

Article

Integral Formulas for a Foliation with a Unit Normal Vector Field

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Abstract: In this article, we prove integral formulas for a Riemannian manifold equipped with a foliation \mathcal{F} and a unit vector field N orthogonal to \mathcal{F} , and generalize known integral formulas (due to Brito-Langevin-Rosenberg and Andrzejewski-Walczak) for foliations of codimension one. Our integral formulas involve Newton transformations of the shape operator of \mathcal{F} with respect to N and the curvature tensor of the induced connection on the distribution $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$, and this decomposition of \mathcal{D} can be regarded as a codimension-one foliation of a sub-Riemannian manifold. We apply our formulas to foliated (sub-)Riemannian manifolds with restrictions on the curvature and extrinsic geometry of the foliation.

Keywords: Riemannian manifold; foliation; harmonic distribution; Newton transformation; shape operator; r -th mean curvature

MSC: 53C12; 53C17



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1. Introduction

Foliations, which are defined as partitions of a manifold into collections of submanifolds of the same dimension, called leaves, appeared in the 1940s in the works of G. Reeb and Ch. Ehresmann, culminating in the book [1]. Since then, the subject has enjoyed a rapid development, see e.g., [2]. Extrinsic geometry of foliations of Riemannian manifolds was developed in recent years, see surveys in [3–9], and has applications in differential geometry and analysis on real and complex manifolds. By extrinsic geometry we mean properties of foliations on Riemannian manifolds which can be expressed in terms of the second fundamental form of the leaves and its invariants (principal curvatures, scalar mean curvature, higher order mean curvatures σ_r 's for $r > 1$ and so on). G. Reeb also published a paper [10] on extrinsic geometry of foliations, in which he proved that the integral of the mean curvature of the leaves of any codimension-one foliation on any closed Riemannian manifold equals zero,

$$\int_M \sigma_1 d\text{vol}_g = 0; \quad (1)$$

thus, either the mean curvature of the leaves $\sigma_1 \equiv 0$ or $\sigma_1(x) \sigma_1(x') < 0$ for some points $x \neq x'$.

Integral formulas in Riemannian geometry can be viewed as “conservation laws” of quantities when the metric changes. Integral formulas are now the centerpiece of extrinsic geometry of foliations and are useful in several geometric situations: characterizing foliations, whose leaves have a given geometric property; prescribing the higher mean curvatures of the leaves of a foliation; minimizing functionals such as volume defined for tensor fields on a foliated manifold. In [4], the Newton transformations $T_r(A_N)$ of the shape operator A_N of the leaves (with a unit normal vector field N) were applied to a codimension-one foliated $(n + 1)$ -dimensional Riemannian manifold (with the curvature

tensor R), and a series of integral formulas with total r -th mean curvatures for $r \in [0, n-2]$ was proved:

$$\int_M ((r+2)\sigma_{r+2} - \operatorname{tr}_{\mathcal{F}}(T_r(A_N)R(\cdot, N)N) - \langle \operatorname{div}_{\mathcal{F}} T_r(A_N), \nabla_N N \rangle) d\operatorname{vol}_g = 0. \quad (2)$$

These integral formulas were obtained by applying the Divergence Theorem to suitable vector fields. The next to (1) integral formula in the series (2) for $r = 0$ and $n \geq 2$ is, see [11],

$$\int_M (2\sigma_2 - \operatorname{Ric}_{N,N}) d\operatorname{vol}_g = 0. \quad (3)$$

By (3), there are no totally umbilical codimension-one foliations of a closed manifold of negative Ricci curvature, and there are no harmonic codimension-one foliations of a closed manifold of positive Ricci curvature. It was proved in [6], and can be deduced from (2), that on a compact space form $M^{n+1}(c)$ the total σ_r 's are independent of \mathcal{F} :

$$\int_M \sigma_r d\operatorname{vol}_g = \begin{cases} c^{r/2} \binom{n/2}{r/2} \operatorname{Vol}(M, g), & n \text{ and } r \text{ even,} \\ 0, & \text{either } n \text{ or } r \text{ odd.} \end{cases} \quad (4)$$

