

Article

Representations of Rectifying Isotropic Curves and Their Centroides in Complex 3-Space

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Abstract: In this work, the rectifying isotropic curves are investigated in three-dimensional complex space \mathbb{C}^3 . The conclusion that an isotropic curve is a rectifying curve if and only if its pseudo curvature is a linear function of its pseudo arc-length is achieved. Meanwhile, the rectifying isotropic curves are expressed by the Bessel functions explicitly. Last but not least, the centroides of rectifying isotropic curves are explored in detail.

Keywords: complex space; isotropic curve; rectifying curve; structure function



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1. Introduction

In 2003, B. Y. Chen first proposed the notion of rectifying curves, which represent a class of space curves whose position vector always lies in its rectifying plane in Euclidean 3-space [1]. The necessary and sufficient conditions of a curve being a rectifying curve are found, for example, a curve is a rectifying curve if and only if the ratio of its curvature and torsion is a non-constant linear function of its arc length. Furthermore, a rectifying curve $r(t)$ can be expressed via a unit speed space curve $y(t)$ as $r(t) = a(\sec t)y(t)$, where a is a positive constant [1].

Motivated by the achievements of Chen on rectifying curves, a series of further research works were carried out in different space-times by many mathematicians [2–4]. For instance, the rectifying curves are generalized into the ones in Minkowski 3-space, which are divided into space-like (resp. time-like, light-like) rectifying curves. The necessary and sufficient conditions of three kinds of space curves being rectifying curves were explored completely in [5,6]. Moreover, the rectifying curves are also defined in Euclidean 4-space, i.e., the position vector of a space curve lies in the orthogonal complement of its principal normal vector field [7]. Some examples of rectifying curves as centroides and extremal curves were discussed in [8,9].

In recent years, isotropic curves have been discussed widely, such as isotropic Bertrand curves, the isotropic helix and k-type isotropic helices [10,11]. In 2015, the isotropic curves in complex space \mathbb{C}^3 were characterized by one of the authors [11] in which the structure function of isotropic curves are defined and the relationship between the pseudo curvature and the structure function is built by a kind of Schwartzian derivative. In this paper, analogous with the definition of rectifying curves in Euclidean 3-space, we focus on the rectifying isotropic curves and their centroides in \mathbb{C}^3 .

This paper is organized as follows. In Section 2, some basic facts about the complex space, the isotropic curve and the structure function are recalled, and the rectifying isotropic curves are defined at the same time. In Section 3, the necessary and sufficient conditions of isotropic curves being rectifying curves are achieved. In addition, the rectifying isotropic curves are expressed explicitly with Bessel functions by solving the Riccati equation of the structure function. Based on the conclusions obtained in Section 3, the centroides of rectifying isotropic curves are explored precisely.

The curves considered in this paper are regular and analytic unless otherwise stated.

2. Preliminaries

Let \mathbb{C}^3 be a three-dimensional complex space with the following standard metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + dx_3^2$$

in terms of the natural coordinate system (x_1, x_2, x_3) .

The norm of a vector is defined by $\| \cdot \| = \sqrt{|\langle \cdot, \cdot \rangle|}$. An arbitrary nonzero vector $v \in \mathbb{C}^3$ is called an isotropic vector if $\langle v, v \rangle = 0$. For a curve $r(t)$ in \mathbb{C}^3 , if its tangent vector is an isotropic vector, then we have the following.

Definition 1. Let $r(s)$ be a curve in \mathbb{C}^3 . If the squared distance between two points of $r(s)$ is equal to zero, then $r(s)$ is called an isotropic curve [11].

The isotropic curves also can be understood as meromorphic mappings $r : t \in U \subset \mathbb{C} \rightarrow \mathbb{C}^3$ with vanishing complex length of their tangent vectors. A regular isotropic curve $r(t)$ is full if, and only if, $\langle r''(t), r''(t) \rangle \neq 0$ [11].

Remark 1. The pseudo arc length parameter for isotropic curves can be defined by normalizing the acceleration vector of isotropic curves, i.e., $\langle r''(s), r''(s) \rangle = -1$ [10,12]. Hereafter, the isotropic curves are always assumed to be parameterized by the pseudo arc length unless stated specifically, and the isotropic geodesic is excluded throughout the paper.

