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Variational Problems with Time Delay and Higher-Order Distributed-Order Fractional Derivatives with Arbitrary Kernels

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Abstract: In this work, we study variational problems with time delay and higher-order distributed-order fractional derivatives dealing with a new fractional operator. This fractional derivative combines two known operators: distributed-order derivatives and derivatives with respect to another function. The main results of this paper are necessary and sufficient optimality conditions for different types of variational problems. Since we are dealing with generalized fractional derivatives, from this work, some well-known results can be obtained as particular cases.

Keywords: fractional calculus; calculus of variations; Euler–Lagrange equations; isoperimetric problems; holonomic problems; higher-order derivatives

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1. Introduction

Fractional calculus is a mathematical area that deals with the generalization of the classical notions of derivative and integral to a noninteger order. This fascinating theory has attracted the interest of the scientific community over the last few decades due to the fact that it is a powerful tool to deal with the dynamics of complex systems. Its importance is notable not only in Mathematics but also in Physics [1], Chemistry [2], Biology [3], Epidemiology [4], Control Theory [5], etc. (for completeness, we also point out that partial differential equations from classical calculus properly fit in the modeling of real problems; see, for instance, Refs. [6–8] for models from mathematical biology).

Since the beginning of the fractional calculus in 1695, numerous definitions of fractional integrals and derivatives were introduced by important mathematicians such as Leibniz, Euler, Fourier, Liouville, Riemann, Letnikov, etc. Many of these fractional derivatives can be related between them by an explicit formula [9,10]. Later on, in 1969, Caputo introduced the distributed-order fractional integrals and derivatives [11,12]. These operators can be seen as a new kind of generalization of the classical fractional operators, since these operators involve a weighted integral of different orders of differentiation. Another way that allows a generalization of the classical fractional operators is considering the notions of fractional integrals and derivatives with respect to another function [9,13,14].

The specificity of fractional calculus that can be considered the cause of its success in applications to real world problems is that the large number of fractional operators allows researchers to choose the most suitable one to model the problem under investigation.

In the recent paper [15], the authors introduced new notions of fractional derivatives combining the distributed-order derivatives and fractional derivatives with respect to an arbitrary smooth function, creating a new type of derivatives: distributed-order fractional derivatives with arbitrary kernels. In this paper, we are going to deal with these kinds of generalized fractional derivatives in order to study different types of problems of the calculus of variations.

The fractional calculus of variations was initiated by Riewe in 1996 [16,17] with the deduction of the Euler–Lagrange equation for problems where the Lagrangian depends on Riemann–Liouville fractional derivatives in order to deal with linear non-conservative forces. Since then, several authors have developed the fractional calculus of variations considering different types of fractional derivatives and different types of variational problems (see, e.g., [18–23] and references therein). For more details on fractional calculus of variations, we refer to the books [24–26].

It is well known that, in real world problems, delays are important to model certain processes and dynamical systems [22,27,28]. However, there are still few works in the literature dedicated to the fractional calculus of variations with time delay. To fill this gap, we will study in this paper time-delayed variational problems involving distributed-order fractional derivatives with arbitrary smooth kernels. We will also study variational problems involving higher-order distributed-order fractional derivatives with arbitrary smooth kernels.

The paper is organized as follows: in Section 2, we recall the new concepts of distributed-order fractional derivatives with respect to another function recently introduced in [15] and then we proceed with the extension to the higher-order case. We finalize Section 2 with the proof of the integration by parts formulae involving the higher-order distributed-order fractional derivatives with arbitrary smooth kernels. Section 3 is devoted to the main results of this paper: necessary and sufficient optimality conditions for variational problems with time delay and higher-order distributed-order fractional derivatives with arbitrary smooth kernels. In Section 4, we present three examples that illustrate the applicability of some of our main results. We finalize the paper with concluding remarks and also mentioning some possibilities for future research.

2. Preliminaries and Notations

We assume that the reader is familiar with the definitions and properties of the Riemann–Liouville and Caputo fractional operators with respect to another function (cf. [9,13], resp.).

In this paper, we consider variational problems involving the new concepts of distributed-order fractional derivatives with respect to an arbitrary smooth kernel recently introduced in [15]. For the reader's convenience, we recall here the definitions introduced in [15].

Let $\phi : [0,1] \to [0,1]$ be a continuous function such that

$$\int_0^1 \phi(\alpha) d\alpha > 0.$$

Definition 1 ([15]). *Let $x : [a,b] \to \mathbb{R}$ be an integrable function and $\psi \in C^1([a,b],\mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$. The left and right Riemann–Liouville distributed-order fractional derivatives of a function x with respect to ψ are defined by:*

$$\mathrm{D}_{a^+}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) \mathrm{D}_{a^+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{D}_{b^-}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) \mathrm{D}_{b^-}^{\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{D}_{a^+}^{\alpha,\psi}$ and $\mathrm{D}_{b^-}^{\alpha,\psi}$ are the left and right ψ-Riemann–Liouville fractional derivatives of order α, respectively.

Definition 2 ([15]). *Let $x, \psi \in C^1([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$. The left and right Caputo distributed-order fractional derivatives of x with respect to ψ are defined by*

$$^{C}\mathrm{D}_{a+}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) {}^{C}\mathrm{D}_{a+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad {}^{C}\mathrm{D}_{b-}^{\phi(\alpha),\psi} x(t) := \int_0^1 \phi(\alpha) {}^{C}\mathrm{D}_{b-}^{\alpha,\psi} x(t) d\alpha,$$

where ${}^{C}\mathrm{D}_{a+}^{\alpha,\psi}$ and ${}^{C}\mathrm{D}_{b-}^{\alpha,\psi}$ are the left and right ψ-Caputo fractional derivatives of order α, respectively.

Now, we will extend the definitions introduced in [15] to the higher-order case.

In the following, we assume that $n \in \mathbb{N}$ and $\phi : [n-1, n] \to [0,1]$ is a continuous function such that

$$\int_{n-1}^{n} \phi(\alpha) d\alpha > 0.$$

To the best of our knowledge, this is the first work that deals with higher-order distributed-order fractional derivatives.

Definition 3. *Let $x : [a,b] \to \mathbb{R}$ be an integrable function and $\psi \in C^n([a,b],\mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$. The left and right Riemann–Liouville distributed-order fractional derivatives of a function x with respect to the kernel ψ are defined by:*

$$\mathrm{D}_{a^+}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{D}_{a^+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{D}_{b^-}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{D}_{b^-}^{\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{D}_{a^+}^{\alpha,\psi}$ and $\mathrm{D}_{b^-}^{\alpha,\psi}$ are the left and right ψ-Riemann–Liouville fractional derivatives of order $\alpha \in [n-1, n]$, respectively.

Definition 4. *Let $x, \psi \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$. The left and right Caputo distributed-order fractional derivatives of x with respect to ψ are defined by*

$${}^{C}\mathrm{D}_{a+}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) {}^{C}\mathrm{D}_{a+}^{\alpha,\psi} x(t) d\alpha \quad \text{and} \quad {}^{C}\mathrm{D}_{b-}^{\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) {}^{C}\mathrm{D}_{b-}^{\alpha,\psi} x(t) d\alpha,$$

where ${}^{C}\mathrm{D}_{a+}^{\alpha,\psi}$ and ${}^{C}\mathrm{D}_{b-}^{\alpha,\psi}$ are the left and right ψ-Caputo fractional derivatives of order $\alpha \in [n-1, n]$, respectively.