The natural question arises: can we find integral formulas similar to (2) and (4) for foliations of arbitrary codimension? In studying this question, we consider a foliation \mathcal{F} of any codimension, whose normal bundle has zero Euler class; thus, \mathcal{F} can be equipped with a unit normal vector field N . Let

$$\mathcal{D} = T\mathcal{F} \oplus \operatorname{span}(N) \quad (5)$$

be a distribution on M (subbundle of the tangent bundle TM) spanned by $T\mathcal{F}$ (the tangent bundle of \mathcal{F}) and N . In the article, we prove integral formulas (in Theorems 1–3 and Corollaries 3 and 4), which generalize (2) and (4) for codimension-one foliations. Our integral formulas involve r -th mean curvatures of \mathcal{F} with respect to N , i.e., symmetric functions of the shape operator A_N of \mathcal{F} , Newton transformations of A_N and the curvature tensor of the induced connection on \mathcal{D} . We apply our formulas to foliations with restrictions on the curvature and the extrinsic geometry.

Please note that (M, \mathcal{D}) with the metric on \mathcal{D} (e.g., the restriction of g) is the object of sub-Riemannian geometry, see [5]. Apparently, a foliated sub-Riemannian manifold $(M, \mathcal{D}, \mathcal{F}, g)$, i.e., the tangent bundle $T\mathcal{F}$ of a foliation \mathcal{F} is a subbundle of \mathcal{D} , is a new (or little-studied) geometrical object. Therefore, the results (in Sections 3 and 4) can be interpreted as integral formulas for codimension-one foliations of a sub-Riemannian manifold.

2. Preliminaries

Here, using the induced linear connection on a distribution, we define the shape operator (with its Newton transformations) and the curvature tensor related to a codimension-one foliated sub-Riemannian manifold, then we prove three auxiliary lemmas.

Let \mathcal{D} be an $(n+1)$ -dimensional distribution on a smooth m -dimensional manifold M , i.e., a subbundle of TM of rank $n+1$ (where $0 < n < m$). In other words, to each point $x \in M$ we assign an $(n+1)$ -dimensional subspace \mathcal{D}_x of the tangent space $T_x M$ smoothly depending on x . A pair (M, \mathcal{D}) , where M is a manifold and \mathcal{D} is a non-integrable distribution on M , is called a *non-holonomic manifold*, see [5]. The concept of a non-holonomic manifold was introduced for a geometric interpretation of constrained systems in classical mechanics. A *sub-Riemannian manifold* is a non-holonomic manifold (M, \mathcal{D}) , equipped with a sub-Riemannian metric $g = \langle \cdot, \cdot \rangle$, i.e., the scalar product $g : \mathcal{D}_x \times \mathcal{D}_x \rightarrow \mathbb{R}$ for all $x \in M$, see [5]. Usually, they assume that the sub-Riemannian metric on the horizontal bundle \mathcal{D} is extended to a Riemannian metric (also denoted by g) on the whole manifold M . This allows us to define the orthogonal distribution \mathcal{D}^\perp (the vertical subbundle) such that $TM = \mathcal{D} \oplus \mathcal{D}^\perp$.

The orthoprojector $P : TM \rightarrow \mathcal{D}$ onto the distribution \mathcal{D} is characterized by the properties

$$P = P^* \text{ (self-adjoint), } P^2 = P,$$

e.g., [12]. A sub-Riemannian manifold equipped with a foliation \mathcal{F} such that the tangent bundle $T\mathcal{F}$ is a subbundle of the horizontal bundle will be called a *foliated sub-Riemannian manifold*. By Frobenius Theorem, e.g., ([5] Theorem 1.7), a foliation is determined by an involutive distribution, i.e., the Lie bracket of any two its vector fields also belongs to this distribution.

