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A New Approach of Some Contractive Mappings on Metric Spaces

Ion Marian Olaru † and Nicolae Adrian Secelean *,†

Department of Mathematics and Computer Science, Lucian Blaga University of Sibiu, Dr. I. Raţiu Street, No. 5–7, 550012 Sibiu, Romania; marian.olaru@ulbsibiu.ro
* Correspondence: nicolae.secelean@ulbsibiu.ro
† These authors contributed equally to this work.

Abstract: In this paper, we introduce a new contraction-type mapping and provide a fixed-point theorem which generalizes and improves some existing results in the literature. Thus, we prove that the Boyd and Wong theorem (1969) and, more recently, the fixed-point results due to Wardowski (2012), Turinici (2012), Piri and Kumam (2016), Secelean (2016), Proinov (2020), and others are consequences of our main result. An application in integral equations and some illustrative examples are indicated.

Keywords: fixed point; Picard operator; contractive mapping

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1. Introduction

The *Banach contraction principle* [1] is a fundamental result in the fixed-point theory. It ensures the existence and uniqueness of fixed points of certain self-maps on metric spaces and provides an iterative method to find the respective fixed points. Therefore, it is a very important and powerful tool in solving the existence problems in pure and applied sciences. More precisely, if T is a self-mapping on a complete metric space (X,d) such that

$$d(Tx,Ty) \leq cd(x,y), \ \ \forall\, x,y \in X$$

for some $c \in (0,1)$, then there exists a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, for each $x_0 \in X$, the sequence $(T^n x_0)_n$ converges to x^*. In this setting, we say that T is a *Banach contraction*.

Since then, many researchers generalized and improved the result of Banach by extending the spaces and the operators. Additionally, new areas of application of these results are being discovered.

A function $T : X \to X$ is called a *Picard operator* [2] if it has a unique fixed point x^* and, for each $x_0 \in X$, $\lim\limits_{n\to\infty} T^n x_0 = x^*$, where T^n is the n-th composition of T.

In 1969, Boyd and Wong [3] generalized the Banach contraction principle by replacing the linear condition with a real-valued map called a *comparison function*. A self-mapping T on a metric space is said to be a φ-contraction if

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big), \ \ \forall\, x,y \in X, \ x \neq y, \tag{1}$$

where $\varphi : (0,\infty) \to (0,\infty)$ is upper semi-continuous from the right mapping and satisfies $\varphi(t) < t$ for every $t > 0$. They proved that if the metric space is complete, then T is a Picard operator.

Later, Wardowski [4] introduced a new type of contractive self-map T on a metric space (X,d), the so-called *F-contraction*. This is defined by the inequality

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)), \ \ \forall\, x,y \in X, \ Tx \neq Ty,$$

where $\tau > 0$ and $F : (0, \infty) \to \mathbb{R}$ satisfies the conditions (F_1)–(F_3) defined as follows:

(F_1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F_2) for each sequence $(\alpha_n)_n$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if, and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Wardowski proved that, whenever (X, d) is complete, every F-contraction is a Picard operator. The above result has been extended to new classes of Picard mappings by weakening the conditions (F_1)–(F_3) or by defining new contractive conditions by many authors (see, for example, [5–13]).

In [14], Jleli and Samet denoted by Θ the family of mappings $\theta : (0, \infty) \to (1, \infty)$, satisfying the following conditions:

(Θ_1) θ is nondecreasing;

(Θ_2) for each sequence $(t_n)_n \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if, and only if $\lim_{n\to\infty} t_n = 0^+$;

(Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\searrow 0} \frac{\theta(t)-1}{t^r} = l$.

Theorem 1. [14] *Let (X, d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$x, y \in X,\ \ d(Tx, Ty) \neq 0 \ \Rightarrow\ \theta\big(d(Tx, Ty)\big) \leq \Big[\theta\big(d(x, y)\big)\Big]^k. \tag{2}$$

Then, T is a Picard operator.

Very recently, Proinov [6] considered a self-mapping T on a complete metric space satisfying a general contractive-type condition of the form

$$\psi\big(d(Tx, Ty)\big) \leq \varphi\big(d(x, y)\big) \tag{3}$$

and proved some fixed-point theorems which extend many earlier results in the literature (see some of them in [6]). In this paper, we generalize the fixed-point result given by Proinov ([6], Th. 3.6) by considering general contractive conditions defined by inequality $G(d(Tx, Ty)) \leq H(d(x, y))$, for each x, y with $Tx \neq Ty$, where $G, H : (0, \infty) \to \mathbb{R}$ satisfy conditions (C_1), (C_2) and (C_3) defined below.