In the following, we use the notations

$$\mathrm{I}_{a+}^{n-\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{I}_{a+}^{n-\alpha,\psi} x(t) d\alpha \quad \text{and} \quad \mathrm{I}_{b-}^{n-\phi(\alpha),\psi} x(t) := \int_{n-1}^{n} \phi(\alpha) \mathrm{I}_{b-}^{n-\alpha,\psi} x(t) d\alpha,$$

where $\mathrm{I}_{a+}^{n-\alpha,\psi}$ and $\mathrm{I}_{b-}^{n-\alpha,\psi}$ are, respectively, the left and right Riemann–Liouville fractional integrals of order $n-\alpha$ with respect to the kernel ψ. In addition, we fix two functions ϕ and ψ satisfying the assumptions above. In order to simplify notation, we will use the abbreviated symbol

$$x_{\psi}^{[m]}(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^{m} x(t).$$

Next, we prove the integration by parts formulae, which are fundamental tools for the proofs of our main results. In our previous work, we proved a similar result when the fractional order is between 0 and 1 [15] [Theorem 3.1]. In this paper, we present a generalization of such result for the case when function ϕ is defined on the interval $[n-1, n]$.

Theorem 1 (Integration by parts formulae)**.** *Let $x : [a,b] \to \mathbb{R}$ be a continuous function and $y \in C^n([a,b],\mathbb{R})$. Then,*

$$\begin{aligned} \int_a^b x(t) {}^{C}\mathrm{D}_{a^+}^{\phi(\alpha),\psi} y(t) dt &= \int_a^b \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{x(t)}{\psi'(t)} \right) \psi'(t) y(t) dt \\ &+ \left[\sum_{k=0}^{n-1} \left(\frac{-1}{\psi'(t)} \frac{d}{dt} \right)^{k} \left(\mathrm{I}_{b^-}^{n-\phi(\alpha),\psi} \frac{x(t)}{\psi'(t)} \right) y_{\psi}^{[n-k-1]}(t) \right]_{t=a}^{t=b} \end{aligned}$$

and

$$\begin{aligned}\int_a^b x(t)^C\mathrm{D}_{b^-}^{\phi(\alpha),\psi}y(t)dt &= \int_a^b \left(\mathrm{D}_{a^+}^{\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)\psi'(t)y(t)dt \\ &+ \left[\sum_{k=0}^{n-1}(-1)^{n-k}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k\left(\mathrm{I}_{a^+}^{n-\phi(\alpha),\psi}\frac{x(t)}{\psi'(t)}\right)y_\psi^{[n-k-1]}(t)\right]_{t=a}^{t=b}.\end{aligned}$$

Proof. Using the definition of the left ψ-Caputo distributed-order fractional derivative, we have

$$\begin{aligned}&\int_a^b x(t)^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi}y(t)dt = \int_a^b x(t)\int_{n-1}^n \phi(\alpha)^C\mathrm{D}_{a^+}^{\alpha,\psi}y(t)d\alpha\, dt \\ &= \int_a^b x(t)\int_{n-1}^n \frac{\phi(\alpha)}{\Gamma(n-\alpha)}\int_a^t\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n y(s)\cdot(\psi(t)-\psi(s))^{n-\alpha-1}\psi'(s)dsd\alpha\, dt \\ &= \int_a^b x(t)\int_{n-1}^n \frac{\phi(\alpha)}{\Gamma(n-\alpha)}\int_a^t\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)y_\psi^{[n-1]}(s)\cdot(\psi(t)-\psi(s))^{n-\alpha-1}\psi'(s)dsd\alpha\, dt \\ &= \int_{n-1}^n \frac{\phi(\alpha)}{\Gamma(n-\alpha)}\int_a^b x(t)\int_a^t \frac{d}{ds}y_\psi^{[n-1]}(s)\cdot(\psi(t)-\psi(s))^{n-\alpha-1}dsdt\, d\alpha.\end{aligned}$$

Applying Dirichlet's formula, we get

$$\begin{aligned}&\int_{n-1}^n \frac{\phi(\alpha)}{\Gamma(n-\alpha)}\int_a^b x(t)\int_a^t \frac{d}{ds}y_\psi^{[n-1]}(s)\cdot(\psi(t)-\psi(s))^{n-\alpha-1}dsdt\, d\alpha \\ &= \int_{n-1}^n \frac{\phi(\alpha)}{\Gamma(n-\alpha)}\int_a^b \frac{d}{ds}y_\psi^{[n-1]}(s)\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dtds\, d\alpha.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}&\int_a^b \frac{d}{ds}y_\psi^{[n-1]}(s)\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\, ds \\ &= \left[\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\cdot y_\psi^{[n-1]}(s)\right]_{s=a}^{s=b} \\ &- \int_a^b y_\psi^{[n-1]}(s)\frac{d}{ds}\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)ds \\ &= \left[\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\cdot y_\psi^{[n-1]}(s)\right]_{s=a}^{s=b} \\ &+ \int_a^b\left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)\cdot\frac{d}{ds}y_\psi^{[n-2]}(s)ds.\end{aligned}$$

Using integration by parts in the last integral, we obtain

$$\begin{aligned}&\int_a^b\left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)\cdot\frac{d}{ds}y_\psi^{[n-2]}(s)ds \\ &= \left[\left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)\cdot y_\psi^{[n-2]}(s)\right]_{s=a}^{s=b} \\ &- \int_a^b y_\psi^{[n-2]}(s)\frac{d}{ds}\left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)ds \\ &= \left[\left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)\cdot y_\psi^{[n-2]}(s)\right]_{s=a}^{s=b} \\ &+ \int_a^b\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^2\left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)\cdot\frac{d}{ds}y_\psi^{[n-3]}(s)ds.\end{aligned}$$

Since

$$
\begin{aligned}
&\int_a^b \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^2 \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right) \cdot \frac{d}{ds} y_\psi^{[n-3]}(s)ds \\
&= \left[\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^2 \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right) \cdot y_\psi^{[n-3]}(s)\right]_{s=a}^{s=b} \\
&- \int_a^b y_\psi^{[n-3]}(s)\frac{d}{ds}\left[\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^2 \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right)\right]ds \\
&= \left[\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^2 \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right) \cdot y_\psi^{[n-3]}(s)\right]_{s=a}^{s=b} \\
&+ \int_a^b \left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)^3 \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right) \cdot \frac{d}{ds} y_\psi^{[n-4]}(s)ds,
\end{aligned}
$$

then we get

$$
\begin{aligned}
&\int_a^b \frac{d}{ds} y_\psi^{[n-1]}(s) \int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\,ds \\
&= \left[\sum_{k=0}^{2}\left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)^k \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right) \cdot y_\psi^{[n-k-1]}(s)\right]_{s=a}^{s=b} \\
&+ \int_a^b \left(\frac{-1}{\psi'(s)}\frac{d}{ds}\right)^3 \left(\int_s^b x(t)(\psi(t)-\psi(s))^{n-\alpha-1}dt\right) \cdot \frac{d}{ds} y_\psi^{[n-4]}(s)ds.
\end{aligned}
$$

Repeating the process of integration by parts $n-3$ more times, we prove the formula. Using similar techniques, we deduce the integration by parts formula involving the operator ${}^C\mathrm{D}_{b^-}^{\phi(\alpha),\psi}$. $\square$

3. Main Results

3.1. Variational Problems with Time Delay

We begin this section by studying variational problems involving distributed-order fractional derivatives with time delay. For clarity of presentation, we restrict ourselves to the case where $\alpha \in [0,1]$, that is, considering the definitions introduced in [15].

Consider two continuous functions $\phi, \varphi : [0,1] \to [0,1]$ satisfying the following conditions

$$
\int_0^1 \phi(\alpha)d\alpha > 0 \qquad \text{and} \qquad \int_0^1 \varphi(\alpha)d\alpha > 0.
$$

In what follows, $a, b \in \mathbb{R}$ are such that $a < b$ and τ is a fixed real number satisfying the condition $0 \le \tau < b - a$.

We are now in position to present the first problem under study.

Problem 1 ((P_τ))**.** *Determine a curve $x \in C^1([a-\tau, b], \mathbb{R})$, subject to $x(t) = \mu(t)$ for all $t \in [a-\tau, a]$, where $\mu \in C^1([a-\tau, a], \mathbb{R})$ is a given initial function, that minimizes or maximizes the following functional:*

$$
\mathcal{J}(x) := \int_a^b L\Big(t, x(t), x(t-\tau), {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} x(t)\Big)dt, \tag{1}
$$

where $L : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ is assumed to be continuously differentiable with respect to the second, third, fourth, and fifth variables. We will consider the variational problem (P_τ) with and without fixed terminal boundary condition, and also with isoperimetric or holonomic constraints.