Proposition 1. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . Then $r(s)$ can be framed by a unique Cartan Frenet frame $\{e_1, e_2, e_3\}$ such that the following is true [10]:

$$\begin{pmatrix} e_1' \\ e_2' \\ e_3' \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 \\ i\kappa & 0 & i \\ 0 & -i\kappa & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{1}$$

where $\langle e_1, e_1 \rangle = \langle e_3, e_3 \rangle = \langle e_1, e_2 \rangle = \langle e_2, e_3 \rangle = 0, \langle e_2, e_2 \rangle = \langle e_1, e_3 \rangle = 1; e_1 \times e_2 = ie_1, e_2 \times e_3 = ie_3, e_3 \times e_1 = ie_2; \det(e_1, e_2, e_3) = i; i^2 = -1$.

In sequence, e_1, e_2, e_3 is called the tangent, principal normal and binormal vector field of $r(s)$, respectively. From Equation (1), it is easy to know that $\kappa(s) = \frac{1}{2} \langle r'''(s), r'''(s) \rangle$. The function $\kappa(s)$ is called the pseudo curvature of $r(s)$.

For an isotropic curve $r(s)$ in \mathbb{C}^3 with Frenet frame $\{e_1, e_2, e_3\}$, there exists a vector field $D = D(s)$ such that the following is true:

$$\begin{cases} e_1'(s) = D \times e_1(s), \\ e_2'(s) = D \times e_2(s), \\ e_3'(s) = D \times e_3(s), \end{cases}$$

which is called the Darboux vector field of $r(s)$. From Proposition 1, the Darboux vector field of $r(s)$ is as follows:

$$D = D(s) = \kappa(s)e_1(s) - e_3(s). \tag{2}$$

Definition 2. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . The curve denoted by the Darboux vector field is called the centrode of $r(s)$.

In [11], the authors introduced the structure function and the representation formula for isotropic curves, namely, the following:

Proposition 2. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . Then $r(s)$ can be written as follows [11]:

$$r(s) = \frac{i}{2} \int f_s^{-1}((f^2 - 1), 2f, -i(f^2 + 1)) ds \tag{3}$$

for some non-constant analytic function $f(s)$, which is called the structure function of $r(s)$.

Proposition 3. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . Then the pseudo curvature $\kappa(s)$ can be expressed by the structure function $f(s)$ as follows [11]:

$$\kappa(s) = -(Sf)(s) = \frac{1}{2} \left(\frac{f_{ss}}{f_s} \right)^2 - \left(\frac{f_{ss}}{f_s} \right)'_s, \tag{4}$$

where $(Sf)(s)$ is the Schwarzian derivative of the structure function $f(s)$.

For an isotropic curve $r(s)$ framed by $\{e_1, e_2, e_3\}$ in \mathbb{C}^3 , the planes spanned by $\{e_1, e_2\}$, $\{e_1, e_3\}$ and $\{e_2, e_3\}$ are known as the osculating, the rectifying and the normal planes, respectively. In Euclidean 3-space, a curve is called a rectifying curve if its position vector always lies in its rectifying plane [1]. Naturally, we have the following definition.

Definition 3. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . If its position vector always lies in its rectifying plane, then $r(s)$ is called a rectifying isotropic curve.

Remark 2. Obviously, the position vector of a rectifying isotropic curve $r(s)$ can be expressed as follows:

$$r(s) = \lambda(s)e_1(s) + \mu(s)e_3(s),$$

where $\lambda(s)$ and $\mu(s)$ are nonzero analytic functions.

3. Rectifying Isotropic Curves

In this part, the properties and expression forms of rectifying isotropic curves are studied.

Theorem 1. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . Then $r(s)$ is congruent to a rectifying isotropic curve if, and only if, the pseudo curvature $\kappa(s)$ of $r(s)$ is a non-constant linear function of the pseudo arc length s , i.e., the following:

$$\kappa(s) = c_1s + c_2, \quad (0 \neq c_1, c_2 \in \mathbb{C}).$$

Proof. Let $r(s)$ be a rectifying isotropic curve framed by $\{e_1, e_2, e_3\}$. From Remark 2, it gives rise to the following:

$$r(s) = \lambda(s)e_1(s) + \mu(s)e_3(s) \tag{5}$$

for some nonzero analytic functions $\lambda(s)$ and $\mu(s)$.