2. Results

Let us consider two mappings $G, H : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(C_1) the set of continuity points of G is dense in $(0, \infty)$;

(C_2) for every $r \geq t > 0$, one has $G(r) > H(t)$;

(C_3) $\liminf_{s\searrow t} \big(G(s) - H(s)\big) > 0$ for each $t > 0$.

We will denote by $\mathcal{G}$ the family of all pairs of functions (G, H) which satisfy conditions (C_1)–(C_3).

The following result is easy to be proved.

Remark 1. *Under hypothesis (C_2), condition (C_3) is equivalent to*

(C_3') for each sequence $(t_n)_n \subset (0, \infty)$, such that $t_n \searrow t > 0$ we have

$$\sum_{n\geq 1} (G(t_n) - H(t_n)) = \infty.$$

Proof. "$(C_3) \Rightarrow (C_3')$" Let $(t_n)_n$ be a sequence of positive numbers such that $t_n \searrow t > 0$. Then,

$$0 < \liminf_{s\searrow t} \big(G(s) - H(s)\big) \leq \liminf_{n\to\infty} \big(G(t_n) - H(t_n)\big)$$

hence, the series of positive terms $\sum_{n\geq 1}(G(t_n)-H(t_n))$ diverges.

"$(C_3') \Rightarrow (C_3)$" We proceed by contradiction. Let us suppose that there exists $t>0$ such that $\liminf_{s\searrow t}\big(G(s)-H(s)\big)=0$. Then, one can find a sequence $(t_k)_k$, $t_k>t$, such that $t_k \searrow t$ and $\lim_{k\to\infty}\big(G(t_k)-H(t_k)\big)=0$. There is no loss of generality in assuming that $(t_k)_k$ is decreasing. By the above, there exists $k_1\in\mathbb{N}$ such that

$$G(t_{k_1})-H(t_{k_1})<\frac{1}{2}.$$

Next, one can find $k_2>k_1$ such that

$$G(t_{k_2})-H(t_{k_2})<\frac{1}{2^2}.$$

Inductively, we obtain a subsequence $(t_{k_n})_n$ of $(t_k)_k$ such that

$$G(t_{k_n})-H(t_{k_n})<\frac{1}{2^n},\ \ \forall\, n\geq 1,$$

and so the series $\sum_n\big(G(t_{k_n})-H(t_{k_n})\big)$ converges. Moreover, $t_{k_n}\searrow t$. This contradicts (C_3'). □

Example 1. *Let us consider* $\alpha,\beta\in(0,1)$, $a>0$, $\tau>0$ *such that* $\alpha a<a-\tau$ *and let us define* $G,H:\mathbb{R}_+\to\mathbb{R}$ *by*

$$G(r)=\beta r,$$

$$H(r)=\begin{cases}\beta r-\tau, & r>\frac{a}{\beta}\\ \alpha\beta r, & 0<r\leq\frac{a}{\beta}.\end{cases}$$

Then, $(G,H)\in\mathcal{G}$.

Proof. (C_1) Obvious.
(C_2) If $r\geq t>0$, then $G(r)\geq G(t)>H(t)$.
(C_3) Let any $t>0$. Then,

$$G(t)-H(t)=\begin{cases}\tau, & t>\frac{a}{\beta}\\ \beta(1-\alpha)t, & 0<t\leq\frac{a}{\beta}\end{cases}$$

and hence, $\liminf_{s\searrow t}\big(G(s)-H(s)\big)>0$.

□

Example 2. *Let us consider* $\alpha<1$ *and* $G,H:(0,\infty)\to\mathbb{R}$ *defined by*

$$G(r)=\ln(1+r),$$

$$H(t)=\ln(1+t)^{\alpha}.$$

Then, $(G,H)\in\mathcal{G}$.

Proof. (C_1) Obvious.
(C_2) Choose $r\geq t>0$. Then

$$G(r)=\ln(1+r)>\alpha\ln(1+r)\geq\alpha\ln(1+t)=H(t).$$

(C_3) If $t>0$, then $\liminf_{s\searrow t}\big(G(s)-H(s)\big)=(1-\alpha)\ln(t+1)>0$.

□

Lemma 1. *Let $F : (0, \infty) \to \mathbb{R}$ be a map and $(t_k)_k$ a sequence of positive real numbers such that $F(t_k) \underset{k}{\longrightarrow} -\infty$. If one of the following conditions holds:*

(a) *F is nondecreasing;*
(b) *F is right-continuous and $(t_k)_k$ is nonincreasing;*
(c) *F is lower semi-continuous and $(t_k)_k$ is nonincreasing,*
then $\lim\limits_{k\to\infty} t_k = 0$.