Let us fix the following notations: by $\partial_i L$, we denote the partial derivative of L with respect to its ith-coordinate and

$$[x]_\tau(t) := \Big(t, x(t), x(t-\tau), {}^C\mathrm{D}_{a+}^{\phi(\alpha),\psi} x(t), {}^C\mathrm{D}_{b-}^{\varphi(\alpha),\psi} x(t)\Big).$$

To simplify the presentation of our results, we consider the following conditions:

$$C_\phi^-[H,i,b-\tau]: \quad t \to \left(\mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi} \frac{\partial_i H[x]_\tau}{\psi'}\right)(t) \text{ is continuous for all } t \in [a, b-\tau]$$

$$C_\phi^-[H,i,b]: \quad t \to \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\partial_i H[x]_\tau}{\psi'}\right)(t) \text{ is continuous for all } t \in [b-\tau, b]$$

$$C_\varphi^+[H,i,a]: \quad t \to \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\partial_i H[x]_\tau}{\psi'}\right)(t) \text{ is continuous for all } t \in [a, b-\tau]$$

$$C_\varphi^+[H,i,b-\tau]: \quad t \to \left(\mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi} \frac{\partial_i H[x]_\tau}{\psi'}\right)(t) \text{ is continuous for all } t \in [b-\tau, b]$$

where H is a function and $i \in \mathbb{N}$.

Theorem 2 (Fractional Euler–Lagrange equations and natural boundary condition for problem (P_τ))**.** *Suppose that L satisfies the conditions $C_\phi^-[L,4,b-\tau]$, $C_\varphi^+[L,5,a]$, $C_\phi^-[L,4,b]$ and $C_\varphi^+[L,5,b-\tau]$. If $x \in C^1([a-\tau,b],\mathbb{R})$ is an extremizer of functional $\mathcal{J}$, then x satisfies the following Euler–Lagrange equations*

$$\begin{aligned}
&\partial_2 L[x]_\tau(t) + \partial_3 L[x]_\tau(t+\tau) + \left(\mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t) \\
&\quad - \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{b-\tau}^{b} (\psi(s)-\psi(t))^{-\alpha} \partial_4 L[x]_\tau(s)\, ds\, d\alpha = 0, \ \forall\, t \in [a, b-\tau]
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
&\partial_2 L[x]_\tau(t) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t) \\
&\quad + \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{b-\tau} (\psi(t)-\psi(s))^{-\alpha} \partial_5 L[x]_\tau(s)\, ds\, d\alpha = 0, \ \forall\, t \in [b-\tau, b].
\end{aligned} \tag{3}$$

If $x(b)$ is free, then the following natural boundary condition holds:

$$\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(b)}{\psi'(b)} = \mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(b)}{\psi'(b)}. \tag{4}$$

Proof. Consider that $h \in C^1([a-\tau,b],\mathbb{R})$ is an arbitrary function such that $h(t) = 0$, $a - \tau \le t \le a$. Define the function j by $j(\epsilon) := \mathcal{J}(x + \epsilon h)$, $\epsilon \in \mathbb{R}$. Since x is an extremizer of $\mathcal{J}$, $j'(0) = 0$, and we have that

$$\begin{aligned}
\int_a^b \Big(&\partial_2 L[x]_\tau(t) \cdot h(t) + \partial_3 L[x]_\tau(t) \cdot h(t-\tau) + \partial_4 L[x]_\tau(t) \cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t) \\
&+ \partial_5 L[x]_\tau(t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t)\Big) dt = 0.
\end{aligned} \tag{5}$$

Since

$$\int_a^b \partial_3 L[x]_\tau(t) \cdot h(t-\tau)dt = \int_{a-\tau}^{a} \partial_3 L[x]_\tau(t+\tau) \cdot h(t)dt + \int_a^{b-\tau} \partial_3 L[x]_\tau(t+\tau) \cdot h(t)dt,$$

and $h(t) = 0$ for $t \in [a-\tau, a]$, then we get

$$\int_a^b \partial_3 L[x]_\tau(t) \cdot h(t-\tau)dt = \int_a^{b-\tau} \partial_3 L[x]_\tau(t+\tau) \cdot h(t)dt. \tag{6}$$

Replacing (6) into (5), we get

$$\begin{aligned}\int_a^{b-\tau} \Big(\partial_2 L[x]_\tau(t) + \partial_3 L[x]_\tau(t+\tau)\Big) \cdot h(t)dt + \int_{b-\tau}^{b} \partial_2 L[x]_\tau(t) \cdot h(t)dt \\ + \int_a^b \Big(\partial_4 L[x]_\tau(t) \cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t) + \partial_5 L[x]_\tau(t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t)\Big)dt = 0.\end{aligned} \tag{7}$$

Note that, for all $t \in [a, b-\tau]$, we have

$$\begin{aligned}\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)} &= \mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)} \\ &- \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) \int_{b-\tau}^{b} (\psi(s) - \psi(t))^{-\alpha} \partial_4 L[x]_\tau(s) ds d\alpha\end{aligned} \tag{8}$$

and, for all $t \in [b-\tau, b]$, we have

$$\begin{aligned}\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)} &= \mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)} \\ &+ \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) \int_a^{b-\tau} (\psi(t) - \psi(s))^{-\alpha} \partial_5 L[x]_\tau(s) ds d\alpha = 0.\end{aligned} \tag{9}$$

Using Theorem 1 and (8), we obtain

$$\begin{aligned}&\int_a^b \partial_4 L[x]_\tau(t) \cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t)dt = \int_a^{b-\tau} \left(\left(\mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\right. \\ &\left. - \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{b-\tau}^{b} (\psi(s) - \psi(t))^{-\alpha} \partial_4 L[x]_\tau(s) ds d\alpha\right) h(t)dt \\ &+ \int_{b-\tau}^{b} \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)h(t)dt + \left[\left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)h(t)\right]_{t=a}^{t=b}.\end{aligned} \tag{10}$$

Once again, by Theorem 1 and (9), we obtain

$$\begin{aligned}&\int_a^b \partial_5 L[x]_\tau(t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t)dt = \int_{b-\tau}^{b} \left(\left(\mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\right. \\ &\left. + \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{b-\tau} (\psi(t) - \psi(s))^{-\alpha} \partial_5 L[x]_\tau(s) ds d\alpha\right) h(t)dt \\ &+ \int_a^{b-\tau} \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)h(t)dt - \left[\left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)h(t)\right]_{t=a}^{t=b}.\end{aligned} \tag{11}$$

Replacing (10) and (11) into (7), we get that

$$\begin{aligned}
&\int_a^{b-\tau}\Bigg(\partial_2 L[x]_\tau(t)+\partial_3 L[x]_\tau(t+\tau)+\left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-}\frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\\
&-\int_0^1\frac{\phi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{b-\tau}^b(\psi(s)-\psi(t))^{-\alpha}\partial_4 L[x]_\tau(s)ds d\alpha+\left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+}\frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\Bigg)h(t)dt\\
&+\int_{b-\tau}^b\Bigg(\partial_2 L[x]_\tau(t)+\left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-}\frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)+\left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+}\frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\\
&+\int_0^1\frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^{b-\tau}(\psi(t)-\psi(s))^{-\alpha}\partial_5 L[x]_\tau(s)ds d\alpha\Bigg)h(t)dt\\
&+\left[\left(\mathrm{I}^{1-\phi(\alpha),\psi}_{b^-}\frac{\partial_4 L[x]_\tau(t)}{\psi'(t)}\right)h(t)\right]_{t=a}^{t=b}-\left[\left(\mathrm{I}^{1-\varphi(\alpha),\psi}_{a^+}\frac{\partial_5 L[x]_\tau(t)}{\psi'(t)}\right)h(t)\right]_{t=a}^{t=b}=0.
\end{aligned} \tag{12}$$

From the arbitrariness of h, we get the desired Equations (2)–(4). □

Next, we consider the case where we add to problem (P_τ) an isoperimetric restriction.