In this article, we assume that \mathcal{F} is a codimension-one foliation relative to \mathcal{D} , i.e., $\dim \mathcal{F} = n$, and there exists a unit vector field N orthogonal to \mathcal{F} and tangent to \mathcal{D} . This means that the Euler characteristic of the subbundle \mathcal{D} is zero and the following orthogonal decomposition is valid, see (5):

$$\mathcal{D} = T\mathcal{F} \oplus \text{span}(N).$$

The Levi-Civita connection ∇ on (M, g) induces a linear connection ∇^P on the distribution $\mathcal{D} = P(TM)$:

$$\nabla_X^P PY = P\nabla_X PY, \quad X, Y \in \Gamma(TM),$$

which is compatible with the metric: $X\langle U, V \rangle = \langle \nabla_X^P U, V \rangle + \langle U, \nabla_X^P V \rangle$ for any sections U and V of \mathcal{D} and $X \in \Gamma(TM)$. Define the horizontal vector field

$$\mathcal{Z} = \nabla_N^P N,$$

and note that $\nabla_N N$ is the curvature vector of N -curves in (M, g) .

The *shape operator* $A_N : T\mathcal{F} \rightarrow T\mathcal{F}$ of the foliation \mathcal{F} with respect to N is defined by

$$A_N(X) = -\nabla_X^P N, \quad X \in T\mathcal{F}. \quad (6)$$

The *elementary symmetric functions* $\sigma_j(A_N)$ of A_N (called the r -th mean curvatures of \mathcal{F} with respect to N in \mathcal{D}) are given by the equality

$$\sum_{r=0}^n \sigma_r(A_N) t^r = \det(\text{id}_{T\mathcal{F}} + tA_N), \quad t \in \mathbb{R}.$$

Note that $\sigma_r(A_N) = \sum_{i_1 < \dots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}$, where $\lambda_1 \leq \dots \leq \lambda_n$ are the eigenvalues of A_N . Let $\tau_j(A_N) = \text{tr } A_N^j$ for $j \in \mathbb{N}$ be the *power sums symmetric functions* of A_N . For example, $\sigma_0(A_N) = 1$, $\sigma_1(A_N) = \tau_1(A_N) = \text{tr } A_N$, $\sigma_n(A_N) = \det A_N$, and

$$2\sigma_2(A_N) = \tau_1^2(A_N) - \tau_2(A_N). \quad (7)$$

For short, we set

$$\sigma_r = \sigma_r(A_N), \quad \tau_r = \tau_r(A_N).$$

Next, we introduce the curvature tensor $R^P : TM \times TM \rightarrow \text{End}(\mathcal{D})$ of the connection ∇^P :

$$R^P(X, Y) = \nabla_X^P \nabla_Y^P - \nabla_Y^P \nabla_X^P - \nabla_{[X, Y]}^P. \quad (8)$$

Set $R^P(X, Y, V, U) = \langle R^P(X, Y)V, U \rangle$ for $U, V \in \mathcal{D}$. The sectional P -curvature of a plane spanned by non-collinear vectors X, Y is $R^P(X, Y, Y, X) / (X^2 Y^2 - \langle X, Y \rangle^2)$.

Obviously, $R^P(Y, X) = -R^P(X, Y)$ (for any linear connection). Since ∇^P is compatible with the metric, then the anti-symmetry for the last pair of vectors is valid, e.g., [13],

$$R^P(X, Y, V, U) = -R^P(X, Y, U, V). \quad (9)$$

Lemma 1. For $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$ on (M, g) , the following Codazzi type equation is valid:

$$(\nabla_X^{\mathcal{F}} A_N)Y - (\nabla_Y^{\mathcal{F}} A_N)X = -R^P(X, Y)N, \quad X, Y \in T\mathcal{F}. \quad (10)$$

Proof. Recall the Codazzi's equation for a foliation (or a submanifold) of (M, g) :

$$(\nabla_X h)(Y, U) - (\nabla_Y h)(X, U) = (R(X, Y)U)^\perp, \quad (11)$$

where $R : TM \times TM \rightarrow \text{End}(TM)$ is the curvature tensor of the Levi-Civita connection,