Proof. (a) ([11], L. 3.2).

Suppose that $(t_k)_k$ is nonincreasing. Then, it is bounded so there is $\alpha \geq 0$ such that $t_k \searrow \alpha$. Assume by contradiction that $\alpha > 0$.

(b) By hypothesis, $F(\alpha) = \lim\limits_{k\to\infty} F(t_k) = -\infty$ which is a contradiction. So $\alpha = 0$.

(c) Let any $\varepsilon > 0$. There exists $\delta > 0$ such that, for every $t > 0$, $|t - \alpha| < \delta$, one has $F(\alpha) \leq F(t) + \varepsilon$. One can find $k(\varepsilon) \in \mathbb{N}$ such that $|t_k - \alpha| < \delta$ for all $k \geq k(\varepsilon)$. Therefore,

$$F(\alpha) \leq F(t_k) + \varepsilon, \ \ \forall\, k \geq k(\varepsilon)$$

contradicting the hypothesis $F(t_k) \to -\infty$. Consequently $\alpha = 0$. □

Definition 1. *We say that a function $F : (0, \infty) \to \mathbb{R}$ satisfies property (P) if, for every nonincreasing sequence $(t_k)_k$ of positive numbers such that $F(t_k) \underset{k}{\longrightarrow} -\infty$, one has $\lim\limits_{k\to\infty} t_k = 0$.*

Note that the previous lemma gives some classes of functions satisfying property (P). At the same time, there exist functions having property (P), but which do not satisfy any of the conditions of Lemma 1 as it follows from the following example.

Example 3. *Let $(r_n)_n$ be a decreasing sequence of positive numbers converging to 0 and $f, g : (0, \infty) \to \mathbb{R}$ be two mappings such that $\lim\limits_{n\to\infty} f(r_n) = -\infty$ and g is bounded from below. Then, the mapping $F : (0, \infty) \to \mathbb{R}$ given by*

$$F(t) = \begin{cases} f(t), & t \in \{r_n,\ n = 1, 2, \dots\} \\ g(t), & t \notin \{r_n,\ n = 1, 2, \dots\} \end{cases}$$

satisfies property (P). If, further, the set of discontinuity points of g is at most countable (in particular if g is monotone on each interval (r_{n+1}, r_n)), then the set of discontinuity points of F is also at most countable.

Proof. Set $M \in \mathbb{R}$ such that $M \leq g(t)$ for all $t > 0$. Let us consider a nonincreasing sequence $(t_k)_k$ of positive numbers such that $F(t_k) \underset{k}{\longrightarrow} -\infty$. Then, there exists $K \in \mathbb{N}$ such that $F(t_k) < M$ for each $k \geq K$. Hence, $F(t_k) = f(t_k)$ for each $k \geq K$, that is, there is $n_k \in \mathbb{N}$ such that $t_k = r_{n_k}$, this means that $(t_k)_{k\geq K}$ is a subsequence of $(r_n)_n$ so $t_k \underset{k}{\longrightarrow} 0$.

For the last assertion, if we denote by Δ_F, Δ_g the sets of discontinuities of F and g, respectively, then

$$\Delta_F \subset \{r_n,\ n = 1, 2, \dots\} \cup \Delta_g,$$

so Δ_F is at most countable. □

Note that one can find easily numerous functions satisfying the conditions of Example 3 such as: $r_n = 1/n$, $f(r) = -1/r$, $g(t) = t$ and so on.

Proposition 1. [12,15] *Let $(x_n)_n$ be a sequence of elements from a metric space (X, d) and Δ be a subset of $(0, \infty)$ such that $(0, \infty) \setminus \Delta$ is dense in $(0, \infty)$. If $d(x_n, x_{n+1}) \underset{n}{\longrightarrow} 0$ and $(x_n)_n$ is not a Cauchy sequence, then there exist $\eta \in (0, \infty) \setminus \Delta$ and the sequences of natural numbers $(m_k)_k, (n_k)_k$ such that*

(1) $d(x_{m_k}, x_{n_k}) \searrow \eta,\ k \to \infty$,
(2) $d(x_{m_k+p}, x_{n_k+q}) \to \eta,\ k \to \infty,\ p, q \in \{0, 1\}$.