Taking the derivative of both sides of (5) with respect to s , we get the following equation system:

$$\begin{cases} \lambda'(s) = 1, \\ \lambda(s) + \mu(s)\kappa(s) = 0, \\ \mu'(s) = 0. \end{cases}$$

From the above equation system, we notice that $\mu(s)$ is a nonzero constant denoted by a and $\lambda(s)$ is a linear function as $\lambda(s) = s + b$, ($b \in \mathbb{C}$). Thus, the pseudo curvature $\kappa(s)$ satisfies the following:

$$\kappa(s) = -\frac{s}{a} - \frac{b}{a} \triangleq c_1s + c_2, \quad (0 \neq c_1, c_2 \in \mathbb{C}).$$

Conversely, assume that $r(s)$ is an isotropic curve in \mathbb{C}^3 such that $\kappa(s) = c_1s + c_2$ for some constants $c_1 \neq 0$ and c_2 . By invoking the Frenet–Serret equation, we obtain the following:

$$\frac{d}{ds}[r(s) - (s + \frac{c_2}{c_1})e_1(s) + \frac{1}{c_1}e_3(s)] = 0,$$

from which and after appropriate translation, $r(s)$ is congruent to a rectifying isotropic curve. \square

Theorem 2. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . Then $r(s)$ is a rectifying isotropic curve if, and only if, one of the following statements holds:

- (1) $\langle r(s), r(s) \rangle = 2a(s + b)$;
- (2) the tangent component $\langle r(s), e_1(s) \rangle = a$;
- (3) the binormal component $\langle r(s), e_3(s) \rangle = s + b$,

where $a \neq 0$ and b are constants.

Proof. Let $r(s)$ be a rectifying isotropic curve framed by $\{e_1, e_2, e_3\}$. According to the proof of Theorem 1, we have the following:

$$r(s) = (s + b)e_1(s) + ae_3(s), \tag{6}$$

where $a \neq 0$ and b are constants.

Taking the scalar product of (6) with the tangent vector $e_1(s)$, the binormal vector $e_3(s)$ and itself successively, we obtain the following:

$$\begin{cases} \langle r(s), e_1(s) \rangle = a, \\ \langle r(s), e_3(s) \rangle = s + b, \\ \langle r(s), r(s) \rangle = 2a(s + b). \end{cases}$$

Conversely, let $r(s)$ be an isotropic curve in \mathbb{C}^3 which satisfies the following statement:

$$\langle r(s), r(s) \rangle = 2a(s + b) \tag{7}$$

for some constants $a \neq 0$ and b . Differentiating (7) twice with respect to s , we have $\langle r(s), e_1'(s) \rangle = 0$, therefore $\langle r(s), e_2(s) \rangle = 0$, which implies that $r(s)$ is a rectifying isotropic curve.

Continuously, suppose that $r(s)$ satisfies $\langle r(s), e_1(s) \rangle = a$. We can easily find that $\langle r(s), e_2(s) \rangle = 0$, thus $r(s)$ is a rectifying isotropic curve. Finally, assume that $r(s)$ satisfies the following:

$$\langle r(s), e_3(s) \rangle = s + b \tag{8}$$

for some constant b . Taking the derivative on both sides of (8) with respect to s , we have $\langle r(s), e_3'(s) \rangle = 0$, thus $\kappa \langle r(s), e_2(s) \rangle = 0$. Since $\kappa \neq 0$, then $\langle r(s), e_2(s) \rangle = 0$, which means $r(s)$ is a rectifying isotropic curve. \square

Next, we explore the expression forms of rectifying isotropic curves via the structure function of isotropic curves.

Theorem 3. Let $r(s)$ be a rectifying isotropic curve in \mathbb{C}^3 . Then $r(s)$ can be represented as follows:

$$r(s) = \frac{i}{2} \int u^2(s) \{ [c_0(\int \frac{1}{u^2(s)} ds)^2 - \frac{1}{c_0}], 2 \int \frac{1}{u^2(s)} ds, -i[c_0(\int \frac{1}{u^2(s)} ds)^2 + \frac{1}{c_0}] \} ds,$$

where

$$u(s) = \sqrt{s} \left[(C_1 + \frac{\sqrt{3}}{3}C_2) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} (\frac{\sqrt{2c}}{6} s^{\frac{3}{2}})^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} C_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} (\frac{\sqrt{2c}}{6} s^{\frac{3}{2}})^{2m - \frac{1}{3}} \right]$$

for some constants $C_1, C_2, cc_0 \neq 0$ and Gamma functions $\Gamma(m + \frac{4}{3}), \Gamma(m + \frac{2}{3})$.

