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Iterative Design for the Common Solution of Monotone Inclusions and Variational Inequalities

Li Wei [1,*], Xin-Wang Shen [1] and Ravi P. Agarwal [2]

[1] School of Mathematics and Statistics, Hebei University of Economics and Business, Shijiazhuang 050061, China; stfanshuxin@heuet.edu.cn
[2] Department of Mathematics, Texas A & M University-Kingsville, Kingsville, TX 78363, USA; Ravi.Agarwal@tamuk.edu
* Correspondence: diandianba@yahoo.com or stweili@heuet.edu.cn

Abstract: Some new forward–backward multi-choice iterative algorithms with superposition perturbations are presented in a real Hilbert space for approximating common solution of monotone inclusions and variational inequalities. Some new ideas of constructing iterative elements can be found and strong convergence theorems are proved under mild restrictions, which extend and complement some already existing work.

Keywords: monotone inclusions; multi-choice iterative algorithm; θ-inversely strongly monotone operator; superposition perturbation;variational inequality

MSC: 47H05; 47H09

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1. Introduction and Preliminaries

Let C be a non-empty closed and convex subset of a real Hilbert space H. Symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner-product of H, respectively. Symbols $\to$ and $\rightharpoonup$ denote the strong and weak convergence in H, respectively.

The classical variational inequality [1] is to find $u \in C$ such that for any $v \in C$,

$$\langle v-u, Tu\rangle \geq 0, \tag{1}$$

where $T : C \to H$ is a nonlinear mapping. We use $VI(C,T)$ to denote the set of solutions of the variational inequality (1).

The theory of variational inequality draws much attention of mathematicians due to its wide application in several branches of pure and applied sciences [1]. Until now, it is still a hot topic (see [2–6] and the references therein).

An operator $A : D(A) \subset H \to 2^H$ is called monotone ([7]) if for each $x, y \in D(A)$, there exist $u \in Ax$ and $v \in Ay$ such that $\langle x-y, u-v\rangle \geq 0$. The monotone operator A is called maximal monotone if $R(I+kA) = H$, for any $k > 0$. In a Hilbert space, a maximal monotone operator can also be called an m-accretive mapping.

A mapping $B : D(B) \subset H \to H$ is called a θ-inversely strongly monotone operator ([8]) if for each $x, y \in D(B)$ and $\theta > 0$, $\langle x-y, Bx-By\rangle \geq \theta\|Bx-By\|^2$.

Let $U : D(U) \subset H \to H$ be a mapping. If $x \in D(U)$ and $Ux = 0$, then x is called a zero point of U. The set of zero points of U is denoted by $N(U)$. If $x \in D(U)$ satisfies that $Ux = x$, then x is called a fixed point of U. The set of fixed points of U is denoted by $F(U)$.

The monotone inclusion problem is to find $u \in H$ such that

$$0 \in Au + Bu, \tag{2}$$

where $A : H \to 2^H$ is maximal monotone and $B : H \to H$ is θ-inversely strongly monotone. The study of monotone inclusions is a hot topic since quite a lot problems appear in minimization problem, convex programming, split feasibility problems, variational inequalities, inverse problem, and image processing can be modeled by it. The construction of iterative algorithms for approximating the solution of (2) has been considered (see [8–14] and the references therein). The forward–backward splitting iterative method is one of them, which means an iteration involves only A as the forward step and B as the backward step, not the sum $A + B$. The classical forward–backward splitting iterative method is as follows:

$$\begin{cases} x_1 \in H \textit{ chosen arbitrarily}, \\ x_{n+1} = (I + r_n A)^{-1}(x_n - r_n B x_n), \ \ n \in N. \end{cases}$$

Some of the related work can be seen in [9,10] (and the references therein).

Recall that $f : H \to H$ is called a contraction with contractive constant k ([15]) if $k \in (0, 1)$ is that $\|f(x) - f(y)\| \leq k\|x - y\|$ for $x, y \in H$.

A mapping $S : H \to H$ is called non-expansive ([15]) if $\|Sx - Sy\| \leq \|x - y\|$, for $x, y \in H$.

A mapping $F : H \to H$ is called a strongly positive mapping with ξ ([15]) if $\xi > 0$ such that $\langle x, Fx\rangle \geq \xi\|x\|^2$ for $x \in H$. Furthermore,

$$\|aI - bF\| = sup_{\|x\|\leq 1}|\langle (aI - bF)x, x\rangle|,$$

where I is the identity mapping, $a \in [0, 1]$, and $b \in [-1, 1]$.

In [15], the study of monotone inclusion (2) is extended to the system of monotone inclusions:

$$0 \in A_i u + B_i u, \tag{3}$$

for $i \in N$, where $A_i : H \to H$ is maximal monotone and $B_i : H \to H$ is θ_i-inversely strongly monotone, for $i \in N$.

Moreover, the iterative algorithm presented in [15] is proved to be strongly convergent to not only the solution of monotone inclusions (3) but also the solution of one kind variational inequality. Specially, the authors constructed the following one by combining the ideas of the splitting method and the midpoint method:

$$\begin{cases} x_0 \in C \subset H \textit{ chosen arbitrarily}, \\ y_n = P_C[(1 - \alpha_n)(x_n + e'_n)], \\ z_n = \delta_n y_n + \beta_n \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1}(I - r_{n,i} B_i)(\frac{y_n + z_n}{2}) + \zeta_n e''_n, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n F) z_n + e'''_n, \ n \in N, \end{cases} \tag{4}$$

where f is a contraction, F is a strongly positive linear bounded mapping, and P_C is the metric projection. Under some conditions, $x_n \to p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$ and p_0 solves the following variational inequality:

$$\langle F p_0 - \eta f(p_0), p_0 - z\rangle, \ \ \forall z \in \bigcap_{i=1}^{\infty} N(A_i + B_i). \tag{5}$$

Recall that $T : D(T) \subset H \to H$ is called ϑ-strongly monotone ([16]) if for each $x, y \in D(T)$,

$$\langle Tx - Ty, x - y\rangle \geq \vartheta\|x - y\|^2,$$

for some $\vartheta \in (0, +\infty)$. Furthermore, $T : D(T) \subset H \to H$ is called μ-strictly pseudo-contractive ([16]) if for each $x, y \in D(T)$,

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \mu\|x - y - (Tx - Ty)\|^2,$$

for some $\mu \in (0, 1)$.

In 2012, Ceng et al. proposed an iterative algorithm with a perturbed operator for approximating a zero point of the maximal monotone operator A in a Hilbert space ([16]).

$$\begin{cases} x_1 \in H \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)(I + r_n A)^{-1} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)[(I + r_n A)^{-1} y_n - \lambda_n \mu_n T((I + r_n A)^{-1} y_n)], \ n \in N, \end{cases} \tag{6}$$

where $T : H \to H$ is a ϑ-strongly monotone and μ-strictly pseudo-contractive mapping with $\vartheta + \mu > 1$, $f : H \to H$ is a contraction, and $A : H \to H$ is maximal monotone. Under some assumptions, $\{x_n\}$ is proved to be convergent strongly to the unique element $p_0 \in N(A)$, which solves the following variational inequality:

$$\langle p_0 - f(p_0), p_0 - u \rangle \leq 0, \ \ \forall u \in N(A). \tag{7}$$

The mapping T, which is called a perturbed operator, only plays a role in the construction of the iterative algorithm (6) for selecting a particular zero point of A, but it is not involved in the variational inequality (7).

Later, in 2017, the work in (6) is extended to approximate the solutions of the systems of monotone inclusions (3). The following is a special case in Hilbert space presented in [17]:

$$\begin{cases} x_1 \in C, \\ u_n = P_C(\alpha_n x_n + \beta_n a_n), \\ v_n = \tau_n u_n + \nu_n \sum_{i=1}^{\infty} \omega_i^{(2)} (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i)(\frac{u_n + v_n}{2}) + \xi_n b_n, \\ x_{n+1} = \delta_n f(x_n) \\ +(1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} \omega_i^{(1)} T_i) \sum_{i=1}^{\infty} \omega_i^{(2)} (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i)(\frac{u_n + v_n}{2}), \ n \in N. \end{cases} \tag{8}$$

In (8), $\sum_{i=1}^{\infty} \omega_i^{(1)} T_i$ is called a superposition perturbation, where $T_i : H \to H$ is perturbed operator in the sense of (6); that is, $T_i : H \to H$ is a ϑ_i-strongly monotone and μ_i-strictly pseudo-contractive mapping, for each $i \in N$.

The iterative sequence $\{x_n\}$ generated by (8) is proved to be strongly convergent to $p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, which solves the variational inequality:

$$\langle p_0 - f(p_0), p_0 - u \rangle \leq 0, \ \ \forall u \in \bigcap_{i=1}^{\infty} N(A_i + B_i). \tag{9}$$

In 2019, Wei et al., proposed some new iterative algorithms. The inertial forward–backward iterative algorithm for approximating the solution of monotone inclusions (3) in [18] is as follows:

$$\begin{cases} x_0, x_1 \in H \text{ chosen arbitrarily}, e_1 \in H \text{ is chosen arbitrarily}, \\ y_n = x_n + k_n(x_n - x_{n-1}), \\ w_n = \alpha_n x_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i} (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) y_n + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq (1-\gamma_n)\|x_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ \qquad + k_n^2 \|x_n - x_{n-1}\|^2 - 2\beta_n k_n \langle x_n - p, x_{n-1} - x_n \rangle \}, \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|x_1 - P_{C_{n+1}}(x_1)\|^2 + \sigma_{n+1}\}, \\ x_{n+1} \in Q_{n+1}, \ n \in N. \end{cases} \tag{10}$$

The result that $x_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, as $n \to \infty$, is proved under some conditions.

