Article

Lagrangian Formulation, Conservation Laws, Travelling Wave Solutions: A Generalized Benney-Luke Equation

Sivenathi Oscar Mbusi [1], Ben Muatjetjeja [1,2] and Abdullahi Rashid Adem [3,*]

[1] Departement of Mathematical Sciences, North-West University, Private Bag X 2046, Mmabatho 2735, South Africa; siveoscar@gmail.com (S.O.M.); muatjetjejab@ub.ac.bw (B.M.)
[2] Department of Mathematics, Faculty of Science, University of Botswana, Private Bag 22, Gaborone, Botswana
[3] Department of Mathematical Sciences, University of South Africa, Pretoria 0003, South Africa
[*] Correspondence: ademar@unisa.ac.za

Abstract: The aim of this paper is to find the Noether symmetries of a generalized Benney-Luke equation. Thereafter, we construct the associated conserved vectors. In addition, we search for exact solutions for the generalized Benney-Luke equation through the extended tanh method. A brief observation on equations arising from a Lagrangian density function with high order derivatives of the field variables, is also discussed.

Keywords: Noether symmetries; conservation laws; Benney-Luke equation; extended tanh method

1. Introduction

In 1964, D.J. Benney and J.C. Luke obtained the following equation [1]

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0. \tag{1}$$

Here α, β are positive constants. Benney-Luke Equation (1) models waves propagating on the surface of a fluid in a shallow channel of constant depth taking into consideration the surface tension effect. The Benney-Luke equation and its generalizations have been extensively investigated [2–6]. The approaches used in the investigation of nonlinear evolution equations include stability analysis, Cauchy problem, existence and analyticity of solutions, etc. We refer the interested reader to references [7–12] and references therein. However in this present work, our goal is to compute conservation laws and exact solutions of Equation (1).

The organization of this paper is twofold. Firstly, we compute conservation laws of Equation (1) by employing Noether's theorem. Secondly, we search for exact solutions of (1) by using the extended tanh method.

2. Noether Symmetries and Conservation Laws

In this section, we recall some essential notions on Noether symmetries and the associated conservation laws. The interested reader is referred to the cited paper for details [13,14].

Let us consider the vector field

$$\mathbf{X} = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{2}$$

with the second-order prolongation

$$\mathbf{X}^{[2]} = \mathbf{X} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{tx}\frac{\partial}{\partial u_{tx}}, \tag{3}$$

where the coefficients of the partial derivatives are defined in [13,14].

The Euler-Lagrange operator is defined by

$$\frac{\delta}{\delta u} \quad = \quad \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_t D_x \frac{\partial}{\partial u_{tx}} + \cdots . \tag{4}$$

2.1. Euler-Lagrange Equation

Let us consider a partial differential equation

$$\Psi(t, x, u, u_t, u_x, u_{tx}, u_{xx}, \ldots) = 0. \tag{5}$$

A function $L(t, x, u, u_t, u_x, u_{tx}, u_{xx})$ is called a Lagrangian of Equation (5) if satisfies the Euler-Lagrange equation given by

$$\frac{\delta L}{\delta u} = 0. \tag{6}$$

2.2. Noether Symmetry

The vector field **X**, defined in Equation (2), is a Noether symmetry corresponding to the Lagrangian $L(t, x, u, u_t, u_x, u_{tx}, u_{xx})$ of Equation (5) if there exist the gauge terms $B^1(t, x, u)$ and $B^2(t, x, u)$ such that

$$\mathbf{X}^{[2]}(L) + \{D_t(\tau) + D_x(\xi)\}L = D_t(B^1) + D_x(B^2). \tag{7}$$

2.3. Noether's Theorem

To every Noether symmetry **X** corresponding to the Lagrangian L of Equation (6), there corresponds a vector $\mathbf{T} = (T^1, T^2)$ with components

$$
\begin{aligned}
T^1 \quad = \quad & L\tau + (\eta - u_t\tau - u_x\xi)\left[\left(\frac{\partial L}{\partial u_t}\right) - D_t\left(\frac{\partial L}{\partial u_{tt}}\right)\right] + (\zeta_1 - u_{tt}\tau - u_{tx}\xi)\left[\left(\frac{\partial L}{\partial u_{tt}}\right)\right] \\
& + (\zeta_2 - u_{xt}\tau - u_{xx}\xi)\left[\left(\frac{\partial L}{\partial u_{tx}}\right)\right] - B^1,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
T^2 \quad = \quad & L\xi + (\eta - u_t\tau - u_x\xi)\left[\left(\frac{\partial L}{\partial u_x}\right) - D_t\left(\frac{\partial L}{\partial u_{tx}}\right) - D_x\left(\frac{\partial L}{\partial u_{xx}}\right)\right] \\
& + (\zeta_2 - u_{xt}\tau - u_{xx}\xi)\left[\left(\frac{\partial L}{\partial u_{xx}}\right)\right] - B^2.
\end{aligned}
\tag{9}
$$

2.4. Conservation Laws

We observe that the generalized Benney-Luke Equation (1) has a variational structure.