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]},$$

$^\perp$ denotes the projection onto the vector bundle orthogonal to \mathcal{F} , and $h : T\mathcal{F} \times T\mathcal{F} \rightarrow (T\mathcal{F})^\perp$ is the second fundamental form of \mathcal{F} in (M, g) defined by

$$h(X, Y) = (\nabla_X Y)^\perp.$$

From (11), for all vectors $X, Y, U \in T\mathcal{F}$ we obtain

$$\begin{aligned} & \langle (\nabla_X^\mathcal{F} A_N)Y - (\nabla_Y^\mathcal{F} A_N)X, U \rangle + \langle h(X, U), \nabla_Y N \rangle \\ & - \langle h(Y, U), \nabla_X N \rangle = -\langle R(X, Y)N, U \rangle. \end{aligned} \quad (12)$$

Applying the orthoprojector on the vector bundle orthogonal to \mathcal{F} , we find

$$\begin{aligned} \langle R(X, Y)N, U \rangle &= \langle R^P(X, Y)N, U \rangle + \langle \nabla_X((\nabla_Y N)^\perp) \\ &- \nabla_Y((\nabla_X N)^\perp) - \nabla_{[X, Y]}^\perp N, U \rangle. \end{aligned} \quad (13)$$

Using the equalities $[X, Y]^\perp = 0$ (since $T\mathcal{F}$ is integrable), (13) and

$$\begin{aligned} \langle h(X, U), \nabla_Y N \rangle &= \langle \nabla_Y N, (\nabla_X U)^\perp \rangle = -\langle \nabla_X((\nabla_Y N)^\perp), U \rangle, \\ \langle h(Y, U), \nabla_X N \rangle &= \langle \nabla_X N, (\nabla_Y U)^\perp \rangle = -\langle \nabla_Y((\nabla_X N)^\perp), U \rangle, \end{aligned}$$

in (12) completes the proof. \square

The following lemma generalizes [4] (Lemma 3.1).

Lemma 2. Let $\{e_i\}$ be a local orthonormal frame of $T\mathcal{F}$ such that at a point $x \in M$:

- $\nabla_X^\mathcal{F} e_i = 0$ ($1 \leq i \leq n$) for any vector $X \in T_x M$;
- $\nabla_\xi^P e_i = 0$ ($1 \leq i \leq n$) for any vector $\xi \in \mathcal{D}_x^\perp M$.

Then the following equality is valid at $x \in M$:

$$\begin{aligned} \langle \nabla_{e_i} \mathcal{Z}, e_j \rangle &= \langle A_N^2 e_i, e_j \rangle + \langle R^P(e_i, N)N, e_j \rangle \\ &- \langle (\nabla_N^\mathcal{F} A_N)e_i, e_j \rangle + \langle \mathcal{Z}, e_i \rangle \langle \mathcal{Z}, e_j \rangle. \end{aligned} \quad (14)$$

Proof. Taking covariant derivative of $\langle \mathcal{Z}, e_j \rangle = -\langle N, \nabla_N e_j \rangle$ with respect to e_i , we find

$$-\langle \mathcal{Z}, \nabla_{e_i} e_j \rangle = \langle \nabla_{e_i} \mathcal{Z}, e_j \rangle + \langle \nabla_{e_i} N, P \nabla_N e_j \rangle + \langle N, \nabla_{e_i} P \nabla_N e_j \rangle. \quad (15)$$

For a foliation \mathcal{F} , we obtain

$$\langle (\nabla_N^\mathcal{F} A_N)e_i, e_j \rangle = \nabla_N \langle N, \nabla_{e_i} e_j \rangle = \langle \mathcal{Z}, \nabla_{e_i} e_j \rangle + \langle N, \nabla_N P \nabla_{e_i} e_j \rangle.$$