The mid-point inertial forward–backward iterative algorithm in [18] is presented as follows:

$$\begin{cases} x_0, x_1 \in H \text{ chosen arbitrarily}, e_1 \in H \text{ chosen arbitrarily}, \\ z_0 = x_0, \\ z_n = \delta_n \lambda f(x_n) + (I - \delta_n F)x_n, \\ v_n = z_n + k_n(z_n - z_{n-1}), \\ w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{v_n + w_n}{2}) + \gamma_n e_n, \\ C_1 = H = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \frac{2\alpha_n + \beta_n}{2 - \beta_n}\|z_n - p\|^2 + \frac{2\gamma_n}{2 - \beta_n}\|e_n - p\|^2, \\ \qquad + \frac{2\alpha_n + \beta_n}{2 - \beta_n} k_n \|z_n - z_{n-1}\|^2 - 2\frac{2\alpha_n + \beta_n}{2 - \beta_n} k_n \langle z_n - p, z_{n-1} - z_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|x_1 - P_{C_{n+1}}(x_1)\|^2 + \sigma_{n+1}\}, \\ x_{n+1} \in Q_{n+1}, \quad n \in N, \end{cases} \tag{11}$$

where f is a contraction and F is a strongly positive linear bounded mapping. The result that $x_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)$, as $n \to \infty$, is proved under some conditions. Furthermore, under the additional assumptions that $\widetilde{x} = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)$ and $\widetilde{x} = P_{\bigcap_{i=1}^{\infty} N(A_i+B_i)}(x_1)[\lambda f(\widetilde{x}) - F(\widetilde{x}) + \widetilde{x}]$, one has that $\widetilde{x}$ solves the variational inequality

$$\langle F\widetilde{x} - \lambda f(\widetilde{x}), \widetilde{x} - z \rangle \leq 0, \ \ \forall z \in \bigcap_{i=1}^{\infty} N(A_i + B_i).$$

The inertial forward–backward iterative algorithm for approximating common solution of monotone inclusions and one kind variational inequalities, where $T : C \to H$ is maximal monotone and τ-Lipschitz continuous, is presented as follows in [18]:

$$\begin{cases} u_0, u_1 \in C \text{ chosen arbitrarily}, e_1 \in H \text{ chosen arbitrarily}, \\ y_0 = P_C(u_0 - \lambda_0 T u_0), \\ y_n = P_C(u_n - \lambda_n T u_n), \\ v_n = y_n + k_n(y_n - y_{n-1}), \\ w_n = \alpha_n v_n + \beta_n \sum_{i=1}^{\infty} \omega_{n,i}(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)P_C(u_n - \lambda_n T u_n) + \gamma_n e_n, \\ C_1 = C = Q_1, \\ C_{n+1} = \{p \in C_n : \|w_n - p\|^2 \leq \alpha_n\|y_n - p\|^2 + \beta_n\|u_n - p\|^2 + \gamma_n\|e_n - p\|^2 \\ \qquad + k_n^2\|y_n - y_{n-1}\|^2 - 2\alpha_n k_n \langle y_n - p, y_{n-1} - y_n \rangle\}, \\ Q_{n+1} = \{p \in C_{n+1} : \|u_1 - p\|^2 \leq \|u_1 - P_{C_{n+1}}(u_1)\|^2 + \sigma_{n+1}\}, \\ u_{n+1} \in Q_{n+1}, \ n \in N, \end{cases} \tag{12}$$

The result that $u_n \to P_{\bigcap_{i=1}^{\infty} N(A_i+B_i) \bigcap VI(C,T)}(u_1)$, as $n \to \infty$, is proved.

Although two sets C_n and Q_n are needed in (10)–(12), infinite choices of iterative sequences can be made from them whose idea is totally different from that in (4) or (8).

Motivated by the above work, in this paper, we construct some new forward–backward multi-choice iterative algorithms with superposition perturbations in a Hilbert space. Furthermore, some strong convergence theorems for approximating common solution of monotone inclusions and variational inequalities are proved under mild conditions.

To begin our study, the following preliminaries are needed.

Definition 1 ([19]). *There exists a unique element $x_0 \in C$ such that $\|x - x_0\| = inf\{\|x - y\| : y \in C\}$, for each $x \in H$. Define the metric projection mapping $P_C : H \to C$ by $P_C x = x_0$, for any $x \in H$.*

Lemma 1 ([20]). *For a contraction $f : H \to H$, there is a unique element $x \in H$ that satisfies $f(x) = x$.*

Lemma 2 ([19]). *For a monotone operator $A : H \to H$ and $r > 0$, one has that $(I + rA)^{-1} : H \to H$ is non-expansive.*

Lemma 3 ([21]). *If $T_i : C \to C$ is non-expansive for $i \in N$ and $\sum_{i=1}^{\infty} a_i = 1$ for $\{a_i\} \subset (0, 1)$, then $\sum_{i=1}^{\infty} a_i T_i$ is non-expansive with $F(\sum_{i=1}^{\infty} a_i T_i) = \bigcap_{i=1}^{\infty} F(T_i)$ under the assumption that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$.*

Lemma 4 ([15]). *If $S : C \to H$ is a single-valued mapping and $T : H \to 2^H$ is maximal monotone, then*

$$F((I + rT)^{-1}(I - rS)) = N(T + S),$$

for $\forall r > 0$.

Definition 2 ([22]). *Suppose $\{K_n\}$ is a sequence of non-empty closed and convex subsets of H. One has:*

(1) The strong lower limit of $\{K_n\}$, $s - liminf K_n$, is defined as the set of all $x \in H$ such that there exists $x_n \in K_n$ for almost all n and it tends to x as $n \to \infty$ in the norm.

(2) The weak upper limit of $\{K_n\}$, $w - limsup K_n$, is defined as the set of all $x \in H$ such that there exists a subsequence $\{K_{n_m}\}$ of $\{K_n\}$ and $x_{n_m} \in K_{n_m}$ for every n_m and it tends to x as $n_m \to \infty$ in the weak topology;.

(3) The limit of $\{K_n\}$, $lim K_n$, is the common value when $s - liminf K_n = w - limsup K_n$.

Lemma 5 ([22]). *Let $\{K_n\}$ be a decreasing sequence of closed and convex subsets of H, i.e., $K_n \subset K_m$ if $n \geq m$. Then, $\{K_n\}$ converges in H and $lim K_n = \bigcap_{n=1}^{\infty} K_n$.*

Lemma 6 ([23]). *If $lim K_n$ exists and is not empty, then $P_{K_n} x \to P_{lim K_n} x$ for every $x \in H$, as $n \to \infty$.*

Lemma 7 ([24]). *Let $r \in (0, +\infty)$. Then, there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, +\infty)$ with $g(0) = 0$ such that $\|kx + (1 - k)y\|^2 \leq k\|x\|^2 + (1 - k)\|y\|^2 - k(1 - k)g(\|x - y\|)$, for $k \in [0, 1]$, $x, y \in H$ with $\|x\| \leq r$ and $\|y\| \leq r$.*

Lemma 8 ([25]). *Let $B : H \to H$ be a ϑ-strongly monotone and μ-strictly pseudo-contractive mapping with $\vartheta + \mu > 1$. Then, for any fixed number $\delta \in (0, 1)$, $I - \delta B$ is a contraction with contractive constant $1 - \delta(1 - \sqrt{\frac{1-\vartheta}{\mu}})$.*

Lemma 9 ([15]). *Suppose $F : H \to H$ is strongly positive bounded mapping with coefficient $\xi > 0$ and $0 < \rho \leq \|F\|^{-1}$, then $\|I - \rho F\| \leq 1 - \rho\xi$.*

Lemma 10 ([15]). *Let $f : H \to H$ be a contraction with contractive constant $k \in (0, 1)$, $F : H \to H$ be strongly positive bounded mapping with coefficient $\xi > 0$ and $U : H \to H$ be a non-expansive mapping. Suppose $0 < \eta \leq \frac{\xi}{2k}$ and $F(U) \neq \emptyset$. If for each $t \in (0, 1)$, define $T_t : H \to H$ by*

$$T_t x := t\eta f(x) + (I - tF)Ux,$$

then T_t has a fixed point x_t, for each $t \in (0, \|F\|^{-1}]$. Moreover, $x_t \to q_0$, as $t \to 0$, where $q_0 \in F(U)$ which satisfies the variational inequality:

$$\langle Fq_0 - \eta f(q_0), q_0 - z\rangle \leq 0, \ \ \forall z \in F(U).$$

Lemma 11 ([15]). *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \ \forall x, y \in H.$$

Lemma 12 ([26]). *Let $\{x_n\}$ and $\{b_n\}$ be two sequences of non-negative real number sequences satisfying*

$$x_{n+1} \leq (1 - t_n)x_n + b_n, \ \ \forall n \in N,$$

where $\{t_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} t_n = +\infty$ and $t_n \to 0$, as $n \to \infty$. If $limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, then $lim_{n\to\infty} x_n = 0$.