Our main result is the following:

Theorem 2. *Let (X, d) be a complete metric space and G, H be two mappings such that $(G, H) \in \mathcal{G}$ and one of them satisfies property (P). Let also consider the map $T : X \to X$ satisfying the following condition*

$$G(d(Tx, Ty)) \le H(d(x, y)),\ \forall\, x, y \in X,\ Tx \neq Ty. \tag{4}$$

Then T is a Picard operator.

Proof. First of all we remark that, from conditions (C_2) and (4), we deduce that T satisfies

$$d(Tx, Ty) < d(x, y),\ \forall\, x, y \in X,\ Tx \neq Ty, \tag{5}$$

which implies that T has at most one fixed point.

In order to show the existence of fixed point of T, let $x_0 \in X$ be fixed. We define a sequence $(x_n)_n$ by $x_n = Tx_{n-1}$, $n \ge 1$, and let us denote $d_n = d(x_{n+1}, x_n)$, $n \ge 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is a fixed point of T. We next suppose that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Then, $d_n > 0$ for all $n \in \mathbb{N}$ and, by (5), the sequence $(d_n)_n$ is decreasing. Thus, one can find $d \ge 0$ such that $d_n \searrow d$.

Next, we will prove that $d = 0$. Indeed, using (4), we get $G(d_n) \le H(d_{n-1})$ for all $n \ge 1$. From the above, we obtain

$$G(d_n) - G(d_{n-1}) \le H(d_{n-1}) - G(d_{n-1}),$$

for every $n \ge 1$. Therefore,

$$\sum_{k=1}^{n} (G(d_k) - G(d_{k-1})) \le \sum_{k=1}^{n} (H(d_{k-1}) - G(d_{k-1}))$$

so

$$G(d_n) \le G(d_0) + \sum_{k=1}^{n} (H(d_{k-1}) - G(d_{k-1})) \xrightarrow[n]{} -\infty,$$

according to condition (C_3') from Remark 1. It follows that $\lim_{n\to\infty} G(d_n) = -\infty$. At the same time, since $d_n < d_{n-1}$, we deduce from (C_2) that $H(d_n) < G(d_{n-1})$ for all $n = 1, 2, \ldots$, hence, $\lim_{n\to\infty} H(d_n) = -\infty$.

We conclude by hypothesis that $\lim_{n\to\infty} d_n = 0$.

Now, assume that the sequence $(x_n)_n$ is not Cauchy, and let Δ be the set of discontinuities of G. Since (G, H) satisfies (C_1), it follows that $(0, \infty) \setminus \Delta$ is dense in $(0, \infty)$.

According to Proposition 1, one can find $\eta \in (0, \infty) \setminus \Delta$ and the sequences $(m_k)_k$, $(n_k)_k$ such that

$$d(x_{m_k}, x_{n_k}) \searrow \eta, d(x_{m_k+1}, x_{n_k+1}) \to \eta, k \to \infty.$$

Since $\eta > 0$, there is $K \in \mathbb{N}$ such that $d(x_{m_k+1}, x_{n_k+1}) > 0$ for all $k \ge K$. Therefore, from (4), for all $k \ge K$, we get

$$G(d(x_{m_k+1}, x_{n_k+1})) \le H(d(x_{m_k}, x_{n_k})). \tag{6}$$

Since G is continuous at η, from the last inequality we obtain letting $k \to \infty$

$$(G - H)\big(d(x_{m_k}, x_{n_k})\big) \le G\big(d(x_{m_k}, x_{n_k})\big) - G\big(d(x_{m_k+1}, x_{n_k+1})\big) \xrightarrow[k]{} G(\eta) - G(\eta) = 0, \tag{7}$$

which contradicts (C_3). Consequently, $(x_n)_n$ is a Cauchy sequence and, X being complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Finally, condition (5) yields

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx_n, x_n) = 0.$$

Thus, $Tx^* = x^*$. $\square$

Example 4. *Let us consider $G, H : (0, \infty) \to \mathbb{R}$ defined by*

$$G(t) = \begin{cases} \ln t, & t \in (0, \frac{1}{2}] \\ \ln(t - \frac{1}{12}), & t \in (\frac{1}{2}, \infty) \end{cases}$$

$$H(t) = \ln \frac{t}{t+1}.$$

Then, G is not monotone, satisfies property (P) and $(G, H) \in \mathcal{G}$. Furthermore, if $X = [0, \infty)$ is endowed with the standard metric $d(x, y) = |x - y|$ and $T : X \to X$, $Tx = \frac{x}{x+1}$, satisfies (4), then T is a Picard operator while it does not satisfy the Banach condition.

Proof. (C_1) Obvious.