Problem 2 ((P_{I_τ}))**.** *The isoperimetric problem with a time delay τ can be formulated in the following way: minimize or maximize the functional $\mathcal{J}$ in (1) subject to an integral constraint of type*

$$\mathcal{I}(x):=\int_a^b G[x]_\tau(t)dt=k, \tag{13}$$

where $k\in\mathbb{R}$ is fixed and $G:[a,b]\times\mathbb{R}^4\to\mathbb{R}$ is a continuously differentiable function with respect to the second, third, fourth, and fifth variables.

The following theorem presents necessary conditions for x to be a solution of the fractional isoperimetric problem (P_{I_τ}) under the assumption that x is not an extremal for G.

Theorem 3 (Necessary optimality conditions for problem (P_{I_τ})—Case I)**.** *Let $x\in C^1([a-\tau,b],\mathbb{R})$ be a curve such that $\mathcal{J}$ attains an extremum at x, when subject to the integral constraint (13). Assume that x does not satisfy the Euler–Lagrange Equation (2) or (3) with respect to G. Moreover, suppose that L satisfies the conditions $C^-_\phi[L,4,b-\tau]$, $C^+_\varphi[L,5,a]$, $C^-_\phi[L,4,b]$ and $C^+_\varphi[L,5,b-\tau]$, and G satisfies the conditions $C^-_\phi[G,4,b-\tau]$, $C^+_\varphi[G,5,a]$, $C^-_\phi[G,4,b]$ and $C^+_\varphi[G,5,b-\tau]$. Then, there exists $\lambda\in\mathbb{R}$ such that x is a solution of the equations*

$$\begin{aligned}
&\partial_2 H[x]_\tau(t)+\partial_3 H[x]_\tau(t+\tau)+\left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-}\frac{\partial_4 H[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)+\left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+}\frac{\partial_5 H[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\\
&\quad-\int_0^1\frac{\phi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{b-\tau}^b(\psi(s)-\psi(t))^{-\alpha}\partial_4 H[x]_\tau(s)ds d\alpha=0,\ \forall\, t\in[a,b-\tau]
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\partial_2 H[x]_\tau(t)+\left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-}\frac{\partial_4 H[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)+\left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+}\frac{\partial_5 H[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\\
&\quad+\int_0^1\frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^{b-\tau}(\psi(t)-\psi(s))^{-\alpha}\partial_5 H[x]_\tau(s)ds d\alpha=0,\ \forall\, t\in[b-\tau,b],
\end{aligned} \tag{15}$$

where $H:=L+\lambda G$.

If $x(b)$ is free, then

$$\mathrm{I}^{1-\phi(\alpha),\psi}_{b^-}\frac{\partial_4 H[x]_\tau(b)}{\psi'(b)}=\mathrm{I}^{1-\varphi(\alpha),\psi}_{a^+}\frac{\partial_5 H[x]_\tau(b)}{\psi'(b)}. \tag{16}$$

Proof. The proof follows from the ideas presented in Theorem 2 and Theorem 3.3 of [15]. □

Now, we present necessary optimality conditions for the case when the solution of the isoperimetric problem is an extremal for the fractional isoperimetric functional (13).

Theorem 4 (Necessary optimality conditions for fractional problem (P_{I_τ})—Case II)**.** *Let x be a curve such that $\mathcal{J}$ attains an extremum at x, when subject to the integral constraint (13). Moreover, suppose that L satisfies the conditions $C_\phi^-[L,4,b-\tau]$, $C_\varphi^+[L,5,a]$, $C_\phi^-[L,4,b]$ and $C_\varphi^+[L,5,b-\tau]$, and G satisfies the conditions $C_\phi^-[G,4,b-\tau]$, $C_\varphi^+[G,5,a]$, $C_\phi^-[G,4,b]$ and $C_\varphi^+[G,5,b-\tau]$. Then, there exists a vector $(\lambda_0,\lambda)\in\mathbb{R}^2\setminus\{(0,0)\}$ such that x is a solution of Equations (14) and (15), with the Hamiltonian H defined as $H := \lambda_0 L + \lambda G$. If $x(b)$ is free, then x must satisfy Equation (16).*

Proof. The result is an immediate consequence of Theorem 3. □

In the following, we study variational problems with a holonomic constraint. For this purpose, we now assume that x is a two-dimensional vector function and $L : [a,b]\times\mathbb{R}^8 \to \mathbb{R}$ is assumed to be continuously differentiable with respect to the ith variable, with $i = 2,\ldots,9$.

Problem 3 ((P_{C_τ}))**.** *Consider the variational problem (P_τ) but in the presence of a holonomic constraint:*

$$g(t,x(t)) = 0, \quad t\in[a,b], \tag{17}$$

where $g : [a,b]\times\mathbb{R}^2 \to \mathbb{R}$ is a C^1 function. The state variable x is a two-dimensional vector function $x = (x_1,x_2)$, where $x_1,x_2 \in C^1([a-\tau,b],\mathbb{R})$. Moreover, the boundary condition

$$x(t) = \mu(t),\ t\in[a-\tau,a], \tag{18}$$

where $\mu \in C^1([a-\tau,a],\mathbb{R})\times C^1([a-\tau,a],\mathbb{R})$ is a given function, is imposed.

Theorem 5 (Necessary optimality conditions for problem (P_{C_τ}))**.** *Consider the functional*

$$\mathcal{J}(x) = \int_a^b L[x]_\tau(t)dt, \tag{19}$$

defined on $C^1([a-\tau,b],\mathbb{R})\times C^1([a-\tau,b],\mathbb{R})$ and subject to the constraints (17) and (18). Suppose that L satisfies the conditions $C_\phi^-[L,i+5,b-\tau]$, $C_\varphi^+[L,i+7,a]$, $C_\phi^-[L,i+5,b]$ and $C_\varphi^+[L,i+7,b-\tau]$, with $i=1,2$.

If x is an extremizer of functional $\mathcal{J}$ and if

$$\partial_3 g(t,x(t)) \neq 0, \quad \forall t\in[a,b],$$

then there exists a continuous function $\lambda : [a,b]\to\mathbb{R}$ such that x is a solution of

$$\begin{gathered}\partial_{i+1}L[x]_\tau(t) + \partial_{i+3}L[x]_\tau(t+\tau) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-}\frac{\partial_{i+5}L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\\ +\left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+}\frac{\partial_{i+7}L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t) - \int_0^1\frac{\phi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{b-\tau}^b(\psi(s)-\psi(t))^{-\alpha}\partial_{i+5}L[x]_\tau(s)dsd\alpha\\ +\lambda(t)\cdot\partial_{i+1}g(t,x(t)) = 0, \quad \forall t\in[a,b-\tau],\ i=1,2\end{gathered} \tag{20}$$

and

$$\begin{gathered}\partial_{i+1}L[x]_\tau(t) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-}\frac{\partial_{i+5}L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+}\frac{\partial_{i+7}L[x]_\tau(t)}{\psi'(t)}\right)\psi'(t)\\ +\int_0^1\frac{\varphi(\alpha)}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^{b-\tau}(\psi(t)-\psi(s))^{-\alpha}\partial_{i+7}L[x]_\tau(s)dsd\alpha + \lambda(t)\cdot\partial_{i+1}g(t,x(t)) = 0,\\ \forall t\in[b-\tau,b],\ i=1,2.\end{gathered} \tag{21}$$

If $x(b)$ is free, then, for $i = 1, 2$,

$$\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\partial_{i+5} L[x]_\tau(b)}{\psi'(b)} = \mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\partial_{i+7} L[x]_\tau(b)}{\psi'(b)}. \tag{22}$$

Proof. The proof follows combining the ideas from Theorem 2 above with Theorem 3.5 from [15]. □

Now, we focus our attention on sufficient optimality conditions for all the variational problems studied previously.