Lemma 13 ([17]). *Let H be a real Hilbert space, $A_i : H \to H$ be maximal monotone, $B_i : H \to H$ be θ_i-inversely strongly monotone, and $W_i : H \to H$ be ϑ_i-strongly monotone and μ_i-strictly pseudo-contractive with $\vartheta_i + \mu_i > 1$ for $i \in N$. Suppose $0 < r_{n,i} \leq 2\theta_i$ for $i \in N$ and $n \in N$, $k_t \in (0,1)$ for $t \in (0,1)$, $\sum_{n=1}^{\infty} c_n \|W_n\| < +\infty$, $\sum_{i=1}^{\infty} a_i = 1 = \sum_{i=1}^{\infty} c_i$ and $\bigcap_{i=1}^{\infty} N(A_i + B_i) \neq \emptyset$. If, for each $t \in (0,1)$, $Z_t^n : H \to H$ is defined by*

$$Z_t^n u = tf(u) + (1-t)(I - k_t \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I - r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) u,$$

then Z_t^n has a fixed point u_t^n. That is,

$$u_t^n = tf(u_t^n) + (1-t)(I - k_t \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I - r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) u_t^n.$$

Moreover, if $\frac{k_t}{t} \to 0$, then $u_t^n \to p_0$, as $t \to 0$, where p_0 is the solution of variational inequality:

$$\langle p_0 - f(p_0), p_0 - z \rangle \leq 0, \ \ \forall z \in \bigcap_{i=1}^{\infty} N(A_i + B_i).$$

2. Strong Convergence Theorems

Our discussion is based on the following assumptions in this section:

- (a) H is a real Hilbert space.
- (b) $A_i : H \to H$ is maximal monotone and $B_i : H \to H$ is θ_i-inversely strongly monotone, for each $i \in N$.
- (c) $f : H \to H$ is a contraction with contractive constant $k \in (0, \frac{1}{2}]$. Furthermore, if $\langle f(x) - x, y - x \rangle = 0$, then $x = 0$ or $y = x$, for $x, y \in H$.
- (d) $F : H \to H$ is a strongly positive linear bounded mapping with $\xi > 0$ and $\langle F(x) - \eta f(x) + f(y) - y, x - y \rangle \geq 0$, for $x, y \in H$.
- (e) $W_i : H \to H$ is ϑ_i-strongly monotone and μ_i-strictly pseudo-contractive, for $i \in N$;.
- (f) $\{e_n\} \subset H$ and $\{\varepsilon_n\} \subset H$ are the computational errors.
- (g) $\{a_i\}$ and $\{c_i\}$ are two real number sequences in $(0,1)$ with $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} c_i = 1$, for $n \in N$.
- (h) $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\omega_n\}$ and $\{\lambda_n\}$ are real number sequences in $(0,1)$, for $n \in N$.
- (i) $\{\sigma_n\}$ and $\{r_{n,i}\}$ are real number sequences in $(0, +\infty)$, for $n, i \in N$.

Theorem 1. *Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_1, y_1 \in H \text{ chosen arbitrarily}, \;\; e_1, \varepsilon_1 \in \; H \text{ chosen arbitrarily}, \\ C_1 = H = Q_1, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ z_n = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ w_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_n, \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F)w_n + e_n, \\ C_{n+1} = \{p \in C_n : 2\langle \alpha_n x_n + (1-\alpha_n)z_n - w_n, p\rangle \\ \qquad \leq \alpha_n \|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|w_n\|^2\} \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \sigma_{n+1}\}, \;\; n \in N, \\ y_{n+1} \in Q_{n+1}, \; n \in N, \end{cases} \tag{13}$$

Under the assumptions that:

(i) $0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$.

(ii) $\mu_i + \vartheta_i > 1$, $\mu_i \in (0,1)$ *and* $\vartheta_i \in (0,1)$, *for* $i \in N$.

(iii) $0 < r_{n,i} \leq 2\theta_i$, *for* $i, n \in N$.

(iv) $\sigma_n \to 0, \alpha_n \to 0, \beta_n \to 0, \delta_n \to 0$, *and* $\zeta_n \to 0$, *as* $n \to \infty$.

(v) $0 < \eta < \frac{\xi}{2k}$.

(vi) $\sum_{i=1}^{\infty} c_i \|W_i\| < +\infty$; $\sum_{n=1}^{\infty} \|e_n\| < +\infty$, $\sum_{n=1}^{\infty} \|\varepsilon_n\| < +\infty$, *and* $\sum_{n=1}^{\infty}(1-\omega_n) < +\infty$.

(vii) $\frac{\delta_n}{\lambda_n} \to 0$, $\frac{\|e_n\|}{\lambda_n} \to 0$, $\frac{\|\varepsilon_n\|}{\lambda_n} \to 0$, $\frac{1-\omega_n}{\lambda_n} \to 0$, $\frac{\zeta_n}{\lambda_n} \to 0$, *as* $n \to \infty$.

$(viii)$ $\sum_{n=1}^{\infty} \lambda_n = +\infty$ *and* $\lambda_n \to 0$,
one has $x_n \to q_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, *as* $n \to \infty$, *where* q_0 *satisfies the following variational inequalities:*

$$\langle Fq_0 - \eta f(q_0), q_0 - y\rangle \leq 0, \;\; \forall y \in \bigcap_{i=1}^{\infty} N(A_i + B_i), \tag{14}$$

and

$$\langle q_0 - f(q_0), q_0 - y\rangle \leq 0, \;\; \forall y \in \bigcap_{i=1}^{\infty} N(A_i + B_i). \tag{15}$$

Moreover, $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, *as* $n \to \infty$, *which means*

$$\begin{cases} \langle Fq_0 - \eta f(q_0), q_0 - P_{\bigcap_{m=1}^{\infty} C_m}(x_1)\rangle \leq 0, \\ \langle q_0 - f(q_0), q_0 - P_{\bigcap_{m=1}^{\infty} C_m}(x_1)\rangle \leq 0. \end{cases}$$

Proof. We split the proof into eleven steps.

Step 1. $\{v_n\}$ is well-defined.

For $s \in (0,1)$, define $U_s : H \to H$ by

$$U_s x := su + (1-s)Tx, \tag{16}$$

for any $x \in H$ and for fixed element $u \in H$, where $T : H \to H$ is any fixed non-expansive mapping.

It is easy to check that $\|U_s x - U_s y\| = (1-s)\|Tx - Ty\| \leq (1-s)\|x-y\|$. Thus, U_s is a contraction, which ensures from Lemma 1 that there exists $x_s \in H$ such that $U_s x_s = x_s$. That is, $x_s = su + (1-s)Tx_s$.

Since $0 < r_{n,i} \leq 2\theta_i$, for $i, n \in N$, for any $x, y \in H$,

$$\begin{aligned}\|(I - r_{n,i}B_i)x - (I - r_{n,i}B_i)y\|^2 &= \|(x-y) - r_{n,i}(B_i x - B_i y)\|^2 \\ &= \|x-y\|^2 - 2r_{n,i}\langle x-y, B_i x - B_i y\rangle + r_{n,i}^2\|B_i x - B_i y\|^2 \\ &\leq \|x-y\|^2 + r_{n,i}(r_{n,i} - 2\theta_i)\|B_i x - B_i y\|^2 \leq \|x-y\|^2.\end{aligned}$$

This ensures that $(I - r_{n,i}B_i) : H \to H$ is non-expansive, for $i, n \in N$. Since $\sum_{i=1}^{\infty} a_i = 1$, from Lemmas 2–4, one has $\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i) : H \to H$ is non-expansive, for $n \in N$. Moreover, $F(\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)) = \bigcap_{i=1}^{\infty} N(A_i + B_i)$.

Considering T in (16) as $\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)$, one can see that $\{v_n\}$ is well-defined.

Step 2. C_n is non-empty closed and convex subset of H, for any $n \in N$.

We can easily know from the construction of C_n that C_n is closed and convex subset of H, for any $n \in N$. We are left to show that $C_n \neq \emptyset$. For this, it suffices to show that $\bigcap_{i=1}^{\infty} N(A_i + B_i) \subset C_n$, for $n \geq 2$.

In fact, for any $p \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, one has

$$\begin{aligned}\|w_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\textstyle\sum_{i=1}^{\infty} a_i\|(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2.\end{aligned}$$

Then,

$$2\langle \alpha_n x_n + (1-\alpha_n)z_n - w_n, p\rangle \leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|w_n\|^2,$$

which implies that $p \in C_n$, for $n \geq 2$. Therefore, $\bigcap_{i=1}^{\infty} N(A_i + B_i) \subset C_n$, for all $n \in N$, and then $C_n \neq \emptyset$, for all $n \in N$.

Step 3. Q_n is a non-empty subset of H, for each $n \in N$, which ensures that $\{y_n\}$ is well-defined.