Lemma 1. *The generalized Benney-Luke Equation (1) satisfies the Euler-Lagrange equations with the Lagrangian function defined by*

$$L = \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 - \frac{1}{2}u_t u_x^2. \tag{10}$$

Proof. Substituting L from Equation (10) into Equation (7) and splitting the monomials yields a linear overdetermined system of partial differential equations. Solving the resulting system of PDEs, Equation (7) admits only the variational Noether symmetries: time translation, space translation, and phase translation. Thus, the application of Noether's theorem leads to the nontrivial conserved vector associated with three Noether point symmetries,

$$
\begin{aligned}
T_1^1 &= \frac{1}{2}u_x^2 + \frac{1}{2}u_t^2 + \frac{1}{2}\alpha u_{xx}^2 + \frac{1}{2}\beta u_{tx}^2, \\
T_1^2 &= -u_t u_x + u_t^2 u_x - \beta u_t u_{ttx} + \alpha u_t u_{xxx} - \alpha u_{tx}u_{xx},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
T_2^1 &= u_x u_t + \frac{1}{2}u_x^3 + \beta u_{tx}u_{xx}, \\
T_2^2 &= -\frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \frac{1}{2}\alpha u_{xx}^2 - \frac{1}{2}\beta u_{tx}^2 + \frac{1}{2}u_t u_x^2 - \beta u_x u_{ttx} + \alpha u_x u_{xxx},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
T_3^1 &= -u_t - \frac{1}{2}u_x^2, \\
T_3^2 &= u_x - u_t u_x + \beta u_{ttx} - \alpha u_{xxx},
\end{aligned}
\tag{13}
$$

respectively. $\square$

Note 1: It is worth pointing out that for equations that admit Lagrangian density depending on higher order derivatives of the field variables, the computations become a bit thorny, this is not because of conceptual complications, but somewhat due to notational difficulties. One must be very observant in expanding the multiple summations to avoid double counting of the appropriate jet coordinates. Nevertheless, variables such as u_{tx} or u_{xt} referred to as mixed derivatives are the same [14]. If one performs some double counting of the jet coordinates then this will leads to incorrect conserved vectors [14]. We now examine and discuss the physical meaning of the derived conservation laws. To begin with, conservation laws are of undisputed significance. They are the keystone for every essential theory of many physical phenomena. In this studying, we notice that the conserved vectors (11) yield a conserved energy because it arises from the time translation symmetry which leaves the Lagrangian (10) invariant. We also notice that the conserved vectors (12) yield a conserved momentum because it arises from the space translation symmetry which leaves the Lagrangian (10) invariant. We further observe that the Lagrangian (10) is invariant under the phase translation symmetry and this yields to charge conserved vectors.

3. Exact Solutions Using the Extended Tanh Method

In this section, we use the extended tanh function method introduced by Wazwaz [15]. We use the following ansatz:

$$
u(x,t) = F(z), \quad z = x - \omega t.
\tag{14}
$$

Making use of (14), Equation (1) is reduced to the following nonlinear ordinary differential equation:

$$
\alpha F''''(z) - \beta \omega^2 F''''(z) + \omega^2 F''(z) - F''(z) - 3\omega F'(z)F''(z) = 0.
\tag{15}
$$

The basic idea in this method is to assume that the solution of (15) can be written in the form

$$
F(z) = \sum_{i=-M}^{M} A_i H(z)^i,
\tag{16}
$$

where $H(z)$ satisfies an auxiliary equation, for example the Riccati equation

$$
H'(z) = 1 - H^2(z),
\tag{17}
$$

whose solution is given by

$$
H(z) = \tanh(z).
\tag{18}
$$

The positive integer M is be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (15). Here A_i are parameters to be determined. In our case, the balancing procedure gives $M = 1$ and so the solutions of (15) are of the form

$$F(z) = A_{-1}H^{-1} + A_0 + A_1H. \tag{19}$$

Substituting (19) into (15) and making use of the Riccati Equation (17) and then equating the coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_i. Solving the resultant system of algebraic equations, with the aid of Mathematica, leads to the following three cases (see Figures 1–3):