Definition 5. *Function $f(t, x_2, x_3, ..., x_n)$ defined on $U \subseteq \mathbb{R}^n$ is called convex (resp. concave) if $\partial_i f(t, x_2, x_3, ..., x_n)$, $i = 2, \ldots, n$, exist and are continuous, and if*

$$f(t, x_2 + h_2, x_3 + h_3, ..., x_n + h_n) - f(t, x_2, x_3, ..., x_n) \geq \text{ (resp. } \leq \text{) } \sum_{i=2}^{n} \partial_i f(t, x_2, x_3, ..., x_n) h_i$$

for all $(t, x_2, x_3, ..., x_n), (t, x_2 + h_2, x_3 + h_3, ..., x_n + h_n) \in U$.

Theorem 6 (Sufficient optimality conditions for problem (P_τ))**.** *Let L be convex (resp. concave) in $[a, b] \times \mathbb{R}^4$. Then, each solution $\overline{x}$ of the fractional Euler–Lagrange Equations (2) and (3) minimizes (resp. maximizes) the functional $\mathcal{J}$ given in (1), subject to the boundary conditions $x(t) = \mu(t)$, $t \in [a - \tau, a]$ and $x(b) = \overline{x}(b)$. If $x(b)$ is free, then each solution $\overline{x}$ of the Equations (2)–(4) minimizes (resp. maximizes) $\mathcal{J}$.*

Proof. We prove the case when L is convex. The other case is similar. Consider $h \in C^1([a - \tau, b], \mathbb{R})$ an arbitrary function. Since L is convex, we can conclude that

$$\begin{aligned}\mathcal{J}(\overline{x} + h) - \mathcal{J}(\overline{x}) \geq \int_a^b \Big(&\partial_2 L[\overline{x}]_\tau(t) \cdot h(t) + \partial_3 L[\overline{x}]_\tau(t) \cdot h(t - \tau) \\ &+ \partial_4 L[\overline{x}]_\tau(t) \cdot {}^C\mathrm{D}_{a^+}^{\phi(\alpha),\psi} h(t) + \partial_5 L[\overline{x}]_\tau(t) \cdot {}^C\mathrm{D}_{b^-}^{\varphi(\alpha),\psi} h(t) \Big) dt.\end{aligned}$$

Using the same techniques used in the proof of Theorem 2, we get

$$\begin{aligned}
&\mathcal{J}(\overline{x} + h) - \mathcal{J}(\overline{x}) \geq \int_a^{b-\tau} \Bigg(\partial_2 L[\overline{x}]_\tau(t) + \partial_3 L[\overline{x}]_\tau(t + \tau) + \left(\mathrm{D}_{(b-\tau)^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t) \\
&- \int_0^1 \frac{\phi(\alpha)}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_{b-\tau}^b (\psi(s) - \psi(t))^{-\alpha} \partial_4 L[\overline{x}]_\tau(s) ds d\alpha + \left(\mathrm{D}_{a^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t) \Bigg) h(t) dt \\
&+ \int_{b-\tau}^b \Bigg(\partial_2 L[\overline{x}]_\tau(t) + \left(\mathrm{D}_{b^-}^{\phi(\alpha),\psi} \frac{\partial_4 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t) + \left(\mathrm{D}_{(b-\tau)^+}^{\varphi(\alpha),\psi} \frac{\partial_5 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t) \\
&+ \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_a^{b-\tau} (\psi(t) - \psi(s))^{-\alpha} \partial_5 L[\overline{x}]_\tau(s) ds d\alpha \Bigg) h(t) dt \\
&+ \left[\left(\mathrm{I}_{b^-}^{1-\phi(\alpha),\psi} \frac{\partial_4 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) h(t) \right]_{t=a}^{t=b} - \left[\left(\mathrm{I}_{a^+}^{1-\varphi(\alpha),\psi} \frac{\partial_5 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) h(t) \right]_{t=a}^{t=b}.
\end{aligned} \tag{23}$$

If $x(b)$ is fixed then $h(a) = h(b) = 0$, and so from (23) we obtain

$$\mathcal{J}(\overline{x}+h) - \mathcal{J}(\overline{x}) \geq \int_a^{b-\tau} \Bigg(\partial_2 L[\overline{x}]_\tau(t) + \partial_3 L[\overline{x}]_\tau(t+\tau) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-} \frac{\partial_4 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t)$$
$$- \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{b-\tau}^b (\psi(s) - \psi(t))^{-\alpha} \partial_4 L[\overline{x}]_\tau(s) ds d\alpha + \left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+} \frac{\partial_5 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t) \Bigg) h(t) dt$$
$$+ \int_{b-\tau}^b \Bigg(\partial_2 L[\overline{x}]_\tau(t) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-} \frac{\partial_4 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t) + \left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+} \frac{\partial_5 L[\overline{x}]_\tau(t)}{\psi'(t)} \right) \psi'(t)$$
$$+ \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{b-\tau} (\psi(t) - \psi(s))^{-\alpha} \partial_5 L[\overline{x}]_\tau(s) ds d\alpha \Bigg) h(t) dt.$$

Since $\overline{x}$ is a solution of the fractional Euler–Lagrange Equations (2) and (3), then we conclude that $\mathcal{J}(\overline{x}+h) - \mathcal{J}(\overline{x}) \geq 0$. The case when $x(b)$ is free follows by considering $h(t) = 0$, $t \in [a-\tau, a]$ and $h(b)$ non-zero in (23). $\square$

Using similar techniques as the ones used in the proof of the last theorem, we can prove the following two results.

Theorem 7 (Sufficient optimality conditions for problem (P_{I_τ}))**.** *Let us assume that, for some constant λ, the functions L and λG are convex (resp. concave) in $[a,b] \times \mathbb{R}^4$ and define the function H as $H = L + \lambda G$. Then, each solution $\overline{x}$ of the fractional Equations (14) and (15) minimizes (resp. maximizes) the functional $\mathcal{J}$ given in (1), subject to the restrictions $x(t) = \mu(t)$, $t \in [a-\tau, a]$ and $x(b) = \overline{x}(b)$, and the integral constraint (13). If $x(b)$ is free, then each solution $\overline{x}$ of the fractional Equations (14)–(16) minimizes (resp. maximizes) $\mathcal{J}$ subject to (13).*

Theorem 8 (Sufficient optimality conditions for problem (P_{C_τ}))**.** *Consider the functional $\mathcal{J}$ defined in (19), where the Lagrangian function L is convex (resp. concave) in $[a,b] \times \mathbb{R}^7$. Define function $\lambda : [a,b] \to \mathbb{R}$ by*

$$\lambda(t) := -\frac{1}{\partial_3 g(t, x(t))} \Bigg(\partial_3 L[x]_\tau(t) + \partial_5 L[x]_\tau(t+\tau) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{(b-\tau)^-} \frac{\partial_7 L[x]_\tau(t)}{\psi'(t)} \right) \psi'(t)$$
$$+ \left(\mathrm{D}^{\varphi(\alpha),\psi}_{a^+} \frac{\partial_9 L[x]_\tau(t)}{\psi'(t)} \right) \psi'(t) - \int_0^1 \frac{\phi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{b-\tau}^b (\psi(s) - \psi(t))^{-\alpha} \partial_7 L[x]_\tau(s) ds d\alpha \Bigg),$$

for $t \in [a, b-\tau]$, and

$$\lambda(t) := -\frac{1}{\partial_3 g(t, x(t))} \Bigg(\partial_3 L[x]_\tau(t) + \left(\mathrm{D}^{\phi(\alpha),\psi}_{b^-} \frac{\partial_7 L[x]_\tau(t)}{\psi'(t)} \right) \psi'(t)$$
$$+ \left(\mathrm{D}^{\varphi(\alpha),\psi}_{(b-\tau)^+} \frac{\partial_9 L[x]_\tau(t)}{\psi'(t)} \right) \psi'(t) + \int_0^1 \frac{\varphi(\alpha)}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^{b-\tau} (\psi(t) - \psi(s))^{-\alpha} \partial_9 L[x]_\tau(s) ds d\alpha \Bigg),$$

for $t \in [b-\tau, b]$, where g is a C^1 function, such that $\partial_3 g(t, x(t)) \neq 0$ for all $t \in [a,b]$. Then, each solution $\overline{x} = (\overline{x}_1, \overline{x}_2)$ of the Equations (20) and (21) minimizes (resp. maximizes) the functional $\mathcal{J}$, subject to the restrictions $x(t) = \mu(t)$, $t \in [a-\tau, a]$ and $x(b) = \overline{x}(b)$, and the holonomic constraint (17). In addition, if $x(b)$ is free, then each solution $\overline{x}$ of the fractional Equations (20)–(22) minimizes (resp. maximizes) $\mathcal{J}$ subject to (17).