It follows from Step 2 and Definition 1 that, for σ_{n+1}, there exists $b_{n+1} \in C_{n+1}$ such that $\|x_1 - b_{n+1}\|^2 \leq (inf_{z \in C_{n+1}}\|x_1 - z\|)^2 + \sigma_{n+1} = \|P_{C_{n+1}}(x_1) - x_1\|^2 + \sigma_{n+1}$. Thus, $Q_{n+1} \neq \emptyset$, for $n \in N$. Then, $\{y_n\}$ is well-defined.

Step 4. $P_{C_n}(x_1) \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1)$, as $n \to \infty$.

It follows from Lemma 5 that $limC_n$ exists and $limC_n = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Then, Lemma 6 implies that $P_{C_n}(x_1) \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1)$, as $n \to \infty$.

Step 5. $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1)$, as $n \to \infty$.

Since $y_{n+1} \in Q_{n+1} \subset C_{n+1}$ and C_n is a convex subset of H, for $\forall t \in (0,1)$, $tP_{C_{n+1}}(x_1) + (1-t)y_{n+1} \in C_{n+1}$, which implies that

$$\|P_{C_{n+1}}(x_1) - x_1\| \le \|tP_{C_{n+1}}(x_1) + (1-t)y_{n+1} - x_1\|. \tag{17}$$

Using Lemma 7, one has:

$$\begin{aligned} &\|tP_{C_{n+1}}(x_1) + (1-t)y_{n+1} - x_1\|^2 = \|t(P_{C_{n+1}}(x_1) - x_1) + (1-t)(y_{n+1} - x_1)\|^2 \\ &\le t\|P_{C_{n+1}}(x_1) - x_1\|^2 + (1-t)\|y_{n+1} - x_1\|^2 - t(1-t)g(\|P_{C_{n+1}}(x_1) - y_{n+1}\|). \end{aligned} \tag{18}$$

From (17) and (18), we have $tg(\|P_{C_{n+1}}(x_1) - y_{n+1}\|) \le \|y_{n+1} - x_1\|^2 - \|P_{C_{n+1}}(x_1) - x_1\|^2 \le \sigma_{n+1}$. Letting $t \to 1$ first and then $n \to \infty$, one has $P_{C_{n+1}}(x_1) - y_{n+1} \to 0$, as $n \to \infty$. Combining with Step 4, $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1)$, as $n \to \infty$.

Step 6. $\{u_n\}$,$\{v_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{x_n\}$ are all bounded.

For $p \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, one has for any $n \in N$,

$$\|u_n - p\| \le \omega_n\|x_n - p\| + (1-\omega_n)\|p\| + \|\varepsilon_n\|. \tag{19}$$

Furthermore, $\|v_n - p\| \le \beta_n\|u_n - p\| + (1-\beta_n)\|v_n - p\|$ implies that for any $n \in N$,

$$\|v_n - p\| \le \|u_n - p\|. \tag{20}$$

In view of Lemma 8 and (20), one has

$$\begin{aligned} &\|z_n - p\| \le \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| \\ &+ (1-\delta_n)\|(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n - p\| \\ &\le \delta_n k\|x_n - p\| + \delta_n\|f(p) - p\| \\ &+ (1-\delta_n)\|(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(v_n - p)\| \\ &+ (1-\delta_n)\|(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)p - p\| \\ &\le \delta_n k\|x_n - p\| + \delta_n\|f(p) - p\| + (1-\delta_n)[1 - \zeta_n(1 - \sum_{i=1}^{\infty} c_i \sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|u_n - p\| \\ &+ (1-\delta_n)\zeta_n \sum_{i=1}^{\infty} c_i\|W_i\|\|p\| \end{aligned} \tag{21}$$

Note that, for any $n \in N$,

$$\|w_n - p\| \le \alpha_n\|x_n - p\| + (1-\alpha_n)\|z_n - p\|. \tag{22}$$

Now, in view of Lemma 9, one has

$$\begin{aligned} &\|x_{n+1} - p\| \le \lambda_n\|\eta f(x_n) - Fp\| + \|(I - \lambda_n F)(w_n - p)\| + \|e_n\| \\ &\le \lambda_n \eta k\|x_n - p\| + \lambda_n\|\eta f(p) - Fp\| + (1 - \lambda_n \xi)\|w_n - p\| + \|e_n\| \end{aligned} \tag{23}$$

Combing with inequalities (19)–(23), by induction, one has

$$
\begin{aligned}
&\|x_{n+1}-p\| \\
&\le \{\lambda_n\eta k+(1-\lambda_n\xi)\alpha_n+(1-\lambda_n\xi)(1-\alpha_n)\delta_n\, k \\
&+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\textstyle\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\omega_n\}\|x_n-p\| \\
&+\lambda_n\|\eta f(p)-F(p)\|+\|e_n\| \\
&+(1-\lambda_n\xi)(1-\alpha_n)\delta_n\|f(p)-p\|+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\zeta_n\textstyle\sum_{i=1}^{\infty}c_i\|W_i\|\|p\| \\
&+(1-\lambda_n\xi)(1-\alpha_n)(1-\omega_n)(1-\delta_n)[1-\zeta_n(1-\textstyle\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|p\| \\
&+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\textstyle\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|\varepsilon_n\| \\
&\le \{\lambda_n\eta k+(1-\lambda_n\xi)\alpha_n+(1-\lambda_n\xi)(1-\alpha_n)\delta_n\, k \\
&+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\textstyle\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\}\|x_n-p\| \\
&+\lambda_n(\xi-\eta k)\frac{\|\eta f(p)-F(p)\|}{\xi-\eta k}+\|e_n\| \\
&+(1-\lambda_n\xi)(1-\alpha_n)\delta_n(1-k)\frac{\|f(p)-p\|}{1-k} \\
&+(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\zeta_n(1-\textstyle\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}} \\
&+(1-\omega_n)\|p\|+\|\varepsilon_n\| \\
&\le max\{\|x_n-p\|,\frac{\|\eta f(p)-F(p)\|}{\xi-\eta k},\frac{\|f(p)-p\|}{1-k},\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}}\}+\|e_n\|+(1-\omega_n)\|p\|+\|\varepsilon_n\| \\
&\dots \\
&\le max\{\|x_1-p\|,\frac{\|\eta f(p)-F(p)\|}{\xi-\eta k},\frac{\|f(p)-p\|}{1-k},\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}}\} \\
&+\textstyle\sum_{i=1}^{n}\|e_i\|+\sum_{i=1}^{n}(1-\omega_i)\|p\|+\sum_{i=1}^{n}\|\varepsilon_i\|
\end{aligned}
$$

Based on the assumptions, one has $\{x_n\}$ is bounded. Following (19)–(22), it is easy to see that $\{u_n\}$,$\{v_n\}$,$\{z_n\}$ and $\{w_n\}$ are all bounded.

Note that, for $p\in\bigcap_{i=1}^{\infty}N(A_i+B_i)$, $\|\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n-p\|\le\|v_n-p\|$. Then, $\{\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\}$ is bounded. Similarly, $\{\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_n\}$ is bounded.
Since $\|\sum_{i=1}^{\infty}c_iW_i\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|\le\sum_{i=1}^{\infty}c_i\|W_i\|\sum_{i=1}^{\infty}\|a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\|$, then $\{\sum_{i=1}^{\infty}c_iW_i\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)v_n\}$ is bounded.

Step 7. There exists $q_0\in\bigcap_{i=1}^{\infty}N(A_i+B_i)$, which is the solution of variational inclusion (14).

It follows from Lemma 10 that there exists z_t such that

$$z_t=t\eta f(z_t)+(I-tF)\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)z_t$$

and $z_t\to q_0$, as $t\to 0$, where q_0 is the solution of (14).

Step 8. $limsup_{n\to\infty}\langle\eta f(q_0)-Fq_0,x_{n+1}-q_0\rangle\le 0$, where q_0 is the same as that in Step 7.

Note that

$$\begin{aligned}
&\|w_n - z_n\| \\
&= \|w_n - \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n \\
&+ \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i)(z_n - v_n) \\
&+ \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n - v_n + v_n - z_n\| \\
&= \|\alpha_n [x_n - \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n] \\
&+ \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i)(z_n - v_n) \\
&+ \beta_n [\sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n - u_n] + v_n - z_n\| \\
&\le \alpha_n \|x_n - \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n\| + 2\|z_n - v_n\| \\
&+ \beta_n \| \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n - u_n\|
\end{aligned} \tag{24}$$

Furthermore,

$$\begin{aligned}
&\|z_n - v_n\| \\
&\le \delta_n \|f(x_n)\| + \zeta_n (1 - \delta_n) \| \sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n\| \\
&+ \beta_n \|u_n - \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n\| \\
&+ \delta_n \| \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n\|
\end{aligned} \tag{25}$$

Since $\{x_n\}$, $\{u_n\}$,$\{v_n\}$,$\{z_n\}$, $\{\sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n\}$, $\{\sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n\}$ and $\{\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) v_n\}$ are bounded, then, based on the assumptions and (24) and (25), $w_n - z_n \to 0$, as $n \to \infty$.