(C_2) Let us consider $r \geq t > 0$. Three cases can occur:

I. $r \leq \frac{1}{2}$. Then

$$G(r) = \ln r \geq \ln t > \ln \frac{t}{t+1} = H(t).$$

II. $r \geq t > \frac{1}{2}$. From the following relations

$$t - \frac{1}{12} > \frac{t}{t+1} \Leftrightarrow \frac{t^2}{t+1} > \frac{1}{12}$$

we deduce that

$$G(r) \geq G(t) = \ln(t - \frac{1}{12}) > \ln \frac{t}{t+1} = H(t).$$

III. $r > \frac{1}{2} \geq t$. One has

$$r - \frac{1}{12} > \frac{5}{12} > \frac{1}{3} \geq \frac{t}{t+1}$$

hence, $G(r) > H(t)$.

(C_3) Let any $t > 0$. If $t < \frac{1}{2}$, then $\liminf_{s \searrow t} \big(G(s) - H(s)\big) = \ln(t+1) > 0$. Additionally, if $t \geq \frac{1}{2}$, then $\liminf_{s \searrow t} \big(G(s) - H(s)\big) = \ln \big(1 - \frac{1}{12t}\big)(t+1) > 0$.

Clearly, G satisfies (P), and it is not monotone.

In order to prove the second part of the statement, set $x, y \in X$ such that $Tx \neq Ty$. Then, $x \neq y$, say $x < y$. The following cases can occur:

I. $0 < \frac{y-x}{(1+x)(1+y)} \leq \frac{1}{2}$. Then

$$\frac{y-x}{(1+x)(1+y)} < \frac{y-x}{1+y-x}$$

$$\Rightarrow G\big(d(Tx, Ty)\big) = \ln \frac{y-x}{(1+x)(1+y)} < \ln \frac{y-x}{1+y-x} = H\big(d(x,y)\big).$$

II. $\frac{y-x}{(1+x)(1+y)} > \frac{1}{2}$. Then

$$G\big(d(Tx, Ty)\big) = \ln \Big(\frac{y-x}{(1+x)(1+y)} - \frac{1}{12}\Big) < \ln \frac{y-x}{1+y-x} = H\big(d(x,y)\big).$$

Therefore, the inequality (4) is fulfilled.

Theorem 2 shows that T is a Picard operator (its unique fixed point being $x = 0$).

For the last sentence, let us consider two sequences $x_n = \frac{1}{n}$, $y_n = \frac{2}{n}$. Then

$$\lim_{n\to\infty} \frac{d(Tx_n, Ty_n)}{d(x_n, y_n)} = \lim_{n\to\infty} \frac{1}{(1+x_n)(1+y_n)} = 1$$

hence, T does not satisfy the Banach condition. □

Corollary 1. *Let (X, d) be a complete metric space and let $T : X \to X$ satisfy*

$$G\big(d(Tx, Ty)\big) \le H\big(d(x, y)\big), \ \ \forall\, x, y \in X,\ Tx \ne Ty, \tag{8}$$

where $G, H : (0, \infty) \to \mathbb{R}$ are two mappings satisfying the following conditions:

(a) *G is nondecreasing;*

(b) *H is upper semi-continuous from the right;*

(c) *$G(t) > H(t)$ for every $t > 0$.*

Then, T is a Picard operator.

Proof. Clearly the set of continuity points of G is dense in $(0, \infty)$ and G satisfies property (P). By (a) and (c), it is also obvious that (G, H) satisfies (C_2).

Let any $t > 0$. Using (b) and (c) one obtains

$$\limsup_{s\searrow t} H(s) \le H(t) < G(t),$$

hence,

$$\liminf_{s\searrow t} \big(G(s) - H(s)\big) \ge \liminf_{s\searrow t} G(s) - \limsup_{s\searrow t} H(s) > G(t) - G(t) = 0.$$

Consequently, (C_3) is verified. The conclusion now follows from Theorem 2. □

Next, we will show that the result of Boyd and Wong [3] can be obtained from Corollary 1. We need first the following elementary lemma.

Lemma 2. *Let $G : (0, \infty) \to \mathbb{R}$, $\varphi : (0, \infty) \to (0, \infty)$ be functions such that:*

(i) *φ is upper semi-continuous from the right at some $a \in (0, \infty)$;*

(ii) *G is nondecreasing and right-continuous at $\varphi(a)$.*

Then, the function $G \circ \varphi$ is upper semi-continuous from the right at a.