Proof. Let $r(s)$ be a rectifying isotropic curve. From Theorem 1, we have $\kappa(s) = c_1s + c_2, (0 \neq c_1, c_2 \in \mathbb{C})$. Through appropriate transformation, we can let $c_2 = 0, c_1 = -c \neq 0$. Then, by Proposition 3, we have the following differential equation:

$$\left(\frac{f_{ss}}{f_s}\right)_s - \frac{1}{2}\left(\frac{f_{ss}}{f_s}\right)^2 = cs, \quad (0 \neq c \in \mathbb{C}). \tag{9}$$

Putting $\frac{f_{ss}}{f_s} = g(s)$ and substituting it into (9), we obtain

$$g'(s) = \frac{1}{2}g^2(s) + cs,$$

which is a Riccati equation. By taking the substitution $u(s) = e^{-\int \frac{1}{2}g(s)ds}$, the Riccati equation can be reduced to a second order linear equation, i.e., $u''(s) + \frac{c}{2}su(s) = 0$. Solving it, there is $u(s) = \sqrt{s} \left[C_1 J_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}) + C_2 Y_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}) \right]$, where $J_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}})$ and $Y_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}})$ are Bessel functions and C_1, C_2 are some constants. Hence, we have $g(s) = -\frac{2u'(s)}{u(s)}$, that is, the following:

$$\frac{f_{ss}}{f_s} = -\frac{2u'(s)}{u(s)}, \tag{10}$$

where $u(s) = \sqrt{s} \left[C_1 J_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}) + C_2 Y_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}) \right]$, $J_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}})$ and $Y_{\frac{1}{3}}(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}})$ are Bessel functions and C_1, C_2 are some constants.

Integrating on both sides of (10) with respect to s , we obtain the following:

$$f_s = \frac{c_0}{u^2(s)}, \quad (c_0 \neq 0), \tag{11}$$

substituting (11) into (3), we have the following

$$r(s) = \frac{i}{2} \int u^2(s) \left\{ [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 - \frac{1}{c_0}], 2 \int \frac{1}{u^2(s)} ds, -i [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 + \frac{1}{c_0}] \right\} ds.$$

Moreover, the Bessel function of first kind can be expressed as follows [13]:

$$J_{\frac{1}{3}}\left(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c}}{6}s^{\frac{3}{2}}\right)^{2m + \frac{1}{3}}$$

and

$$J_{-\frac{1}{3}}\left(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c}}{6}s^{\frac{3}{2}}\right)^{2m - \frac{1}{3}}.$$

By the relationship between the Bessel function of first kind and the Bessel function of second kind, there is the following:

$$Y_{\frac{1}{3}}\left(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}\right) = \frac{\sqrt{3}}{3} J_{\frac{1}{3}}\left(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}\right) - \frac{2\sqrt{3}}{3} J_{-\frac{1}{3}}\left(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}\right),$$

i.e.,

$$Y_{\frac{1}{3}}\left(\frac{\sqrt{2c}}{3}s^{\frac{3}{2}}\right) = \frac{\sqrt{3}}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c}}{6}s^{\frac{3}{2}}\right)^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c}}{6}s^{\frac{3}{2}}\right)^{2m - \frac{1}{3}}.$$

Thus, $u(s)$ can be rewritten as follows:

$$u(s) = \sqrt{s} \left[(C_1 + \frac{\sqrt{3}}{3}C_2) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} (\frac{\sqrt{2c}}{6} s^{\frac{3}{2}})^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} C_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} (\frac{\sqrt{2c}}{6} s^{\frac{3}{2}})^{2m - \frac{1}{3}} \right],$$

where $C_1, C_2, c \neq 0$ are constants and $\Gamma(m + \frac{4}{3}), \Gamma(m + \frac{2}{3})$ are Gamma functions. \square

Theorem 4. Let $r(s)$ be an isotropic curve in \mathbb{C}^3 . Then, it is a rectifying curve if, and only if, it can be represented as follows:

$$r(s) = \sqrt{2as}\psi(s), \quad (0 \neq a \in \mathbb{C}),$$

where $\psi(s)$ satisfies the following:

$$\begin{cases} \langle \psi(s), \psi(s) \rangle = 1, \\ \langle \psi'(s), \psi'(s) \rangle = -\frac{1}{4s^2}, \\ \langle \psi''(s), \psi''(s) \rangle = -\frac{3}{16s^4} - \frac{1}{2as}. \end{cases}$$