Let z_t be the same as that in Step 7, then $\|z_t\| \le \|z_t - q_0\| + \|q_0\|$, which implies that $\{z_t\}$ is bounded.

Note that

$$\begin{aligned}
&\| \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) w_n - w_n\| \\
&\le \| \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i)(w_n - z_n)\| \\
&+ \| \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n - w_n\| \\
&\le \|w_n - z_n\| + \| \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n - w_n\| \\
&= \|w_n - z_n\| + \alpha_n \|x_n - \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) z_n\|
\end{aligned} \tag{26}$$

Since $\alpha_n \to 0$ and $w_n - z_n \to 0$, as $n \to \infty$, then $\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n \to 0$, as $n \to \infty$. In view of Lemma 11,

$$
\begin{aligned}
&\|z_t - w_n\|^2 \\
&= \|z_t - \textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n + \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n\|^2 \\
&\le \|z_t - \textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n\|^2 \\
&+2\langle z_t - w_n, \textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n\rangle \\
&\le \|z_t - w_n\|^2 \\
&+2\langle z_t - \textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n, t\eta f(z_t) - tF(\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_t)\rangle \\
&+2\|z_t - w_n\|\|\textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n\|
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&t\langle z_t - \textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n, F(\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_t) - \eta f(z_t)\rangle \\
&\le \|z_t - w_n\|\|\textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n\|,
\end{aligned}
$$

which implies that

$$
lim_{t\to 0} limsup_{n\to\infty}\langle z_t - \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n, F(\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_t) - \eta f(z_t)\rangle \le 0.
$$

Since $z_t \to q_0$ as $t \to 0$, then

$$
limsup_{n\to\infty}\langle q_0 - \textstyle\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n, Fq_0 - \eta f(q_0)\rangle \le 0.
$$

Since $\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)w_n - w_n \to 0$, $w_n - z_n \to 0$ and $x_{n+1} - w_n = \lambda_n(\eta f(x_n) - Fw_n) + e_n \to 0$, then $limsup_{n\to\infty}\langle q_0 - x_{n+1}, Fq_0 - \eta f(q_0)\rangle \le 0$.

Step 9. $x_n \to q_0$, as $n \to \infty$, where q_0 is the same as that in Steps 7 and 8.

In fact, using Lemma 11 again, one has

$$
\begin{aligned}
\|u_n - q_0\|^2 &= \|\omega_n(x_n - q_0) + (\omega_n - 1)q_0 + \varepsilon_n\|^2 \\
&\le \omega_n\|x_n - q_0\|^2 + 2\langle \varepsilon_n, u_n - q_0\rangle + 2(1 - \omega_n)\langle q_0, q_0 - u_n\rangle \\
&\le \omega_n\|x_n - q_0\|^2 + 2\|\varepsilon_n\|\|u_n - q_0\| + 2(1 - \omega_n)\|q_0\|\|u_n - q_0\|.
\end{aligned} \tag{27}
$$

Furthermore, $\|v_n - q_0\|^2 \le \beta_n\|u_n - q_0\|^2 + (1 - \beta_n)\|v_n - q_0\|^2$ ensures that

$$
\|v_n - q_0\|^2 \le \|u_n - q_0\|^2. \tag{28}
$$

In view of Lemma 11 again, one has

$$
\begin{aligned}
&\|z_n - q_0\|^2 \\
&= \|\delta_n(f(x_n) - q_0) + (1 - \delta_n)[(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n - q_0]\|^2 \\
&\le (1 - \delta_n)\|v_n - q_0\|^2 + 2\delta_n\langle z_n - q_0, f(x_n) - q_0\rangle \\
&- 2(1 - \delta_n)\zeta_n\langle \sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, z_n - q_0\rangle \\
&\le (1 - \delta_n)\|v_n - q_0\|^2 + 2\delta_n\langle z_n - q_0, f(x_n) - f(q_0)\rangle + 2\delta_n\langle z_n - q_0, f(q_0) - q_0\rangle \\
&+ 2(1 - \delta_n)\zeta_n\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\|\|z_n - q_0\| \\
&\le (1 - \delta_n)\|v_n - q_0\|^2 + 2\delta_n k\|z_n - x_n\|\|x_n - q_0\| + 2\delta_n k\|x_n - q_0\|^2 + 2\delta_n\langle z_n - q_0, f(q_0) - q_0\rangle \\
&+ 2\zeta_n\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\|\|z_n - q_0\|
\end{aligned} \tag{29}
$$

Note that

$$\|w_n - q_0\|^2 \le \alpha_n \|x_n - q_0\|^2 + (1-\alpha_n)\|z_n - q_0\|^2. \tag{30}$$

Now, in view of Lemma 9 and using (27)–(30), one has

$$\begin{aligned}
&\|x_{n+1} - q_0\|^2 = \|\lambda_n \eta f(x_n) + (I - \lambda_n F)w_n + e_n - q_0\|^2 \\
&= \|\lambda_n(\eta f(x_n) - F(q_0)) + (I - \lambda_n F)(w_n - q_0) + e_n\|^2 \\
&\le (1-\lambda_n\xi)\|w_n - q_0\|^2 + 2\langle e_n, x_{n+1} - q_0\rangle + 2\lambda_n\langle \eta f(x_n) - Fq_0, x_{n+1} - q_0\rangle \\
&\le (1-\lambda_n\xi)\|w_n - q_0\|^2 + 2\|e_n\|\|x_{n+1} - q_0\| + 2\lambda_n\eta\langle f(x_n) - f(q_0), x_{n+1} - q_0\rangle \\
&+ 2\lambda_n\langle \eta f(q_0) - Fq_0, x_{n+1} - q_0\rangle \\
&\le (1-\lambda_n\xi)\|w_n - q_0\|^2 + 2\|e_n\|\|x_{n+1} - q_0\| + 2\lambda_n\eta k\|x_n - q_0\|\|x_{n+1} - q_0\| \\
&+ 2\lambda_n\langle \eta f(q_0) - F(q_0), x_{n+1} - q_0\rangle \\
&\le \{(1-\lambda_n\xi)\alpha_n + \lambda_n\eta k + 2(1-\lambda_n\xi)(1-\alpha_n)\delta_n k \\
&+ (1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\omega_n\}\|x_n - q_0\|^2 \\
&+ 2\|e_n\|\|x_{n+1} - q_0\| + 2(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\|\varepsilon_n\|\|u_n - q_0\| \\
&+ 2(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)(1-\omega_n)\|q_0\|\|u_n - q_0\| \\
&+ 2\delta_n k(1-\lambda_n\xi)(1-\alpha_n)\|x_n - q_0\|\|x_n - z_n\| \\
&+ 2(1-\lambda_n\xi)(1-\alpha_n)\delta_n\|z_n - q_0\|\|f(q_0) - q_0\| \\
&+ 2(1-\lambda_n\xi)(1-\alpha_n)\zeta_n\|z_n - q_0\|\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\| \\
&+ \lambda_n\eta k\|x_{n+1} - q_0\|^2 + 2\lambda_n\langle \eta f(q_0) - F(q_0), x_{n+1} - q_0\rangle.
\end{aligned} \tag{31}$$

Let $M_1 = sup_n\{2\|x_{n+1} - q_0\|, 2\|u_n - q_0\|, 2k\|x_n - q_0\|\|x_n - z_n\|, 2\|u_n - q_0\|\|q_0\|, 2\|f(q_0) - q_0\|\|z_n - q_0\|, 2\|z_n - q_0\|\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n\| : n \in N\}$. Then, from Step 6, one has $M_1 < +\infty$.

Therefore, it follows from (31) that

$$\begin{aligned}
&(1-\lambda_n\eta k)\|x_{n+1} - q_0\|^2 \\
&\le \{(1-\lambda_n\xi)\alpha_n + \lambda_n\eta k + 2(1-\lambda_n\xi)(1-\alpha_n)\delta_n k + (1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\}\|x_n - q_0\|^2 \\
&+ [\|e_n\| + \|\varepsilon_n\| + (1-\omega_n) + 2\delta_n + \zeta_n]M_1 \\
&+ 2\lambda_n\langle \eta f(q_0) - F(q_0), x_{n+1} - q_0\rangle
\end{aligned} \tag{32}$$

If we set $b_n^{(1)} = \frac{\lambda_n(\xi - 2\eta k)}{1-\lambda_n\eta k}$, $b_n^{(2)} = \frac{M_1}{1-\lambda_n\eta k}[\|e_n\| + \|\varepsilon_n\| + (1-\omega_n) + 2\delta_n + \zeta_n] + \frac{2\lambda_n}{1-\lambda_n\eta k}\langle \eta f(q_0) - Fq_0, x_{n+1} - q_0\rangle$, then (32) can be reduced as follows:

$$\|x_{n+1} - q_0\|^2 \le (1 - b_n^{(1)})\|x_n - q_0\|^2 + b_n^{(2)}.$$

Based on the assumptions and Step 8, we know that $b_n^{(1)} \to 0$, $\sum_{n=1}^{\infty} b_n^{(1)} = +\infty$ and $limsup_{n\to\infty}\frac{b_n^{(2)}}{b_n^{(1)}} \le 0$. Then, from Lemma 12, $x_n \to q_0$, as $n \to \infty$.