3.2. Higher-Order Variational Problems

In this section, we consider the general case with respect to fractional orders. Thus, the distributions ϕ_i, φ_i have domain $[i-1, i]$, $i = 1, \dots, n$, where $n \in \mathbb{N}$ is fixed, with

$$\int_{i-1}^{i} \phi_i(\alpha) d\alpha > 0 \quad \text{and} \quad \int_{i-1}^{i} \varphi_i(\alpha) d\alpha > 0.$$

The problem is formulated as follows:

Problem 4 ((P_n))**.** *Find a curve $x \in C^n([a,b], \mathbb{R})$ for which the functional*

$$\mathcal{J}(x) := \int_a^b L\Big(t, x(t), {}^C\mathrm{D}_{a^+}^{\phi_1(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi_1(\alpha),\psi} x(t), \dots, {}^C\mathrm{D}_{a^+}^{\phi_n(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi_n(\alpha),\psi} x(t)\Big) dt, \tag{24}$$

attains a minimum or a maximum value, where $L : [a,b] \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is a continuously differentiable function. In addition, the following boundary conditions

$$x^{(i)}(a) = x_a^i \quad \text{and} \quad x^{(i)}(b) = x_b^i, \text{ with } x_a^i, x_b^i \in \mathbb{R}, \ i = 0, \dots, n-1 \tag{25}$$

may be assumed.

We will consider the variational problem (P_n) with and without fixed boundary conditions (25), and also with isoperimetric or holonomic constraints.

As done previously, we use the abbreviations

$$[x]_n(t) := \Big(t, x(t), {}^C\mathrm{D}_{a^+}^{\phi_1(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi_1(\alpha),\psi} x(t), \dots, {}^C\mathrm{D}_{a^+}^{\phi_n(\alpha),\psi} x(t), {}^C\mathrm{D}_{b^-}^{\varphi_n(\alpha),\psi} x(t)\Big)$$

and

$$C_{\phi_i}^-[H,j]: \quad t \to \left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi} \frac{\partial_j H[x]_n}{\psi'}\right)(t) \text{ is continuous for all } t \in [a,b]$$

$$C_{\varphi_i}^+[H,j]: \quad t \to \left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi} \frac{\partial_j H[x]_n}{\psi'}\right)(t) \text{ is continuous for all } t \in [a,b]$$

where H is a function and $i, j \in \mathbb{N}$.

Theorem 9 (Fractional Euler–Lagrange equation and natural boundary conditions for problem (P_n))**.** *Let $x \in C^n([a,b], \mathbb{R})$ be an extremizer of functional $\mathcal{J}$ defined by (24). If conditions $C_{\phi_i}^-[L, 2i+1]$ and $C_{\varphi_i}^+[L, 2i+2]$ hold, for all $i \in \{1, \dots, n\}$, then x satisfies the following Euler–Lagrange equation:*

$$\partial_2 L[x]_n(t) + \sum_{i=1}^n \left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi} \frac{\partial_{2i+1} L[x]_n(t)}{\psi'(t)}\right)\psi'(t) + \left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi} \frac{\partial_{2i+2} L[x]_n(t)}{\psi'(t)}\right)\psi'(t)\right] = 0, \tag{26}$$

for all $t \in [a,b]$. In addition, if $x^{(i)}(a)$ are free, for $i = 0, \dots, n-1$, then

$$\sum_{k=i+1}^{n} \left[\left(\left(-\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k-i-1} \left(\mathrm{I}_{b^-}^{k-\phi_k(\alpha),\psi} \frac{\partial_{2k+1} L[x]_n(t)}{\psi'(t)}\right) + (-1)^{i+1}\left(\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k-i-1} \left(\mathrm{I}_{a^+}^{k-\varphi_k(\alpha),\psi} \frac{\partial_{2k+2} L[x]_n(t)}{\psi'(t)}\right)\right)\right] = 0, \quad \text{at } t = a, \tag{27}$$

and if $x^{(i)}(b)$ *are free, for* $i = 0, ..., n-1$, *then*

$$\sum_{k=i+1}^{n}\left[\left(\left(-\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k-i-1}\left(\mathrm{I}_{b^-}^{k-\phi_k(\alpha),\psi}\frac{\partial_{2k+1}L[x]_n(t)}{\psi'(t)}\right)\right.\right.$$
$$\left.\left.+(-1)^{i+1}\left(\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k-i-1}\left(\mathrm{I}_{a^+}^{k-\varphi_k(\alpha),\psi}\frac{\partial_{2k+2}L[x]_n(t)}{\psi'(t)}\right)\right)\right]=0,\quad at\ t=b. \quad (28)$$

Proof. Let $h \in C^n([a,b],\mathbb{R})$ be a function. Observe that, given $i \in \{0, ..., n-1\}$, if $x^{(i)}(a)$ or $x^{(i)}(b)$ are fixed, then we need to assume that $h^{(i)}(a) = 0$ or $h^{(i)}(b) = 0$, respectively, and so

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{i}h(t)=0, \text{ at } t=a \text{ or } t=b,$$

respectively. Defining j as $j(\epsilon) := \mathcal{J}(x+\epsilon h)$, $\epsilon \in \mathbb{R}$, then $j'(0) = 0$, and so

$$\int_a^b\left(\partial_2 L[x]_n(t)\cdot h(t)+\sum_{i=1}^{n}\left(\partial_{2i+1}L[x]_n(t)\cdot{}^{C}\mathrm{D}_{a^+}^{\phi_i(\alpha),\psi}h(t)\right.\right.$$
$$\left.\left.+\partial_{2i+2}L[x]_n(t)\cdot{}^{C}\mathrm{D}_{b^-}^{\varphi_i(\alpha),\psi}h(t)\right)\right)dt=0.$$

Using Theorem 1, we obtain, for each $i \in \{1, ..., n\}$,

$$\int_a^b \partial_{2i+1}L[x]_n(t)\cdot{}^{C}\mathrm{D}_{a^+}^{\phi_i(\alpha),\psi}h(t)dt=\int_a^b\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi}\frac{\partial_{2i+1}L[x]_n(t)}{\psi'(t)}\right)\psi'(t)h(t)dt$$
$$+\sum_{k=0}^{i-1}\left[\left(-\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k}\left(\mathrm{I}_{b^-}^{i-\phi_i(\alpha),\psi}\frac{\partial_{2i+1}L[x]_n(t)}{\psi'(t)}\right)\cdot h_\psi^{[i-k-1]}(t)\right]_{t=a}^{t=b}$$

and

$$\int_a^b \partial_{2i+2}L[x]_n(t)\cdot{}^{C}\mathrm{D}_{b^-}^{\varphi_i(\alpha),\psi}h(t)dt=\int_a^b\left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}L[x]_n(t)}{\psi'(t)}\right)\psi'(t)h(t)dt$$
$$+\sum_{k=0}^{i-1}\left[(-1)^{i-k}\left(\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k}\left(\mathrm{I}_{a^+}^{i-\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}L[x]_n(t)}{\psi'(t)}\right)\cdot h_\psi^{[i-k-1]}(t)\right]_{t=a}^{t=b}.$$