Proof. Let $r(s)$ be a rectifying isotropic curve framed by $\{e_1, e_2, e_3\}$. Based on (6), making appropriate transformation with $b = 0$, $r(s)$ can be expressed as follows:

$$r(s) = se_1(s) + ae_3(s).$$

Assuming that $\psi(s) = \frac{r(s)}{\sqrt{2as}}$, then the following is true:

$$r(s) = \sqrt{2as}\psi(s). \tag{12}$$

From now on, we study the properties of $\psi(s)$. Firstly, by differentiating (12) with respect to s , we have the following:

$$e_1(s) = \frac{a}{\sqrt{2as}}\psi(s) + \sqrt{2as}\psi'(s). \tag{13}$$

Taking the scalar product on both sides of (13) with $\psi(s), \psi'(s)$ and $e_1(s)$ respectively, we obtain, after arrangement, the following:

$$\langle \psi'(s), \psi'(s) \rangle = -\frac{1}{4s^2}. \tag{14}$$

Furthermore, taking the derivative of (13) with respect to s , then taking the scalar product with itself, we obtain the following:

$$-\langle e_2(s), e_2(s) \rangle = \frac{a}{(2s)^3} \langle \psi(s), \psi(s) \rangle - \frac{a}{s^2} \langle \psi(s), \psi'(s) \rangle - \frac{a}{s} \langle \psi(s), \psi''(s) \rangle + \frac{2a}{s} \langle \psi'(s), \psi'(s) \rangle + 4a \langle \psi'(s), \psi''(s) \rangle + 2as \langle \psi''(s), \psi''(s) \rangle,$$

which yields the following:

$$\langle \psi''(s), \psi''(s) \rangle = -\frac{3}{16s^4} - \frac{1}{2as}. \tag{15}$$

Conversely, let $r(s)$ be an isotropic curve in \mathbb{C}^3 with the form $r(s) = \sqrt{2as}\psi(s)$ for a nonzero constant a , and a curve $\psi(s)$ satisfies (14), (15) and $\langle \psi(s), \psi(s) \rangle = 1$. Through direct computations, we have the following:

$$r'(s) = \sqrt{2as} \left(\frac{1}{2s} \psi(s) + \psi'(s) \right). \tag{16}$$

Differentiating (16) together with (14) and (15), we can obtain $\langle r''(s), r''(s) \rangle = -1$, i.e., s is the pseudo arc length parameter of $r(s)$. Thus, $r'(s) = e_1(s)$ and $\langle r(s), e_1(s) \rangle = a$, that is, the tangent component of the position vector of $r(s)$ is a nonzero constant. From Theorem 2, $r(s)$ is a rectifying isotropic curve. \square

From Theorems 3 and 4, the following conclusion is straightforward.

Corollary 1. *Let $r(s)$ be a rectifying isotropic curve in \mathbb{C}^3 expressed as follows:*

$$r(s) = \sqrt{2as}\psi(s)$$

for nonzero constant a and analytic function $\psi(s)$ such that the following is true:

$$\begin{cases} \langle \psi(s), \psi(s) \rangle = 1, \\ \langle \psi'(s), \psi'(s) \rangle = -\frac{1}{4s^2}, \\ \langle \psi''(s), \psi''(s) \rangle = -\frac{3}{16s^4} - \frac{1}{2as}. \end{cases}$$

Then, $\psi(s)$ can be represented by the following:

$$\psi(s) = \frac{i}{\sqrt{8as}} \int u^2(s) \left\{ [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 - \frac{1}{c_0}], 2 \int \frac{1}{u^2(s)} ds, -i [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 + \frac{1}{c_0}] \right\} ds,$$

where

$$u(s) = \sqrt{s} \left[\left(C_1 + \frac{\sqrt{3}}{3} C_2 \right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c}}{6} s^{\frac{3}{2}} \right)^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} C_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c}}{6} s^{\frac{3}{2}} \right)^{2m - \frac{1}{3}} \right],$$

$C_1, C_2, c_0 \neq 0 \in \mathbb{C}$, $\Gamma(m + \frac{4}{3})$ and $\Gamma(m + \frac{2}{3})$ are Gamma functions.

In the following, we reparameterize $r(s)$ with a proper parameter t , such that $\frac{dt}{ds} := \frac{i}{2s}$, i.e., the following:

$$t = \frac{i}{2} \ln s + c_1$$

for some constant c_1 . Without loss of generality, we can let $c_1 = 0$, then $s = e^{-2it}$. By the Euler formula, we have the following:

$$s = \cos 2t - i \sin 2t.$$

From Theorem 4, $r(s)$ can be expressed by parameter t as follows:

$$r(t) = \sqrt{2a}(\cos t - i \sin t)\psi(t),$$

where $\langle \psi(t), \psi(t) \rangle = 1$.