Step 10. There exists $p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, which is the solution of the variational inclusion

$$\langle p_0 - f(p_0), p_0 - y\rangle \le 0, \ \ \forall y \in \bigcap_{i=1}^{\infty} N(A_i + B_i). \tag{33}$$

In fact, it follows from Lemma 13 that there exists u_t^n such that

$$u_t^n = tf(u_t^n) + (1-t)(I - k_t \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) u_t^n$$

and $u_t^n \to p_0$, as $t \to 0$, where p_0 is the solution of (33).

Step 11. $x_n \to p_0$, as $n \to \infty$, where p_0 is the same as that in Step 10.

It suffices to show that $p_0 = q_0$.

Since $p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$,

$$\langle Fq_0 - \eta f(q_0), q_0 - p_0 \rangle \le 0. \tag{34}$$

Since F is strongly positive linear bounded, f is a contraction, and $0 < \eta < \frac{\xi}{2k}$,

$$\begin{aligned} &\langle (Fq_0 - \eta f(q_0)) - (Fp_0 - \eta f(p_0)), q_0 - p_0 \rangle \\ &= \langle F(q_0 - p_0), q_0 - p_0 \rangle + \eta \langle f(p_0) - f(q_0), q_0 - p_0 \rangle \\ &\ge \xi \|q_0 - p_0\|^2 - \eta k \|q_0 - p_0\|^2 \ge 0. \end{aligned} \tag{35}$$

Therefore, (34) ensures that

$$\langle Fp_0 - \eta f(p_0), q_0 - p_0 \rangle \le \langle Fq_0 - \eta f(q_0), q_0 - p_0 \rangle \le 0. \tag{36}$$

On the other hand, it follows from (33) that

$$\langle f(p_0) - p_0, q_0 - p_0 \rangle \le 0. \tag{37}$$

Combining with (34), one has $\langle Fq_0 - \eta f(q_0) + f(p_0) - p_0, q_0 - p_0 \rangle \le 0$. Following Condition (d), we know that $\langle Fq_0 - \eta f(q_0) + f(p_0) - p_0, q_0 - p_0 \rangle = 0$. Then, (34) and (37) ensure that $\langle Fq_0 - \eta f(q_0), q_0 - p_0 \rangle = \langle f(p_0) - p_0, q_0 - p_0 \rangle = 0$.

Since $0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, from Condition (c), we know that $p_0 = 0$ or $p_0 = q_0$. If $p_0 = q_0$, then the result follows. If $p_0 = 0$, then $\langle Fq_0 - \eta f(q_0), q_0 - p_0 \rangle = 0$ implies that $\langle Fq_0 - \eta f(q_0), q_0 \rangle = 0$. Therefore, $\xi \|q_0\|^2 \le \langle Fq_0, q_0 \rangle = \eta \langle f(q_0), q_0 \rangle \le \eta k \|q_0\|^2$. Since $\xi > 2\eta k$, then $q_0 = 0$, which means that $p_0 = q_0 = 0$. Therefore, $x_n \to p_0 = q_0$, as $n \to \infty$.

This completes the proof. □

Theorem 2. *Let $\{x_n\}$ be generated by the following iterative algorithm:*

$$\begin{cases} x_1, y_1 \in H \text{ chosen arbitrarily}, \ \ \varepsilon_1, e_1 \in H \text{ chosen arbitrarily}, \\ C_1 = H = Q_1, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1-\beta_n) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) (\frac{u_n + v_n}{2}), \\ z_n = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) (\frac{u_n + v_n}{2}), \\ w_n = \alpha_n x_n + (1-\alpha_n) \sum_{i=1}^{\infty} a_i (I + r_{n,i} A_i)^{-1} (I - r_{n,i} B_i) (\frac{u_n + z_n}{2}), \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F) w_n + e_n, \\ C_{n+1} = \{ p \in C_n : 2\langle \alpha_n x_n + (1-\alpha_n) z_n - w_n, p \rangle \\ \qquad \le \alpha_n \|x_n\|^2 + (1-\alpha_n) \|z_n\|^2 - \|w_n\|^2 \} \\ Q_{n+1} = \{ p \in C_{n+1} : \|x_1 - p\|^2 \le \|P_{C_{n+1}}(x_1) - x_1\|^2 + \sigma_{n+1} \}, \ \ n \in N, \\ y_{n+1} \in Q_{n+1}, \ n \in N. \end{cases} \tag{38}$$

Under the assumptions of Theorem 1, one has

$x_n \to q_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, as $n \to \infty$, where q_0 is the unique solution of the system of variational inclusions (14) and (15). Moreover, $y_n \to P_{\bigcap_{m=1}^{\infty} C_m}(x_1) \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, as $n \to \infty$.

Proof. The proof is split into eleven steps. Copy Steps 2–5, 7, 10 and 11 in Theorem 1. Furthermore, modify the other steps in Theorem 1 as follows

Step 1. $\{v_n\}$ is well-defined.

For $s \in (0,1)$, define $U_s : H \to H$ by $U_s x := su + (1-s)T(\frac{u+x}{2})$, for any $x \in H$ and for fixed $u \in H$, where $T : H \to H$ is any fixed non-expansive mapping.

It is easy to check that $\|U_s x - U_s y\| = (1-s)\|T(\frac{u+x}{2}) - T(\frac{u+y}{2})\| \le \frac{1-s}{2}\|x-y\|$. Thus, U_s is a contraction, which ensures from Lemma 1 that there exists $x_s \in H$ such that $U_s x_s = x_s$. That is, $x_s = su + (1-s)T(\frac{u+x_s}{2})$.

Considering T here as $\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)$, similar to Step 1 of Theorem 1, one can see that $\{v_n\}$ is well-defined.

Step 6. $\{u_n\}$,$\{v_n\}$, $\{z_n\}$, $\{w_n\}$, and $\{x_n\}$ are all bounded.

For $p \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, one has

$\|v_n - p\| \le \beta_n\|u_n - p\| + (1-\beta_n)\frac{\|u_n - p\|}{2} + (1-\beta_n)\frac{\|v_n - p\|}{2}$. This implies that (20) is still true.

In view of Lemma 8 and (20), one has

$$\begin{aligned}
&\|z_n - p\| \le \delta_n\|f(x_n) - f(p)\| + \delta_n\|f(p) - p\| \\
&+(1-\delta_n)\|(I - \zeta_n \textstyle\sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+v_n}{2}) - p\| \\
&\le \delta_n k\|x_n - p\| + \delta_n\|f(p) - p\| \\
&+(1-\delta_n)\|(I - \zeta_n \textstyle\sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+v_n}{2} - p)\| \\
&+(1-\delta_n)\|(I - \zeta_n \textstyle\sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)p - p\| \\
&\le \delta_n k\|x_n - p\| + \delta_n\|f(p) - p\| + (1-\delta_n)[1 - \zeta_n(1 - \textstyle\sum_{i=1}^{\infty} c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|\frac{u_n - p}{2} + \frac{v_n - p}{2}\| \\
&+(1-\delta_n)\zeta_n \textstyle\sum_{i=1}^{\infty} c_i\|W_i\|\|p\| \\
&\le \delta_n k\|x_n - p\| + \delta_n\|f(p) - p\| + (1-\delta_n)[1 - \zeta_n(1 - \textstyle\sum_{i=1}^{\infty} c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|u_n - p\| \\
&+(1-\delta_n)\zeta_n \textstyle\sum_{i=1}^{\infty} c_i\|W_i\|\|p\|,
\end{aligned}$$

which ensures that (21) is still true.

Note that

$$\|w_n - p\| \le \alpha_n\|x_n - p\| + (1-\alpha_n)\frac{\|u_n - p\|}{2} + (1-\alpha_n)\frac{\|z_n - p\|}{2}. \tag{39}$$