Proof. One has

$$\limsup_{t\searrow a} \varphi(t) = \inf_{\varepsilon>0} \Big(\sup_{t\in(a,a+\varepsilon)} \varphi(t)\Big) = \lim_{\varepsilon\searrow 0} \Big(\sup_{t\in(a,a+\varepsilon)} \varphi(t)\Big) \le \varphi(a).$$

We need to show that

$$\limsup_{t\searrow a} G\big(\varphi(t)\big) \le G\big(\varphi(a)\big).$$

Two cases can occur.

Case I. $\limsup\limits_{t\searrow a} \varphi(t) < \varphi(a)$. Then, there exists $\varepsilon_0 > 0$ such that

$$0 < \sup_{t\in(a,a+\varepsilon_0)} \varphi(t) < \varphi(a).$$

Thus, by the monotonicity of G, one has

$$G\Big(\sup_{t\in(a,a+\varepsilon_0)}\varphi(t)\Big) \le G\big(\varphi(a)\big) \;\Rightarrow\; \inf_{\varepsilon>0} G\Big(\sup_{t\in(a,a+\varepsilon)}\varphi(t)\Big) \le G\big(\varphi(a)\big)$$

hence,

$$\limsup_{t\searrow a} G\big(\varphi(t)\big) = \inf_{\varepsilon>0}\Big(\sup_{t\in(a,a+\varepsilon)} G(\varphi(t))\Big) \le \inf_{\varepsilon>0} G\Big(\sup_{t\in(a,a+\varepsilon)}\varphi(t)\Big) \le G\big(\varphi(a)\big).$$

Case II. $\limsup\limits_{t\searrow a}\varphi(t) = \varphi(a)$. Then, using (i), (ii),

$$\limsup_{t\searrow a} G\big(\varphi(t)\big) \le \lim_{\varepsilon\searrow 0} G\Big(\sup_{t\in(a,a+\varepsilon)}\varphi(t)\Big) = G\Big(\lim_{\varepsilon\searrow 0}\Big(\sup_{t\in(a,a+\varepsilon)}\varphi(t)\Big)\Big) = G\big(\varphi(a)\big).$$

□

Remark 2. *If we take in Corollary 1 a nondecreasing and right-continuous function $G : (0,\infty) \to \mathbb{R}$ and $H = G\circ\varphi$, where φ is a comparison function, then every self-mapping T on a complete metric space satisfying (1) is a Picard operator.*

Proof. By the monotonicity of G, it is obvious that (8) is equivalent to (1). From Lemma 2, it follows that H is upper semi-continuous from the right function. The rest of the conditions from Corollary 1 are clearly verified. □

In the next two corollaries, we will highlight that the results given by Secelean and Wardowski [8], Secelean [9], Wardowski [4], and Piri and Kumam [5] can be obtained as particular cases of Theorem 2.

For every $\mu \in \overline{\mathbb{R}}_+$ we denote by Ψ_μ the family of all nondecreasing functions $\psi : (-\infty,\mu) \to (-\infty,\mu)$ such that $\psi(t) < t$ for all $t \in (-\infty,\mu)$.

Corollary 2. *Let us consider (X,d) a complete metric space, and $T : X \to X$. We suppose that there exists a nondecreasing function $F : (0,\nu) \to \mathbb{R}$, $\nu > diam(X) := \sup\limits_{x,y\in X} d(x,y)$, and a right-continuous map $\psi \in \Psi_\mu$, $\mu = \sup\limits_{t\in(0,\nu)} F(t)$, and*

$$(\forall)x,y\in X\;[d(Tx,Ty) > 0 \Rightarrow F(d(Tx,Ty)) \le \psi\big(F(d(x,y))\big)].$$

Then, T is a Picard operator.

Proof. Let us consider $G,H : (0,\infty) \to \mathbb{R}$ defined by

$$G = F,\quad H = \psi\circ F.$$

We will state that G,H satisfy conditions (C_1), (C_2) and (C_3).

(C_1) This condition is clearly verified due to the monotonicity of F.

(C_2) Fix $r \ge t > 0$. Then, by the property of ψ, one has

$$H(t) = \psi\big(F(t)\big) < F(t) \le F(r) = G(r).$$

(C_3) Let us consider a sequence of positive real numbers $(t_n)_n$ such that $t_n \searrow t > 0$. Then, the sequence $\big(F(t_n)\big)_n$ is non-increasing and $F(t) \le F(t_n)$ for every $n = 1,2,\ldots$ hence, one can find $\lambda \in [F(t),\mu)$ such that $F(t_n) \searrow \lambda$. Since ψ is right-continuous at λ, one has

$$\lim_{n\to\infty}\big(G(t_n) - H(t_n)\big) = \lim_{n\to\infty}\big(F(t_n) - \psi(F(t_n))\big) = \lambda - \psi(\lambda) > 0,$$

so $\sum_n (G(t_n) - H(t_n)) = \infty$, that is (G, H), satisfies (C'_3). From Remark 1, we deduce that (C_3) is also satisfied.