Furthermore, through direct calculations, we have the following:

$$\begin{cases} \psi'(s) = \psi'(t) \frac{dt}{ds} = \frac{i}{2s} \psi'(t), \\ \psi''(s) = \psi''(t) \left(\frac{dt}{ds} \right)^2 + \psi'(t) \frac{d^2t}{ds^2} = -\frac{\psi''(t)}{4s^2} - \frac{i\psi'(t)}{2s^2}. \end{cases}$$

Since $\langle \psi'(s), \psi'(s) \rangle = -\frac{1}{4s^2}$, we know $\langle \psi'(t), \psi'(t) \rangle = 1$. What is more, from $\langle \psi''(s), \psi''(s) \rangle = -\frac{3}{16s^4} - \frac{1}{2as}$, we have the following:

$$\langle \psi''(t), \psi''(t) \rangle = 1 - \frac{8}{a}e^{-i6t} = 1 - \frac{8}{a}(\cos 6t - i \sin 6t).$$

Theorem 5. Let $r(t)$ be an isotropic curve with proper parameter t in \mathbb{C}^3 . Then, it is a rectifying isotropic curve if, and only if, it can be represented as follows:

$$r(t) = \sqrt{2a}(\cos t - i \sin t)\psi(t), \quad (0 \neq a \in \mathbb{C}),$$

where $\psi(t)$ satisfies the following:

$$\begin{cases} \langle \psi(t), \psi(t) \rangle = \langle \psi'(t), \psi'(t) \rangle = 1, \\ \langle \psi''(t), \psi''(t) \rangle = 1 - \frac{8}{a}(\cos 6t - i \sin 6t). \end{cases}$$

From Theorems 3 and 5, the following conclusion is straightforward.

Corollary 2. Let $r(t)$ be a rectifying isotropic curve in \mathbb{C}^3 expressed as follows:

$$r(t) = \sqrt{2a}(\cos t - i \sin t)\psi(t)$$

for nonzero constant a and analytic function $\psi(t)$ such that the following is true:

$$\begin{cases} \langle \psi(t), \psi(t) \rangle = \langle \psi'(t), \psi'(t) \rangle = 1, \\ \langle \psi''(t), \psi''(t) \rangle = 1 - \frac{8}{a}(\cos 6t - i \sin 6t). \end{cases}$$

Then $\psi(t)$ can be represented by the following:

$$\begin{aligned} \psi(t) = & \frac{e^{it}}{\sqrt{2a}} \int \frac{u^2(t)}{e^{2it}} \left\{ [-4c_0 \left(\int \frac{1}{e^{2it}u^2(t)} dt \right)^2 - \frac{1}{c_0}], -4i \int \frac{1}{e^{2it}u^2(t)} dt, \right. \\ & \left. i[4c_0 \left(\int \frac{1}{e^{2it}u^2(t)} dt \right)^2 - \frac{1}{c_0}] \right\} dt, \end{aligned}$$

where

$$\begin{aligned} u(t) = & e^{-it} \left[\left(C_1 + \frac{\sqrt{3}}{3} C_2 \right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c}}{6e^{3it}} \right)^{2m + \frac{1}{3}} - \right. \\ & \left. \frac{2\sqrt{3}}{3} C_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c}}{6e^{3it}} \right)^{2m - \frac{1}{3}} \right], \end{aligned}$$

$C_1, C_2, c_0 \neq 0 \in \mathbb{C}$, $\Gamma(m + \frac{4}{3})$ and $\Gamma(m + \frac{2}{3})$ are Gamma functions.

4. The Centroides of Rectifying Isotropic Curves

Suppose that $r(s)$ is an isotropic curve framed by $\{e_1(s), e_2(s), e_3(s)\}$ with the pseudo curvature $\kappa(s)$. Then from Definition 2, the centroide $D(s)$ of $r(s)$ is as follows:

$$D(s) = \kappa(s)e_1(s) - e_3(s). \tag{17}$$

Rearranging (2), (17) can be rewritten as follows:

$$D(s) = 2\kappa(s)e_1(s) + ie'_2(s). \tag{18}$$

Differentiating (3) three times with respect to s , we have the following:

$$\begin{cases} r' = e_1, \\ r'' = -ie_2, \\ r''' = -ie'_2. \end{cases}$$

Then by simple calculations and arrangements, we can make out the following:

$$\begin{cases} e_1 = \frac{i}{2}f_s^{-1}(f^2 - 1, 2f, -i(f^2 + 1)), \\ e'_2 = \left(\frac{1}{2}f_s^{-2}f_{sss} - f_s^{-3}f_{ss}^2 \right) (f^2 - 1, 2f, -i(f^2 + 1)) + \\ f_s^{-1}f_{ss}(f, 1, -if) - (f_s, 0, -if_s). \end{cases} \tag{19}$$

Substituting (4) and (19) into (18), we can rewrite $D(s)$ with the structure function f as the following conclusion.