Combining with inequalities (19)–(21) and (39), one has

$$
\begin{aligned}
&\|x_{n+1}-p\| \\
&\leq \{\lambda_n\eta k+(1-\lambda_n\xi)\alpha_n+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)\delta_n k}{2} \\
&+\frac{(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\omega_n}{2}+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)\omega_n}{2}\}\|x_n-p\| \\
&+\lambda_n\|\eta f(p)-F(p)\|+\|e_n\|+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)(1-\omega_n)}{2}\|p\|+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)\|\varepsilon_n\|}{2} \\
&+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)\delta_n}{2}\|f(p)-p\|+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)\zeta_n\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{2} \\
&+\frac{(1-\lambda_n\xi)(1-\alpha_n)(1-\omega_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|p\|}{2} \\
&+\frac{(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]\|\varepsilon_n\|}{2} \\
&\leq \{\lambda_n\eta k+(1-\lambda_n\xi)\alpha_n+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)}{2}+\tfrac{(1-\lambda_n\xi)(1-\alpha_n)\delta_n k}{2} \\
&+\frac{(1-\lambda_n\xi)(1-\alpha_n)(1-\delta_n)[1-\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}})]}{2}\}\|x_n-p\| \\
&+\lambda_n(\xi-\eta k)\tfrac{\|\eta f(p)-F(p)\|}{\xi-\eta k}+\|e_n\| \\
&+(1-\lambda_n\xi)\tfrac{(1-\alpha_n)}{2}\delta_n(1-k)\tfrac{\|f(p)-p\|}{1-k} \\
&+(1-\lambda_n\xi)\tfrac{(1-\alpha_n)}{2}(1-\delta_n)\zeta_n(1-\sum_{i=1}^{\infty}c_i\sqrt{\tfrac{1-\vartheta_i}{\mu_i}})\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}} \\
&+2(1-\omega_n)\|p\|+2\|\varepsilon_n\| \\
&\leq max\{\|x_n-p\|,\tfrac{\|\eta f(p)-p\|}{\xi-\eta k},\tfrac{\|f(p)-p\|}{1-k},\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}}\}+\|e_n\|+2(1-\omega_n)\|p\|+2\|\varepsilon_n\| \\
&\cdots \\
&\leq max\{\|x_1-p\|,\tfrac{\|\eta f(p)-p\|}{\xi-\eta k},\tfrac{\|f(p)-p\|}{1-k},\frac{\sum_{i=1}^{\infty}c_i\|W_i\|\|p\|}{1-\sum_{i=1}^{\infty}c_i\sqrt{\frac{1-\vartheta_i}{\mu_i}}}\} \\
&+\sum_{i=1}^{n}\|e_i\|+2\sum_{i=1}^{n}(1-\omega_i)\|p\|+2\sum_{i=1}^{n}\|\varepsilon_i\|
\end{aligned}
$$

Therefore, $\{x_n\}$ is bounded. Similar to Step 6 in Theorem 1, $\{u_n\}$,$\{v_n\}$,$\{z_n\}$, $\{w_n\}$, $\{\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{u_n+v_n}{2})\}$, $\{\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{u_n+z_n}{2})\}$, and $\{\sum_{i=1}^{\infty}c_iW_i\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{u_n+v_n}{2})\}$ are all bounded.

Step 8. $limsup_{n\to\infty}\langle\eta f(q_0)-Fq_0,x_{n+1}-q_0\rangle\leq 0$, where q_0 is the same as that in Step 7.
Note that

$$
\begin{aligned}
&\|w_n-z_n\| \\
&\leq \|w_n-\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)\frac{u_n+z_n}{2}\| \\
&+\|\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{u_n+z_n}{2}-\frac{u_n+v_n}{2})\| \\
&+\|\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)\frac{u_n+v_n}{2}-v_n\|+\|v_n-z_n\| \\
&\leq \|\alpha_n[x_n-\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)\frac{u_n+z_n}{2}]\| \\
&+\beta_n\|\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)\frac{u_n+v_n}{2}-u_n\|+\frac{3}{2}\|v_n-z_n\| \\
&=\alpha_n\|x_n-\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)\frac{u_n+z_n}{2}\|+\frac{3}{2}\|z_n-v_n\| \\
&+\beta_n\|\sum_{i=1}^{\infty}a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)\frac{u_n+v_n}{2}-u_n\|
\end{aligned}
\tag{40}
$$

Furthermore,

$$\begin{aligned}
&\|z_n - v_n\| \\
&\le \delta_n\|f(x_n)\| + \zeta_n(1-\delta_n)\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n + v_n}{2})\| \\
&+ \beta_n\|u_n - \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n + v_n}{2})\| \\
&+ \delta_n\|\sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n + v_n}{2})\|
\end{aligned} \tag{41}$$

Based on the assumptions, (40) and (41), and Step 6, one has $w_n - z_n \to 0$, as $n \to \infty$. Copying the corresponding part of Step 8 in Theorem 1, one can see that $limsup_{n\to\infty}\langle \eta f(q_0) - Fq_0, x_{n+1} - q_0\rangle \le 0$.

Step 9. $x_n \to q_0$, as $n \to \infty$, where q_0 is the same as that in Steps 7 and 8.

Similar to Step 9 in Theorem 1, we can easily see that both (27) and (28) are still true.

In view of Lemma 11 and (28), one has

$$\begin{aligned}
&\|z_n - q_0\|^2 \\
&= \|\delta_n(f(x_n) - q_0) + (1-\delta_n)[(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i) \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)\frac{u_n + v_n}{2} - q_0]\|^2 \\
&\le (1-\delta_n)\|\frac{u_n + v_n}{2} - q_0\|^2 + 2\delta_n\langle z_n - q_0, f(x_n) - q_0\rangle \\
&- 2(1-\delta_n)\zeta_n\langle \sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)\frac{u_n + v_n}{2}, z_n - q_0\rangle \\
&\le (1-\delta_n)\|\frac{u_n + v_n}{2} - q_0\|^2 + 2\delta_n\langle z_n - q_0, f(x_n) - f(q_0)\rangle + 2\delta_n\langle z_n - q_0, f(q_0) - q_0\rangle \\
&+ 2(1-\delta_n)\zeta_n\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)\frac{u_n + v_n}{2}\|\|z_n - q_0\| \\
&\le (1-\delta_n)\|u_n - q_0\|^2 + 2\delta_n k\|z_n - x_n\|\|x_n - q_0\| + 2\delta_n k\|x_n - q_0\|^2 + 2\delta_n\langle z_n - q_0, f(q_0) - q_0\rangle \\
&+ 2\zeta_n\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)\frac{u_n + v_n}{2}\|\|z_n - q_0\|
\end{aligned} \tag{42}$$

Note that

$$\|w_n - q_0\|^2 \le \alpha_n\|x_n - q_0\|^2 + (1-\alpha_n)\frac{\|u_n - q_0\|^2}{2} + (1-\alpha_n)\frac{\|z_n - q_0\|^2}{2}. \tag{43}$$

Now, in view of Lemma 9 and using (27), (28), (42), and (43), one has

$$\begin{aligned}
&\|x_{n+1} - q_0\|^2 = \|\lambda_n(\eta f(x_n) - Fq_0) + (I - \lambda_n F)(w_n - q_0) + e_n\|^2 \\
&\le (1-\lambda_n\xi)\|w_n - q_0\|^2 + 2\|e_n\|\|x_{n+1} - q_0\| + 2\lambda_n\eta\langle f(x_n) - f(q_0), x_{n+1} - q_0\rangle \\
&+ 2\lambda_n\langle \eta f(q_0) - Fq_0, x_{n+1} - q_0\rangle \\
&\le \{(1-\lambda_n\xi)\alpha_n + \lambda_n\eta k + (1-\lambda_n\xi)(1-\alpha_n)\delta_n k + (1-\lambda_n\xi)(1-\alpha_n)(1 - \frac{\delta_n}{2})\omega_n\}\|x_n - q_0\|^2 \\
&+ 2\|e_n\|\|x_{n+1} - q_0\| + (1-\lambda_n\xi)(1-\alpha_n)(2-\delta_n)\|\varepsilon_n\|\|u_n - q_0\| \\
&+ (1-\lambda_n\xi)(1-\alpha_n)(2-\delta_n)(1-\omega_n)\|q_0\|\|u_n - q_0\| \\
&+ \delta_n\eta k(1-\lambda_n\xi)(1-\alpha_n)\|x_n - q_0\|\|x_n - z_n\| \\
&+ (1-\lambda_n\xi)(1-\alpha_n)\delta_n\|z_n - q_0\|\|f(q_0) - q_0\| \\
&+ (1-\lambda_n\xi)(1-\alpha_n)\zeta_n\|z_n - q_0\|\|\sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n + v_n}{2}) - q_0\| \\
&+ \lambda_n\eta k\|x_{n+1} - q_0\|^2 + 2\lambda_n\langle \eta f(q_0) - F(q_0), x_{n+1} - q_0\rangle
\end{aligned} \tag{44}$$

Therefore,

$$
\begin{aligned}
&(1-\lambda_n\eta k)\|x_{n+1}-q_0\|^2 \\
&\leq \{(1-\lambda_n\xi)\alpha_n+\lambda_n\eta k+(1-\lambda_n\xi)(1-\alpha_n)\delta_n k+(1-\lambda_n\xi)(1-\alpha_n)(1-\frac{\delta_n}{2})\}\|x_n-q_0\|^2 \\
&+[\|e_n\|+\|\varepsilon_n\|+(1-\omega_n)+2\delta_n+\zeta_n]M_2+2\lambda_n\langle\eta f(q_0)-F(q_0),x_{n+1}-q_0\rangle,
\end{aligned} \tag{45}
$$

where $M_2 = sup_n\{2\|x_{n+1}-q_0\|, \|x_n-q_0\|\|z_n-x_n\|, \|z_n-q_0\|\|f(q_0)-q_0\|, \|z_n-q_0\| \| \sum_{i=1}^{\infty} c_i W_i \sum_{i=1}^{\infty} a_i(I+r_{n,i}A_i)^{-1}(I-r_{n,i}B_i)(\frac{u_n+v_n}{2})-q_0\|, \|u_n-q_0\|, 2\|q_0\|\|u_n-q_0\| : n \in N\} < +\infty$.