By Lemma 1, we deduce that G satisfies property (P).

Now, the conclusion follows from Theorem 2. □

If, in the previous corollary, we take $\psi(t) = t - \tau$ for some $\tau > 0$, one obtains an improvement of the results from [4,9], where F satisfies only condition (F_1).

Corollary 3. *Let us consider (X, d) a complete metric space and $T : X \to X$. We suppose that there exist nondecreasing $F : (0, \infty) \to \mathbb{R}$ and $\tau > 0$, such that*

$$\forall\, x, y \in X\ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))].$$

Then, T is a Picard operator.

Inspired by [16], we can formulate an improvement of Wardowski's result given in the previous corollary.

Corollary 4. *Let us consider a complete metric space (X, d) and two functions $F : (0, \infty) \to \mathbb{R}$, $\theta : (0, \infty) \to (0, \infty)$. Assume that F has property (P) and the set of its continuity points is dense in $(0, \infty)$. Suppose further that $\sum_n \theta(t_n) = \infty$ for each decreasing sequence of real numbers $(t_n)_n$ with a positive limit. If $T : X \to X$ is such that*

$$\theta\big(d(x, y)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big) \ \ \forall\, x, y \in X \text{ with } Tx \neq Ty,$$

then T is a Picard operator.

Proof. Set $G, H : (0, \infty) \to \infty$, $G = F$, $H = F - \theta$. On account of the hypothesis, it follows immediately that (G, H) satisfies (C_1), and G has the property (P).

Since conditions (C_2), (C'_3) are also obviously verified, one can apply Theorem 2. □

Corollary 5. [13] *Let us consider a complete metric space (X, d) and three functions $F : (0, \infty) \to \mathbb{R}$, $\varphi : (0, \infty) \to (0, \infty)$, $T : X \to X$ satisfying the following conditions:*

(α) F is nondecreasing;

(β) $\liminf_{s \searrow t} \varphi(s) > 0$ for each $t > 0$;

(γ) $\varphi\big(d(x, y)\big) + F\big(d(Tx, Ty)\big) \leq F\big(d(x, y)\big)$ for all $x, y \in X$ with $Tx \neq Ty$.

Then, T is a Picard operator.

Proof. The conclusion follows easily from Theorem 2 by taking $G = F$ and $H = F - \varphi$. □

Remark 3. *The following corollary shows that Theorem 1 can be obtained from Theorem 2 without imposing on the function θ conditions (Θ_2) and (Θ_3). We will also answer the open question formulated in [17].*

Corollary 6. *Let us consider a nondecreasing function $\theta : (0, \infty) \to (1, \infty)$ and $k \in (0, 1)$. Assume that T is a self-mapping on a complete metric space (X, d) such that (2) holds. Then, T is a Picard operator.*

Proof. Define $G, H : (0, \infty) \to \mathbb{R}$, $G = \frac{1}{1-\theta}$, $H = \frac{1}{1-\theta^k}$. Since θ and k satisfy (2), it follows that G, H satisfy (4). We will show that $(G, H) \in \mathcal{G}$.

Since every continuity point of θ is a continuity point of both G and H and θ is monotonic, it follows that G, H satisfy (C_1). Next, if $r \geq t > 0$, then

$$G(r) \geq G(t) = \frac{1}{1 - \theta(t)} > \frac{1}{1 - \theta(t)^k} = H(t)$$

hence (C_2) holds. Let us consider (t_n), $t_n \searrow t > 0$. Then, the sequence $\big(\theta(t_n)\big)$ is nonincreasing and bounded, hence, there exists $\lambda \geq \theta(t)$ such that $\theta(t_n) \searrow \lambda$. Thus, $\lim\limits_{n\to\infty}\big(\frac{1}{1-\theta(t_n)} - \frac{1}{1-\theta(t_n)^k}\big) = \frac{1}{1-\lambda} - \frac{1}{1-\lambda^k} > 0$. Consequently,

$$\sum_n \big(G(t_n) - H(t_n)\big) = \infty$$

hence, (C_3'). Next, we apply Remark 1.

According to Lemma 1, G satisfies property (P). □

In the following, we will show that one of the main theorems of Proinov can be obtained as a consequence of our results.