Therefore,

$$\int_a^b\left(\partial_2 L[x]_n(t)+\sum_{i=1}^{n}\left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi}\frac{\partial_{2i+1}L[x]_n(t)}{\psi'(t)}\right)\psi'(t)\right.\right.$$
$$\left.\left.+\left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}L[x]_n(t)}{\psi'(t)}\right)\psi'(t)\right]\right)h(t)dt$$
$$+\sum_{i=1}^{n}\sum_{k=0}^{i-1}\left[\left(\left(-\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k}\left(\mathrm{I}_{b^-}^{i-\phi_i(\alpha),\psi}\frac{\partial_{2i+1}L[x]_n(t)}{\psi'(t)}\right)\right.\right.$$
$$\left.\left.+(-1)^{i-k}\left(\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k}\left(\mathrm{I}_{a^+}^{i-\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}L[x]_n(t)}{\psi'(t)}\right)\right)h_\psi^{[i-k-1]}(t)\right]_{t=a}^{t=b}=0.$$

Since

$$\sum_{i=1}^{n}\sum_{k=0}^{i-1}\left[\left(\left(-\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k}\left(\mathrm{I}_{b^-}^{i-\phi_i(\alpha),\psi}\frac{\partial_{2i+1}L[x]_n(t)}{\psi'(t)}\right)\right.\right.$$
$$\left.\left.+(-1)^{i-k}\left(\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k}\left(\mathrm{I}_{a^+}^{i-\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}L[x]_n(t)}{\psi'(t)}\right)\right)h_{\psi}^{[i-k-1]}(t)\right]_{t=a}^{t=b}$$
$$=\sum_{i=0}^{n-1}h_{\psi}^{[i]}(t)\sum_{k=i+1}^{n}\left[\left(\left(-\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k-i-1}\left(\mathrm{I}_{b^-}^{k-\phi_k(\alpha),\psi}\frac{\partial_{2k+1}L[x]_n(t)}{\psi'(t)}\right)\right.\right.$$
$$\left.\left.+(-1)^{i+1}\left(\frac{1}{\psi'(t)}\frac{1}{dt}\right)^{k-i-1}\left(\mathrm{I}_{a^+}^{k-\varphi_k(\alpha),\psi}\frac{\partial_{2k+2}L[x]_n(t)}{\psi'(t)}\right)\right)\right]_{t=a}^{t=b},$$

from the arbitrariness of h, we prove (26), (27), and (28). □

When in the presence of an isoperimetric or holonomic contraint, similar results are proven for this new variational problem. To simplify, we will assume that the boundary conditions (25) hold. In addition, the proofs will be omitted since they follow the same pattern as the ones presented before.

Problem 5 ((P_{In})). *The isoperimetric problem can be formulated as follows: minimize or maximize the functional $\mathcal{J}$ in (24) assuming the boundary conditions (25) and also an integral restriction*

$$\mathcal{I}(x)=\int_a^b G[x]_n(t)dt=k,\ \ k\in\mathbb{R}, \tag{29}$$

where $G:[a,b]\times\mathbb{R}^{2n+1}\to\mathbb{R}$ is a C^1 function.

Theorem 10 (Necessary optimality conditions for problem (P_{In})—Case I). *Let $x\in C^n([a,b],\mathbb{R})$ be a solution of problem (P_{In}). Suppose that there exists some $t\in[a,b]$ such that*

$$\partial_2 G[x]_n(t)+\sum_{i=1}^{n}\left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi}\frac{\partial_{2i+1}G[x]_n(t)}{\psi'(t)}\right)\psi'(t)+\left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}G[x]_n(t)}{\psi'(t)}\right)\psi'(t)\right]\neq 0. \tag{30}$$

If conditions $C^-_{\phi_i}[L,2i+1]$, $C^+_{\varphi_i}[L,2i+2]$, $C^-_{\phi_i}[G,2i+1]$, and $C^+_{\varphi_i}[G,2i+2]$ hold, for all $i\in\{1,...,n\}$, then there exists a real number λ such that x is a solution of the equation

$$\partial_2 H[x]_n(t)+\sum_{i=1}^{n}\left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi}\frac{\partial_{2i+1}H[x]_n(t)}{\psi'(t)}\right)\psi'(t)+\left(\mathrm{D}_{a^+}^{\varphi_i(\alpha),\psi}\frac{\partial_{2i+2}H[x]_n(t)}{\psi'(t)}\right)\psi'(t)\right]=0, \tag{31}$$

for all $t\in[a,b]$, where $H:=L+\lambda G$.

Theorem 11 (Necessary optimality conditions for problem (P_{In})—Case II). *Let $x\in C^n([a,b],\mathbb{R})$ be a solution of problem (P_{In}). If conditions $C^-_{\phi_i}[L,2i+1]$, $C^+_{\varphi_i}[L,2i+2]$, $C^-_{\phi_i}[G,2i+1]$, and $C^+_{\varphi_i}[G,2i+2]$ hold, for all $i\in\{1,...,n\}$, then there exists a vector $(\lambda_0,\lambda)\in\mathbb{R}^2\setminus\{(0,0)\}$ such that x is a solution of Equation (31) for all $t\in[a,b]$, with the Hamiltonian H defined as $H:=\lambda_0 L+\lambda G$.*

To finish this section, we will study problem (P_n) with a holonomic constraint.

Problem 6 ((P_{Cn})). *The objective is to find $x\in C^n([a,b],\mathbb{R})\times C^n([a,b],\mathbb{R})$ that minimizes or maximizes the functional*

$$\mathcal{J}(x)=\int_a^b L[x]_n(t)dt, \tag{32}$$

defined on $C^n([a,b],\mathbb{R}) \times C^n([a,b],\mathbb{R})$ *and subject a constraint*

$$g(t,x(t)) = 0, \quad t \in [a,b], \tag{33}$$

where $g : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ *is a* C^n *function. In addition, boundary conditions*

$$x^{(i)}(a) = x_a^{(i)} \ \text{ and } \ x^{(i)}(b) = x_b^{(i)}, \ \ x_a^i, x_b^i \in \mathbb{R}^2 \ \text{ for } \ i = 0, \ldots, n-1 \tag{34}$$

are imposed on the variational problem.

Theorem 12 (Necessary optimality conditions for problem (P_{Cn}))**.** *Let* x *be an extremizer of functional* $\mathcal{J}$ *defined by* (32) *and subject to the constraints* (33)–(34)*. If conditions* $C^-_{\phi_i}[L, 4i+j-1]$ *and* $C^+_{\phi_i}[L, 4i+j+1]$ *hold for all* $i \in \{1, \ldots, n\}$ *and* $j = 1, 2$*, and if*

$$\partial_3 g(t,x(t)) \neq 0, \ \ \forall\, t \in [a,b],$$

then there exists a continuous function $\lambda : [a,b] \to \mathbb{R}$ *such that* x *is a solution of*

$$\begin{gathered}\partial_{j+1}L[x]_n(t) + \textstyle\sum_{i=1}^n \left[\left(\mathrm{D}_{b^-}^{\phi_i(\alpha),\psi} \frac{\partial_{4i+j-1}L[x]_n(t)}{\psi'(t)} \right) \psi'(t) + \left(\mathrm{D}_{a^+}^{\phi_i(\alpha),\psi} \frac{\partial_{4i+j+1}L[x]_n(t)}{\psi'(t)} \right) \psi'(t) \right] \\ + \lambda(t) \partial_{j+1} g(t,x(t)) = 0,\end{gathered} \tag{35}$$

for all $t \in [a,b]$ *and* $j = 1, 2$.

Remark 1. *In a similar way, we can prove that, in case function* L *is convex (resp. concave), then the conditions given in Theorems* 9–12 *are also sufficient conditions to ensure that the candidates of extremizers are indeed minimizers (resp. maximizers) of the functional.*

4. Illustrative Examples

Some illustrative examples are provided to demonstrate the applicability of our results.