If we set $b_n^{(3)} = b_n^{(1)}, b_n^{(4)} = \frac{[\|e_n\|+\|\varepsilon_n\|+(1-\omega_n)+2\delta_n+\zeta_n]M_2+2\lambda_n\langle\eta f(q_0)-F(q_0),x_{n+1}-q_0\rangle}{1-\lambda_n\eta k}$, then

$$
\|x_{n+1}-q_0\|^2 \leq (1-b_n^{(3)})\|x_n-q_0\|^2+b_n^{(4)}.
$$

Similar to Step 9 in Theorem 1, in view of Lemma 12, we have $x_n \to q_0$, as $n \to \infty$. This completes the proof. □

Remark 1. *The restrictions imposed on the mappings $f(x)$ and $F(x)$ are available. For example, take $F(x) = \frac{3}{4}x$ and $f(x) = \frac{x}{2}$, for $x \in (-\infty,+\infty)$. Take $\eta = \frac{1}{2}, k = \frac{1}{2}, \xi = \frac{3}{4}$. Then, we can easily see that F is a strongly positive linear bounded mapping with ξ, f is a contraction, and $\xi > 2\eta k$. Moreover, $\langle F(x)-\eta f(x)+f(y)-y, x-y\rangle = \frac{1}{2}\|x-y\|^2 \geq 0$, for $x,y \in (-\infty,+\infty)$. Furthermore, if $\langle f(x)-x, y-x\rangle = 0$, then $\frac{x}{2}(y-x) = 0$, which implies that $x = 0$ or $y = x$.*

Remark 2. *In both (13) and (38), the idea of forward–backward splitting method is embodied, the superposition perturbation is considered and multi-choice sets are constructed, which extends and complements the corresponding studies.*

Remark 3. *From Theorems 1 and 2, we may find that the limit q_0 of the iterative sequence $\{x_n\}$ is not only the solution of the system of monotone inclusions (3) but also the solution of variational inequalities (14) and (15). That is, the study on iterative construction of the solution of (14) in [18] and the solution of (15) in [17] are unified in our paper.*

Remark 4. *From Theorems 1 and 2, we may find that the relationship between the metric projection $P_{\bigcap_{m=1}^{\infty} C_m}(x_1)$ and the common solution of variational inequalities and monotone inclusions q_0 is set up in our paper.*

3. Applications

In this section, one kind capillarity system discussed in [18] is employed again to demonstrate the application of Theorems 1 and 2.

The discussion begins under the following assumptions:

(1) Ω is a bounded conical domain in R^n ($n \in N$) with its boundary $\Gamma \in C^1$.
(2) ϑ is the exterior normal derivative of Γ.
(3) λ_i is a positive number, for $i \in N$.
(4) $p_i \in (\frac{2n}{n+1}, +\infty)$, for $i \in N$. Moreover, if $p_i \geq n$, then suppose $1 \leq q_i, r_i < +\infty$, for $i \in N$. If $p_i < n$, then suppose $1 \leq q_i, r_i \leq \frac{np_i}{n-p_i}$, for $i \in N$.
(5) $|\cdot|$ denotes the norm in R^n and $<\cdot,\cdot>$ the inner-product.

Now, examine the capillarity systems:

$$
\begin{cases}
-div[(1+\frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}] \\
+\lambda_i(|u^{(i)}|^{q_i-2}u^{(i)}+|u^{(i)}|^{r_i-2}u^{(i)})+u^{(i)}(x) = f_i(x), \quad x \in \Omega \\
-<\vartheta, (1+\frac{|\nabla u^{(i)}|^{p_i}}{\sqrt{1+|\nabla u^{(i)}|^{2p_i}}})|\nabla u^{(i)}|^{p_i-2}\nabla u^{(i)}> = 0, \quad x \in \Gamma, \ i \in N.
\end{cases} \tag{46}
$$

Lemma 14. *(see [18]) For $i \in N$, define $A_i : L^2(\Omega) \to L^2(\Omega)$ by*

(1) $D(A_i) = \{u \in L^2(\Omega) | \ \exists f \in L^2(\Omega)$ *such that* $f \in \widetilde{A_i}u\}$, *where* $\widetilde{A_i} : W^{1,p_i}(\Omega) \to (W^{1,p_i}(\Omega))^*$ *is defined by*

$$\begin{aligned}\langle w, \widetilde{A_i}u\rangle = \int_\Omega < (1+\frac{|\nabla u|^{p_i}}{\sqrt{1+|\nabla u|^{2p_i}}})|\nabla u|^{p_i-2}\nabla u, \nabla v > dx \\ +\lambda_i \int_\Omega |u(x)|^{q_i-2}u(x)v(x)dx + \lambda_i \int_\Omega |u(x)|^{r_i-2}u(x)v(x)dx,\end{aligned}$$

for any $u, w \in W^{1,p_i}(\Omega)$;

(2) $A_i u = \{f \in L^2(\Omega) \mid f \in \widetilde{A_i}u\}$.

Then, $A_i : L^2(\Omega) \to L^2(\Omega)$ *is maximal monotone, for each* $i \in N$.

Lemma 15. *(see [18]) Define* $B_i : L^2(\Omega) \to L^2(\Omega)$ *by*

$$(B_i u)(x) = u(x) - f_i(x), \quad \text{for all } u(x) \in D(B_i),$$

and then B_i *is* θ_i*-inversely strongly accretive, for* $\theta_i \in (0,1]$ *and* $i \in N$.

Lemma 16. *(see [18]) If, in (46),* $f_i(x) \equiv \lambda_i(|k|^{q_i-1} + |k|^{r_i-1})sgnk + k$, *where k is a constant, then* $\{u^{(i)} \equiv k : i \in N\}$ *is the solution of capillarity system (46). Furthermore,* $\{k\} = \bigcap_{i=1}^\infty N(A_i + B_i)$.

Theorem 3. *Suppose* $f_i(x) \equiv \lambda_i(|k|^{q_i-1} + |k|^{r_i-1})sgnk + k$, A_i *and* B_i *are the same as those in Lemmas 14 and 15,* $F : L^2(\Omega) \to L^2(\Omega)$ *is a strongly positive linear bounded operator with coefficient* $\xi > 0$, $f : L^2(\Omega) \to L^2(\Omega)$ *is a contraction with coefficient* $k \in (0,1)$ *and* $W_i : L^2(\Omega) \to L^2(\Omega)$ *is* ϑ_i*-strongly monotone and* μ_i*-strictly pseudo-contractive mapping, for* $i \in N$.

Two iterative algorithms are constructed as follows:

$$\begin{cases} x_1, y_1 \in L^2(\Omega) \text{ chosen arbitrarily}, \ \ e_1, \varepsilon_1 \in L^2(\Omega) \text{ chosen arbitrarily}, \\ C_1 = L^2(\Omega) = Q_1, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^\infty a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ z_n = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^\infty c_i W_i)\sum_{i=1}^\infty a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)v_n, \\ w_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^\infty a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)z_n, \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F)w_n + e_n, \\ C_{n+1} = \{p \in C_n : 2\langle \alpha_n x_n + (1-\alpha_n)z_n - w_n, p\rangle \\ \qquad \leq \alpha_n \|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|w_n\|^2\} \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \sigma_{n+1}\}, \ \ n \in N, \\ y_{n+1} \in Q_{n+1}, \ \ n \in N, \end{cases} \tag{47}$$

and

$$\begin{cases} x_1, y_1 \in L^2(\Omega) \text{ chosen arbitrarily}, \ \ \varepsilon_1, e_1 \in L^2(\Omega) \text{ chosen arbitrarily}, \\ C_1 = L^2(\Omega) = Q_1, \\ u_n = \omega_n x_n + \varepsilon_n, \\ v_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+v_n}{2}), \\ z_n = \delta_n f(x_n) + (1-\delta_n)(I - \zeta_n \sum_{i=1}^{\infty} c_i W_i)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+v_n}{2}), \\ w_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{\infty} a_i (I + r_{n,i}A_i)^{-1}(I - r_{n,i}B_i)(\frac{u_n+z_n}{2}), \\ x_{n+1} = \lambda_n \eta f(x_n) + (I - \lambda_n F) w_n + e_n, \\ C_{n+1} = \{p \in C_n : 2\langle \alpha_n x_n + (1-\alpha_n) z_n - w_n, p\rangle \\ \qquad \leq \alpha_n \|x_n\|^2 + (1-\alpha_n)\|z_n\|^2 - \|w_n\|^2\} \\ Q_{n+1} = \{p \in C_{n+1} : \|x_1 - p\|^2 \leq \|P_{C_{n+1}}(x_1) - x_1\|^2 + \sigma_{n+1}\}, \ \ n \in N, \\ y_{n+1} \in Q_{n+1}, \ n \in N. \end{cases} \tag{48}$$

Under the assumptions of Theorems 1 and 2, one has $x_n \to q_0(x) \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, where $q_0(x)$ is common solution of the capillarity system (46) and the system of variational inclusions (14) and (15).

4. Conclusions

Some new forward–backward multi-choice iterative algorithm with superposition perturbations are presented in a real Hilbert space. The iterative sequences are proved to be strongly convergent to not only the solution of monotone inclusions but also the solution of variational inequalities. In the near future, more work can be done to weaken the restrictions imposed on the contraction f and the strongly positive linear bounded mapping F.

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