Case 1

$$\omega = k, \tag{20a}$$

$$A_{-1} = 0, \tag{20b}$$

$$A_1 = -\frac{-4\,k\alpha + 4\,k\beta}{4\,\alpha - 1}; \tag{20c}$$

Case 2

$$\omega = k, \tag{21a}$$

$$A_{-1} = -\frac{-4\,k\alpha + 4\,k\beta}{4\,\alpha - 1}, \tag{21b}$$

$$A_1 = 0; \tag{21c}$$

Case 3

$$\omega = p, \tag{22a}$$

$$A_{-1} = -\frac{-4\,p\alpha + 4\,p\beta}{16\,\alpha - 1}, \tag{22b}$$

$$A_1 = -\frac{-4\,p\alpha + 4\,p\beta}{16\,\alpha - 1}, \tag{22c}$$

where k and p are any roots of $(4\,\beta - 1)k^2 - 4\,\alpha + 1 = 0$ and $(-1 + 16\,\beta)p^2 - 16\,\alpha + 1 = 0$ respectively. As a result, a solution of (1) is

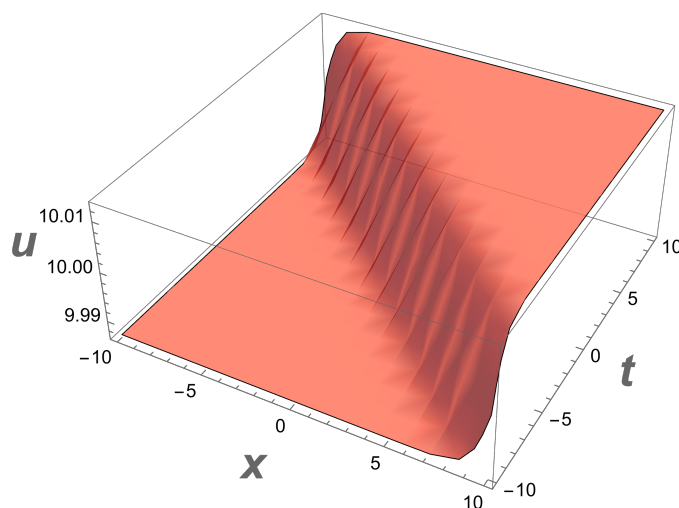$$u(x,t) = A_{-1}\coth(z) + A_0 + A_1\tanh(z), \tag{23}$$

where $z = x - \omega t$.

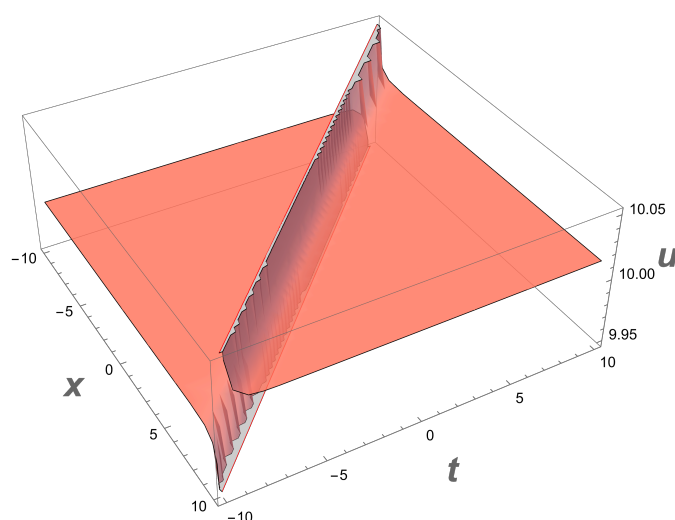**Figure 1.** Evolution of the solution of (20) for case 1.

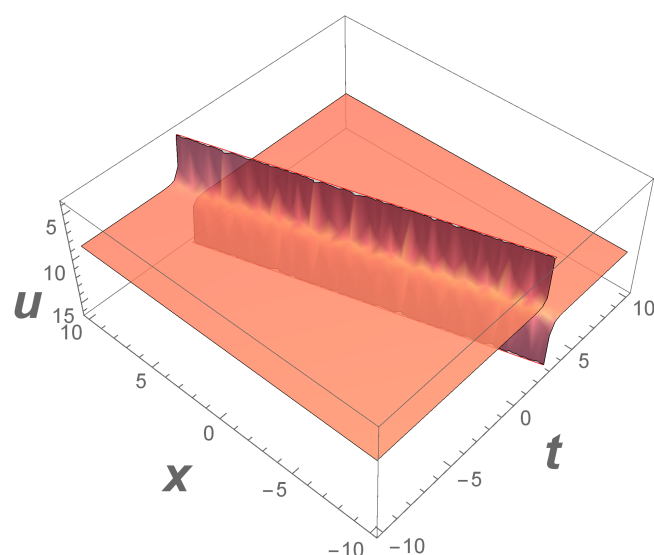**Figure 2.** Evolution of the solution of (21) for case 2.