Example 1. *Suppose we intend to find a function* $\overline{x} \in C^3([0,1],\mathbb{R})$*, subject to the initial conditions* $x(0) = (\psi(1) - \psi(0))^5$, $x'(0) = -5\psi'(0)(\psi(1) - \psi(0))^4$, $x''(0) = -5\psi''(0)(\psi(1) - \psi(0))^4 + 20(\psi'(0))^2(\psi(1) - \psi(0))^3$*, and terminal conditions* $x(1) = x'(1) = x''(1) = 0$*, that extremizes the functional*

$$\begin{aligned}\mathcal{J}(x) = \int_0^1 \Big({}^C\mathrm{D}_{0^+}^{\phi_3(\alpha),\psi} x(t) \cdot (\psi(1) - \psi(t))^5 \psi'(t) \\ - x(t) \cdot \frac{(\psi(1) - \psi(t))^3 - (\psi(1) - \psi(t))^2}{\ln(\psi(1) - \psi(t))} \psi'(t) \Big) dt,\end{aligned}$$

where $\phi_3 : [2,3] \to [0,1]$ *is defined by*

$$\phi_3(\alpha) = \frac{\Gamma(6-\alpha)}{5!}.$$

The Euler–Lagrange equation associated is the following (cf. Theorem 9*):*

$$-\frac{(\psi(1) - \psi(t))^3 - (\psi(1) - \psi(t))^2}{\ln(\psi(1) - \psi(t))} + \mathrm{D}_{1^-}^{\phi_3(\alpha),\psi}\big((\psi(1) - \psi(t))^5\big) = 0.$$

By ([14] Lemma 14),

$$\mathrm{D}_{1^-}^{\alpha,\psi}\big((\psi(1) - \psi(t))^5\big) = \frac{5!}{\Gamma(6-\alpha)} (\psi(1) - \psi(t))^{5-\alpha},$$

and so

$$\mathrm{D}_{1^-}^{\phi_3(\alpha),\psi}\left((\psi(1)-\psi(t))^5\right)=\frac{(\psi(1)-\psi(t))^3-(\psi(1)-\psi(t))^2}{\ln(\psi(1)-\psi(t))},$$

proving that the function $\overline{x}(t)=(\psi(1)-\psi(t))^5$, $t\in[0,1]$, *is a candidate to be an extremizer of the proposed problem.*

Example 2. *We want to find a curve* $\overline{x}\in C^1([-1,2],\mathbb{R})$, *subject to the condition* $x(t)=\mu(t),\ t\in[-1,0]$, *where* $\mu\in C^1([-1,0],\mathbb{R})$ *is a fixed initial function with* $\mu(0)=(\psi(2)-\psi(0))^2$, *that minimizes the following functional:*

$$\begin{aligned}\mathcal{J}(x) &= \int_0^2\left(\left(x(t-1)-(\psi(2)-\psi(t-1))^2\right)^2\right.\\ &\qquad\left.+\left({}^C\mathrm{D}_{2^-}^{\varphi(\alpha),\psi}x(t)-\frac{\psi(t)-\psi(2)+(\psi(2)-\psi(t))^2}{\ln(\psi(2)-\psi(t))}\right)^2\right)dt,\end{aligned}$$

where $\varphi:[0,1]\to[0,1]$ *is defined by*

$$\varphi(\alpha)=\frac{\Gamma(3-\alpha)}{2}.$$

By Lemma 1 in [13], if $\overline{x}:[-1,2]\to\mathbb{R}$ *is defined by* $\overline{x}(t)=(\psi(2)-\psi(t))^2$ *if* $t\in[0,2]$, *and* $\overline{x}(t)=\mu(t)$, *if* $t\in[-1,0]$, *then*

$${}^C\mathrm{D}_{2^-}^{\alpha,\psi}\overline{x}(t)=\frac{2}{\Gamma(3-\alpha)}(\psi(2)-\psi(t))^{2-\alpha},\quad t\in[0,2],$$

and so the distributed-order derivative with respect to ψ *is given by*

$${}^C\mathrm{D}_{2^-}^{\varphi(\alpha),\psi}\overline{x}(t)=\int_0^1\varphi(\alpha){}^C\mathrm{D}_{2^-}^{\alpha,\psi}\overline{x}(t)d\alpha=\frac{\psi(t)-\psi(2)+(\psi(2)-\psi(t))^2}{\ln(\psi(2)-\psi(t))}.$$

Note that $\overline{x}$ *satisfies the assumptions of Theorem 2 and also the Euler–Lagrange Equations (2) and (3), as well as the transversality condition (4), proving that* $\overline{x}$ *is a candidate to be a local minimizer of* $\mathcal{J}$. *Since the Lagrangian function is convex, we conclude by Theorem 6 that* $\overline{x}$ *is a minimizer of* $\mathcal{J}$.

Example 3. *Determine* $\overline{x}$ *that minimizes the functional*

$$\begin{aligned}\mathcal{J}(x)=\int_0^1\left(\left({}^C\mathrm{D}_{0^+}^{\phi_2(\alpha),\psi}x(t)-\frac{\psi(t)-\psi(0)-1}{\ln(\psi(t)-\psi(0))}\right)^2\right.\\ \left.+\left({}^C\mathrm{D}_{1^-}^{\varphi_2(\alpha),\psi}x(t)-\frac{\psi(1)-\psi(t)-1}{\ln(\psi(1)-\psi(t))}\right)^2\right)dt,\end{aligned}$$

in the class of functions $C^2([0,1],\mathbb{R})$ *subject to the boundary conditions* $x(0)=x'(0)=0$, *where* $\phi_2,\varphi_2:[1,2]\to[0,1]$ *are defined by*

$$\phi_2(\alpha)=\frac{\Gamma(3-\alpha)}{2}=\varphi_2(\alpha).$$

Again, by [13, Lemma 1], if $\overline{x}(t)=(\psi(t)-\psi(0))^2$, $t\in[0,1]$, *then*

$${}^C\mathrm{D}_{0^+}^{\alpha,\psi}\overline{x}(t)=\frac{2}{\Gamma(3-\alpha)}(\psi(t)-\psi(0))^{2-\alpha},$$

and so

$$ {}^{C}\mathrm{D}_{0^+}^{\phi_2(\alpha),\psi}\overline{x}(t) = \int_1^2 \phi_2(\alpha)\, {}^{C}\mathrm{D}_{0^+}^{\alpha,\psi}\overline{x}(t)d\alpha = \frac{\psi(t)-\psi(0)-1}{\ln(\psi(t)-\psi(0))}. $$

In addition, observe that

$$ \begin{aligned} {}^{C}\mathrm{D}_{1^-}^{\alpha,\psi}\overline{x}(t) &= {}^{C}\mathrm{D}_{1^-}^{\alpha,\psi}((\psi(1)-\psi(t))+(\psi(0)-\psi(1)))^2 \\ &= {}^{C}\mathrm{D}_{1^-}^{\alpha,\psi}(\psi(1)-\psi(t))^2 = \frac{2}{\Gamma(3-\alpha)}(\psi(1)-\psi(t))^{2-\alpha}, \end{aligned} $$

and therefore

$$ {}^{C}\mathrm{D}_{1^-}^{\varphi_2(\alpha),\psi}\overline{x}(t) = \int_1^2 \varphi_2(\alpha)\, {}^{C}\mathrm{D}_{1^-}^{\alpha,\psi}\overline{x}(t)d\alpha = \frac{\psi(1)-\psi(t)-1}{\ln(\psi(1)-\psi(t))}. $$

We can easily verify that $\overline{x}$ satisfies assumptions of Theorem 9, the Euler–Lagrange Equation (26)*, and the natural boundary condition* (28)*, proving that $\overline{x}$ is a candidate to be a local minimizer of $\mathcal{J}$. Since the Lagrangian function is convex, we conclude that $\overline{x}$ is a minimizer of $\mathcal{J}$.*

5. Conclusions and Future Work

In this article, we continue the study started in [15], considering now new problems in the calculus of variations. Namely, two distinct types are considered: when the Lagrangian function involves a time delay and derivatives of order greater than 1. Necessary and sufficient optimization conditions are proved, for the basic problem and when in the presence of additional constraints to the problem. The study is formulated in the context of fractional calculus, where the derivative of the state curve is of the fractional type involving distributed-orders and the kernel involves an arbitrary smooth function.

In the future, we intend to study variational problems of Herglotz type and some generalizations involving distributed-order fractional derivatives with arbitrary smooth kernels.

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