Theorem 6. Let $r(s)$ be an isotropic curve with the structure function f in \mathbb{C}^3 and $D(s)$ its centrode. Then, the centrode $D(s)$ can be expressed as follows:

$$D(s) = \frac{f_{ss}^2 - f_s f_{sss}}{2f_s^3} (i(f^2 - 1), 2if, f^2 + 1) + \frac{f_{ss}}{f_s} (if, i, f) - (if_s, 0, f_s).$$

By differentiating (17) with respect to s , we have $D'(s) = \kappa'(s)e_1(s)$ and $\langle D'(s), D'(s) \rangle = 0$. Hence, we have Theorem 7.

Theorem 7. The centrode of an isotropic curve is an isotropic curve.

Assume that the centrode $D(s)$ is reparameterized by the pseudo arc-length $s^D = s^D(s)$ and framed by $\{e_1^D, e_2^D, e_3^D\}$. Differentiating (17) twice with respect to s , we have the following:

$$\begin{cases} e_1^D \frac{ds^D}{ds} = \kappa' e_1, \\ -ie_2^D \left(\frac{ds^D}{ds} \right)^2 + e_1^D \frac{d^2s^D}{ds^2} = \kappa'' e_1 - i\kappa' e_2. \end{cases} \tag{20}$$

Taking the scalar product on both sides of the second equation in (20) with itself, we have the following:

$$\frac{ds^D}{ds} = \sqrt{\kappa'}, \tag{21}$$

from which κ is non-constant and (20) can be rewritten as follows:

$$\begin{cases} e_1^D = \sqrt{\kappa'} e_1(s), \\ e_2^D = \frac{i\kappa''}{2\kappa'} e_1 + e_2. \end{cases} \tag{22}$$

Differentiating the second equation in (22) with respect to s , we easily obtain the following:

$$(\kappa^D e_1^D + e_3^D) \frac{ds^D}{ds} = \left[\frac{1}{2} \left(\frac{\kappa''}{\kappa'} \right)' + \kappa \right] e_1 - \frac{i\kappa''}{2\kappa'} e_2 + e_3, \tag{23}$$

where κ^D is the pseudo curvature of the centrode $D(s)$. Taking the scalar product on both sides of (23) with itself, then substituting (21) into it, we have the following:

$$\kappa^D = \frac{\kappa}{\kappa'} + \frac{1}{2\kappa'} \left(\frac{\kappa''}{\kappa'} \right)' - \frac{1}{8\kappa'} \left(\frac{\kappa''}{\kappa'} \right)^2. \tag{24}$$

Substituting (21), (22) and (24) into (23), we have the following:

$$e_3^D = \frac{1}{8\sqrt{\kappa'}} \left(\frac{\kappa''}{\kappa'} \right)^2 e_1 - \frac{i\kappa''}{2(\kappa')^{\frac{3}{2}}} e_2 + \frac{1}{\sqrt{\kappa'}} e_3.$$

From the deduction process above, we can get the following conclusions.

Theorem 8. Let $r(s)$ be an isotropic curve with the pseudo curvature κ framed by $\{e_1, e_2, e_3\}$ in \mathbb{C}^3 , $D(s)$ its centrode framed by $\{e_1^D, e_2^D, e_3^D\}$. Then, they satisfy the following:

$$\begin{pmatrix} e_1^D \\ e_2^D \\ e_3^D \end{pmatrix} = \begin{pmatrix} (\kappa')^{\frac{1}{2}} & 0 & 0 \\ i(2\kappa')^{-1}\kappa'' & 1 & 0 \\ \frac{1}{8}(\kappa')^{-\frac{5}{2}}(\kappa'')^2 & -\frac{i}{2}(\kappa')^{-\frac{3}{2}}\kappa'' & (\kappa')^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Theorem 9. Let $r(s)$ be an isotropic curve with the pseudo curvature κ in \mathbb{C}^3 , $D(s)$ its centrode with the pseudo curvature κ^D . Then, they satisfy the following:

$$\kappa^D = \frac{\kappa}{\kappa'} + \frac{1}{2\kappa'}\left(\frac{\kappa''}{\kappa'}\right)' - \frac{1}{8\kappa'}\left(\frac{\kappa''}{\kappa'}\right)^2.$$

From now on, the centrode of a rectifying isotropic curve is discussed. Suppose that $r(s)$ is a rectifying isotropic curve with the pseudo curvature κ and the structure function f , $D(s)$ is the centrode of $r(s)$ with the pseudo curvature κ^D and the structure function f^D .