Figure 3. Evolution of the solution of (22) for case 3.

Note 2: It should be noted that the three cases given by the extended tanh method rise to kink solutions [15–17].

4. Concluding Remarks

The Noether symmetries of a generalized Benney-Luke equation were computed. Thereafter, we constructed the associated conservation laws. Moreover, we derived exact solutions for the generalized Benney-Luke equation via the extended tanh method. An observation on equations arising from a Lagrangian density function with high order derivatives of the field variables was also elaborated. It is expected that the derived conservation laws and exact solutions computed in this paper can be used as yardsticks for numerical simulations of the underlying equation and these will be reported elsewhere.

Author Contributions: Conceptualization, B.M.; Formal analysis, S.O.M.; Validation, A.R.A. All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

References

1. Benney, D.J.; Luke, J.C. Interactions of permanent waves of finite amplitude. *J. Math. Phys.* **1964**, *43*, 309–313. [CrossRef]
2. Carlos, J.; Grajales, M. Instability and long-time evolution of cnoidal wave solutions for a Benney–Luke equation. *Internat. J. Non-Linear Mech.* **2009**, *44*, 999–1010.
3. Wang, S.; Xu, G.; Chen, G. Cauchy problem for the generalized Benney-Luke equation. *J. Math. Phys.* **2007**, *48*, 073521. [CrossRef]
4. Li, J.B. Exact traveling wave solutions to 2D-generalized Benney-Luke equation. *Appl. Math. Mech.* **2008**, *29*, 1391–1398. [CrossRef]
5. Quintero, J.R. Nonlinear stability of solitary waves for a 2-D Benney-Luke equation. *Discrete Contin. Dyn. Syst.* **2005**, *13*, 203–218. [CrossRef]
6. Quintero, J.R.; Grajales, J.C.M. Instability of solitary waves for a generalized Benney-Luke equation. *Nonlinear Anal.* **2008**, *68*, 3009–3033. [CrossRef]
7. Zhu, J.T.; Lü, X.; Cheng, J.H. Finding the interaction solutions to the dimensionally reduced equations based on computerized symbolic computation. *Modern Phys. Lett. B* **2020**, *34*, 2050157. [CrossRef]
8. Wazwaz, A.M. Painlevé analysis for Boiti-Leon-Manna-Pempinelli equation of higher dimensions with time-dependent coefficients: Multiple soliton solutions. *Phys. Lett. A* **2020**, *384*, 126310. [CrossRef]
9. Wazwaz, A.M. New (3+1)-dimensional Date-Jimbo-Kashiwara-Miwa equations with constant and time-dependent coefficients: Painlevé integrability. *Phys. Lett. A* **2020**, *384*, 126787. [CrossRef]
10. Lü, X.; Chen, S.J. Interaction solutions to nonlinear partial differential equations via Hirota bilinear forms: One-lump-multi-stripe and one-lump-multi-soliton types. *Nonlinear Dyn.* **2021**, *103*, 947–977. [CrossRef]
11. Wazwaz, A.M. New integrable (2+1)-and (3+1)-dimensional sinh-Gordon equations with constant and time-dependent coefficients. *Phys. Lett. A* **2020**, *384*, 126529. [CrossRef]
12. He, X.J.; Lü, X.; Li, M.G. Bäcklund transformation, Pfaffian, Wronskian and Grammian solutions to the (3+1)-dimensional generalized Kadomtsev–Petviashvili equation. *Anal. Math. Phys.* **2020**, *11*, 4. [CrossRef]
13. Noether, E. Invariante Variationsprobleme. *König Gesell Wissen Göttingen Math-Phys Kl Heft* **1918**, *2*, 235–257.
14. Adem, A.R.; Muatjetjeja, B. Conservation laws and exact solutions for a 2D Zakharov–Kuznetsov equation. *Appl. Math. Lett.* **2015**, *48*, 109–117. [CrossRef]
15. Wazwaz, A.M. New solitary wave solutions to the Kuramoto-Sivashinsky and the Kawahara equations. *Appl. Math. Comput.* **2006**, *182*, 1642–1650. [CrossRef]
16. Wazwaz, A.M. The extended tanh method for the Zakharov-Kuznetsov (ZK) equation, the modified ZK equation, and its generalized forms. *Commun. Nonlinear Sci. Numer. Simul.* **2008**, *13*, 1039–1047. [CrossRef]
17. Wazwaz, A.M. The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations. *Appl. Math. Comput.* **2007** , *184*, 1002–1014. [CrossRef]