lfloor \frac{-1+\sqrt{4ft+4t+1}}{2t} \right\rfloor\right] \mid d \text{ odd } \}$;
  while $\mathcal{B} \neq \emptyset$ **do**
    $d = \mathrm{First}(\mathcal{B})$;
    $\mathcal{B} = \mathcal{B} \setminus \{d\}$;
    $\gamma = \frac{f-dt}{d-1}$;
    if $\big((\gamma \in \mathbb{N}) \wedge (\gamma > dt) \wedge (\gcd(d,\gamma) = 1)\big)$ **then**
      return $S_t \oplus_{d,\gamma} \mathbb{N}$;
return $\mathcal{A}$;

5. Conclusions

The present paper introduces a new family of numerical semigroups called *generalized strongly increasing semigroups*, using the technique of gluing of numerical semigroups. Our definition of GSI-semigroup is motivated by the already established notion of SI-semigroups, which are the semigroups associated with singular plane branches, and that it is indeed a generalization: every SI-semigroup is a GSI-semigroup. The SI-semigroups determine the topological classification of singular plane branches, and the knowledge of their gaps allows us to study the analytical classification of these curves.

Our main result is an accurate description of the set of gaps of a GSI-semigroup. As an application of this result, some algorithms for finding GSI-semigroups with certain Frobenius numbers are formulated.

For future research, it would be interesting to deepen in the study of this new family of semigroups, determining new properties and invariants, and the geometric varieties associated with.

Author Contributions: Conceptualization, E.R.G.B., J.I.G.-G. and A.V.-T.; methodology, E.R.G.B., J.I.G.-G. and A.V.-T.; software, E.R.G.B., J.I.G.-G. and A.V.-T.; validation, E.R.G.B., J.I.G.-G. and A.V.-T.; investigation, E.R.G.B., J.I.G.-G. and A.V.-T.; resources, E.R.G.B., J.I.G.-G. and A.V.-T.; writing—original draft preparation, E.R.G.B., J.I.G.-G. and A.V.-T.; writing—review and editing, E.R.G.B., J.I.G.-G. and A.V.-T.; project administration, E.R.G.B., J.I.G.-G. and A.V.-T.; funding acquisition, E.R.G.B., J.I.G.-G. and A.V.-T. All authors have read and agreed to the published version of the manuscript.

Funding: The first-named author was partially supported by the Spanish Projects MTM2016-80659-P and PID2019-105896GB-I00. The second and third-named authors were partially supported by the Spanish Project MTM2017-84890-P and Junta de Andalucía groups FQM-343 and FQM-366.

Acknowledgments: The authors thank the referees for their helpful observations.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. GAP Code

```
# The inputs is a NumericalSemigroup S
# The function returns true if S is a SI-semigroup
IsStronglyIncreasing:=function(S)
    local k, lEs, i, smg;
    smg:=MinimalGeneratingSystemOfNumericalSemigroup(S);
    for i in [2..Length(smg)] do
        if(Gcd(smg{[1..i-1]})<=Gcd(smg{[1..i]})) then
                return false;
        fi;
    od;
    for k in [1..(Length(smg)-2)] do
        if (Gcd(smg{[1..k]})*smg[k+1]>=
                    Gcd(smg{[1..(k+1)]})*smg[k+2]) then
                return false;
        fi;
    od;
    return true;
end;

# The input is a NumericalSemigroup S
# The function returns true if S is a GSI-semigroup
IsGeneralizedStronglyIncreasing:=function(S)
    local smg,d,gamma,aux,S1,fn1;
    smg:=MinimalGeneratingSystemOfNumericalSemigroup(S);
    gamma:=smg[Length(smg)];
    d:=Gcd(smg{[1..Length(smg) -1]});
    aux:=(1/d)*smg{[1..Length(smg) -1]};
    S1:=NumericalSemigroup(aux);
    fn1:=FrobeniusNumber(S1);
    if(gamma<=d*fn1) then
```

```
        return false;
    fi;
    if(gamma<=d*aux[Length(aux)]) then
        return false;
    fi;
    return true;
end;

# The inputs are:
# fS (value of F(S)), d (value of d),
# and b (maximum of the Frobenius of listF)
# The function returns listF:
# the list of Frobenius numbers lower than b
# that are Frobenius number of at least a GSI-semigroup
# for a given value of d
# obtained using a semigroup with Frobenius equal to fS
ListOfFrobenius:=function(fS,d,b)
    local f,listF,gamma,lowerBound;
    listF:=[]; f:=0;
    lowerBound:=d*fS;
    if(fS=2) then lowerBound:=fS*5; fi;
    if(fS=4) then lowerBound:=fS*7; fi;
    if(fS=6) then lowerBound:=fS*7; fi;
    if(fS=8) then lowerBound:=fS*9; fi;
    for gamma in [(lowerBound+1)..(b-1)] do
        if(GcdInt(gamma,d)=1) then
            f:=d*fS+(d-1)*gamma;
            if(f<b) then Append(listF,[f]);
            fi;
        fi;
    od;
    return listF;
end;

# The inputs are: fS (value of F(S)),
# boundD (maximum value of d we want to use),
# and b (maximum of the Frobenius of listF)
# The function returns listF:
# the list of Frobenius numbers lower than b
# that are Frobenius number of at least a GSI-semigroup
# obtained using a semigroup with Frobenius equal to fS
ListOfFrobeniusD:=function(fS,boundD,b)
    local listF,d;
    d:=3; listF:=[];
    for d in List([1..Int((boundD-1)/2)],k->2*k+1) do
        listF:=Union(listF,ListOfFrobenius(fS,d,b));
    od;
    return listF;
end;
```