Corollary 7. ([6], Th. 3.6) *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying condition (3), where the functions $\psi, \varphi : (0,\infty) \to \mathbb{R}$ has the following properties:*

(i) *ψ is nondecreasing;*
(ii) *$\varphi(t) < \psi(t)$ for every $t > 0$;*
(iii) *$\limsup\limits_{s\searrow t} \varphi(s) < \lim\limits_{s\searrow t} \psi(s)$ for every $t > 0$.*

Then, T is a Picard operator.

Proof. Let us denote $G = \psi$, $H = \varphi$.

We first note that (i) implies that G satisfies property (P). Next, from (i) and (ii) we deduce that, for some $r \geq t > 0$, we have

$$G(r) \geq G(t) > H(t)$$

hence, (G,H) satisfies (C_2).

Additionally, due to the monotonicity of ψ, we deduce that there exists $\lim\limits_{s\searrow t} \psi(s)$ and $\lim\limits_{s\searrow t} \psi(s) = \liminf\limits_{s\searrow t} \psi(s) \in \mathbb{R}$, for every $t > 0$. Consequently, using (iii), one has

$$\liminf_{s\searrow t} \big(\psi(s) - \varphi(s)\big) \geq \liminf_{s\searrow t} \psi(s) - \limsup_{s\searrow t} \varphi(s) > \liminf_{s\searrow t} \psi(s) - \lim_{s\searrow t} \psi(s) = 0.$$

Thus, (G,H) satisfies (C_3).

The conclusion follows from Theorem 2. □

Notice that Corollary 1 can be obtained as a particular case of the previous corollary.

Remark 4. *Example 4 proves that the result of Proinov pointed out in Corollary 7 can be obtained from Theorem 2 without imposing the monotonicity of the function ψ.*

3. Application

Next, we use our main results in order to give an existence and uniqueness result for the solution of a certain integral equation.

Proposition 2. *Let us consider G, H defined in Example 1, and the integral equation*

$$x(t) = \int_0^t K(t,s,x(s))ds + f(t),\ t \in [0,1] \tag{9}$$

under the following conditions:

(H_0) $K \in C([0,1] \times [0,1] \times \mathbb{R}, \mathbb{R})$, $f \in C([0,1],\mathbb{R})$;
(H_1) $|K(t,s,u) - K(t,s,v)| \leq l(u,v)$ *for all $t,s \in [0,1]$ and $u,v \in \mathbb{R}$, where*

(i)

$$l : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$l(u, v) = \begin{cases} |u - v| - m\frac{\tau}{\beta}, & |u - v| > \frac{a}{\beta} \\ \gamma \cdot |u - v|, & |u - v| \leq \frac{a}{\beta} \end{cases}$$

(ii) $a - m \cdot \tau < \alpha \cdot a < a - \tau$
(iii) $\gamma = \frac{a - m \cdot \tau}{a}$

Then, the Equation (9) has a unique solution in $C([0, 1], \mathbb{R})$ *(the class of continuous functions* $x : [0, 1] \to \mathbb{R}$*).*

Proof. Let us endow $C([0, 1], \mathbb{R})$ with $\|x\|_\infty = \sup_{t \in [0,1]} |x(t)|$, and let

$$T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R}),$$

defined by

$$Tx(t) = \int_0^t K(t, s, x(s))ds + f(t).$$

According to Example 1 and Remark 1, the applications G, H satisfy the conditions $(C_1), (C_2)$ and (C_3). On the other hand, for each $x, y \in C([0, 1], \mathbb{R})$ and $t \in [0, 1]$, we have

$$\beta|Tx(t) - Ty(t)| \leq \beta \int_0^t |K(t, s.x(s)) - K(t, s.y(s))|ds$$

$$\leq \beta \int_0^t l(x(s), y(s))ds \leq \int_0^t H(|x(s) - y(s)|)ds \leq H(\|x - y\|_\infty).$$

Therefore, for every $x, y \in C([0, 1], \mathbb{R})$, we get

$$G(\|Tx - Ty\|_\infty) = \beta\|Tx - Ty\|_\infty = \beta \sup_{t \in [0,1]} |Tx(t) - Ty(t)| \leq H(\|x - y\|_\infty).$$

The conclusion now follows from Theorem 2 applied to operator T. □

4. Conclusions

We introduce a new type of contractive function on a metric space that generalizes and extends some of the contractions studied in the literature, and we provide a fixed-point theorem that improves many of the known results. Some examples and an application to integral equations are also given. The result we proved can be extended to more general metric spaces.

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