From Theorem 1, we have $\kappa(s) = c_1s + c_2$, ($0 \neq c_1, c_2 \in \mathbb{C}$). Through appropriate transformation, we can let $c_2 = 0$, i.e., $\kappa(s) = c_1s$, ($0 \neq c_1 \in \mathbb{C}$). Substituting it into (24), we obtain the following:

$$\kappa^D = s. \tag{25}$$

Taking (4) into consideration together with (25), we have the following:

$$\frac{1}{2} \left(\frac{f_{s^D s^D}^D}{f_{s^D}^D} \right)^2 - \left(\frac{f_{s^D s^D}^D}{f_{s^D}^D} \right)_{s^D} = s. \tag{26}$$

Putting $\frac{f_{s^D s^D}^D}{f_{s^D}^D} = g^D(s)$ and substituting it into (26), we obtain the following:

$$(g^D)' = \left(\frac{1}{2}(g^D)^2 - s \right) \frac{ds^D}{ds}. \tag{27}$$

Substituting (21) into (27), we have

$$(g^D)' = \frac{\sqrt{c_1}}{2}(g^D)^2 - \sqrt{c_1}s, \quad (0 \neq c_1 \in \mathbb{C})$$

which is a Riccati equation. Solving this Riccati equation, we obtain the following:

$$g^D = -\frac{2u'(s)}{\sqrt{c_1}u(s)},$$

whence we have the following:

$$(\ln f_{s^D}^D)' = -\frac{2u'(s)}{u(s)}, \tag{28}$$

where $u(s) = \sqrt{s} \left[C_1 J_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) + C_2 Y_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) \right]$, $J_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right)$ and $Y_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right)$ are Bessel functions and $0 \neq c_1, C_1, C_2 \in \mathbb{C}$.

Integrating on both sides of (28) with respect to s , we obtain the following:

$$f_{s^D}^D = \frac{c_0}{u^2(s)}, \quad (0 \neq c_0 \in \mathbb{C}). \tag{29}$$

On the other hand, from Proposition 2, $D(s^D)$ can be expressed as follows:

$$D(s^D) = \frac{i}{2} \int (f_{s^D}^D)^{-1} ((f^D)^2 - 1, 2f^D, -i((f^D)^2 + 1)) ds^D. \tag{30}$$

Substituting (21) and (29) into (30), we have the following:

$$D(s) = \frac{i}{2\sqrt{c_1}} \int u^2(s) \left\{ [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 - \frac{1}{c_0}], 2 \int \frac{1}{u^2(s)} ds, -i [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 + \frac{1}{c_0}] \right\} ds.$$

Moreover, the Bessel function of first kind can be expressed as follows:

$$J_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m + \frac{1}{3}}$$

and

$$J_{-\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m - \frac{1}{3}}.$$

By the relationship between the Bessel function of the first kind and the Bessel function of the second kind, there is the following:

$$Y_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) = \frac{\sqrt{3}}{3} J_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) - \frac{2\sqrt{3}}{3} J_{-\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right),$$

i.e.,

$$Y_{\frac{1}{3}} \left(\frac{\sqrt{2c_1}}{3} s^{\frac{3}{2}} i \right) = \frac{\sqrt{3}}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m - \frac{1}{3}}.$$

Thus, $u(s)$ can be rewritten as follows:

$$u(s) = \sqrt{s} \left[\left(C_1 + \frac{\sqrt{3}}{3} C_2 \right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} C_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m - \frac{1}{3}} \right],$$

where $C_1, C_2, c_1 \neq 0$ are some constants and $\Gamma(m + \frac{4}{3}), \Gamma(m + \frac{2}{3})$ are Gamma functions.

Theorem 10. Let $r(s)$ be a rectifying isotropic curve in \mathbb{C}^3 and $D(s)$ its centrode. Then, the centrode $D(s)$ can be written as follows:

$$D(s) = \frac{i}{2\sqrt{c_1}} \int u^2(s) \left\{ [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 - \frac{1}{c_0}], 2 \int \frac{1}{u^2(s)} ds, -i [c_0 \left(\int \frac{1}{u^2(s)} ds \right)^2 + \frac{1}{c_0}] \right\} ds,$$

where

$$u(s) = \sqrt{s} \left[\left(C_1 + \frac{\sqrt{3}}{3} C_2 \right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{4}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m + \frac{1}{3}} - \frac{2\sqrt{3}}{3} C_2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{2}{3})} \left(\frac{\sqrt{2c_1}}{6} s^{\frac{3}{2}} i \right)^{2m - \frac{1}{3}} \right]$$

for some constants $C_1, C_2, c_0 c_1 \neq 0$ and Gamma functions $\Gamma(m + \frac{4}{3}), \Gamma(m + \frac{2}{3})$.

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