References

1. Rosales, J.C.; Garcia-Sanchez, P.A. *Finitely Generated Commutative Monoids*; Nova Science Pub Inc: New York, NY, USA, 1999.
2. Barucci, V. *Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains*; American Mathematical Society: Providence, RI, USA, 1997.
3. Campillo, A. On saturations of curve singularities (any characteristic). In *Singularities, Part 1 (Arcata, Calif., 1981), Volume 40 of Proc. Sympos. Pure Math.*; American Mathematical Society: Providence, RI, USA, 1983; pp. 211–220.
4. Blanco, V.; Puerto, J. An application of integer programming to the decomposition of numerical semigroups. *SIAM J. Discrete Math.* **2012**, *26*, 1210–1237. [CrossRef]
5. Assi, A.; Garcia-Sanchez, P.A. *Numerical Semigroups and Applications*; Springer: Cham, Switzerland, 2016.
6. García Barroso, E.R.; Płoski, A. An approach to plane algebroid branches. *Rev. Mat. Complut.* **2015**, *28*, 227–252. [CrossRef]
7. Rosales, J.C.; Branco, M.B. Irreducible numerical semigroups with arbitrary multiplicity and embedding dimension. *J. Algebra* **2003**, *264*, 305–315. [CrossRef]
8. Assi, A.; García-Sánchez, P.A. Constructing the set of complete intersection numerical semigroups with a given Frobenius number. *Appl. Algebra Engrg. Comm. Comput.* **2013**, *24*, 133–148. [CrossRef]
9. Eliahou, S. Wilf's conjecture and Macaulay's theorem. *J. Eur. Math. Soc. (JEMS)* **2018**, *20*, 2105–2129. [CrossRef]
10. Rosales, J.C.; Branco, M.B.; Torrão, D. The Frobenius problem for Mersenne numerical semigroups. *Math. Z.* **2017**, *286*, 741–749. [CrossRef]
11. Sylvester, J.J. Mathematical questions with their solutions. *Educ. Times* **1884**, *41*, 21.
12. Ramírez Alfonsín, J.L. *The Diophantine Frobenius Problem, Oxford Lectures Series in Mathematics and its Applications 30*; Oxford University Press: Oxford, UK, 2005.
13. Robles-Pérez, A.; Rosales, J.C. The Frobenius problem for some numerical semigroups with embedding dimension equal to three. *Hacet. J. Math. Stat.* **2015**, *44*, 901–908.
14. Farrán, J.I.; García-Sánchez, P.A.; Heredia, B.A.; Leamer, M.J. The second Feng–Rao number for codes coming from telescopic semigroups. *Des. Codes Cryptogr.* **2018**, *86*, 1849–1864. [CrossRef]
15. Rosales, J.C.; Garcia-Sanchez, P.A. *Numerical Semigroups*; Springer: Berlin/Heidelberg, Germany, 2009.
16. Barrolleta, R.D.; García Barroso, E.R.; Płoski, A. On the Abhyankar-Moh inequality. *Univ. Iagel. Acta Math.* **2015**, *52*, 7–14.
17. Herzog, J. Generators and relations of Abelian semigroup rings. *Manuscr. Math.* **1970**, *3*, 748–751. [CrossRef]
18. Delorme, C. Sous-monoïdes d'intersection complète de $\mathbb{N}$. *Ann. Sci. Éc. Norm. Super.* **1976**, *4*, 145–154. [CrossRef]
19. D'Anna, M.; Micale, V.; Sammartano, A. Classes of complete intersection numerical semigroups. *Semigroup Forum* **2014**, *88*, 453–467. [CrossRef]
20. Delgado, M.; Garcia-Sanchez, P.A.; Morais, J. NumericalSgps, A Package for Numerical Semigroups, Version 1.2.0. Available online: https://gap-packages.github.io/numericalsgps (accessed on 12 June 2021).