Therefore, using (8), we calculate at the point $x \in M$:

$$\begin{aligned} & \langle A_N^2 e_i, e_j \rangle + \langle R^P(e_i, N)N, e_j \rangle - \langle (\nabla_N^\mathcal{F} A_N)e_i, e_j \rangle \\ &= \langle A_N^2 e_i, e_j \rangle - \langle R^P(e_i, N)e_j, N \rangle + N \langle \nabla_{e_i} N, e_j \rangle \\ &= \langle A_N^2 e_i, e_j \rangle - \langle \mathcal{Z}, \nabla_{e_i} e_j \rangle - \langle \nabla_{e_i} P \nabla_N e_j, N \rangle + \langle \nabla_{[e_i, N]} e_j, N \rangle. \end{aligned} \quad (16)$$

By conditions at $x \in M$, we rewrite the last term in (16) as

$$\langle \nabla_{P[e_i, N]} e_j, N \rangle = \langle \nabla_{[e_i, N]} e_j, N \rangle.$$

Then, using (15) and the following equalities at $x \in M$:

$$\begin{aligned} P\nabla_{e_i} N &= \sum_{1 \leq j \leq n} \langle \nabla_{e_i} N, e_j \rangle e_j, \quad P\nabla_N e_i = \langle \nabla_N e_i, N \rangle N, \\ \langle A_N^2 e_i, e_j \rangle &= \langle \nabla_{e_i} N, N \rangle \langle \nabla_N e_j, N \rangle, \end{aligned}$$

we simplify the last line in (16) as

$$\langle \nabla_{e_i} \mathcal{Z}, e_j \rangle - \langle \mathcal{Z}, e_i \rangle \langle \mathcal{Z}, e_j \rangle.$$

From the above, the claim follows. \square

Many authors investigated r -th mean curvatures of foliations and hypersurfaces of Riemannian manifolds using the Newton transformations of the shape operator, see [4].

Definition 1. The Newton transformations $T_r(A_N)$ of the shape operator A_N of an n -dimensional foliation \mathcal{F} of a sub-Riemannian manifold (M, \mathcal{D}, g) are defined recursively or explicitly by

$$\begin{aligned} T_0(A_N) &= \text{id}_{T\mathcal{F}}, \quad T_r(A_N) = \sigma_r \text{id}_{T\mathcal{F}} - A_N T_{r-1}(A_N), \quad 1 \leq r \leq n, \\ T_r(A_N) &= \sum_{j=0}^r (-1)^j \sigma_{r-j} A_N^j = \sigma_r \text{id}_{T\mathcal{F}} - \sigma_{r-1} A_N + \dots + (-1)^r A_N^r. \end{aligned}$$

For example, $T_1(A_N) = \sigma_1 \text{id}_{T\mathcal{F}} - A_N$ and $T_n(A_N) = 0$. Notice that A_N and $T_r(A_N)$ commute. The following properties of $T_r(A_N)$ are proved similarly as for codimension-one foliations of a Riemannian manifold, e.g., [4] (or [7] Lemma 1.3).

Lemma 3. For the shape operator A_N we have

$$\begin{aligned} \text{tr}_{\mathcal{F}} T_r(A_N) &= (n-r) \sigma_r, \\ \text{tr}_{\mathcal{F}} (A_N \cdot T_r(A_N)) &= (r+1) \sigma_{r+1}, \\ \text{tr}_{\mathcal{F}} (A_N^2 \cdot T_r(A_N)) &= \sigma_1 \sigma_{r+1} - (r+2) \sigma_{r+2}, \\ \text{tr}_{\mathcal{F}} (T_{r-1}(A_N) (\nabla_X^{\mathcal{F}} A_N)) &= X(\sigma_r), \quad X \in T\mathcal{F}, \quad r > 0. \end{aligned}$$

On the other hand, since the $(1,1)$ -tensors A_N and $T_r(A_N)$ are self-adjoint, we have

$$\langle (\nabla_X^{\mathcal{F}} T_r(A_N)) Y, V \rangle = \langle (\nabla_X^{\mathcal{F}} T_r(A_N)) V, Y \rangle, \quad X, Y, V \in T\mathcal{F}. \quad (17)$$

3. Main Results

Here, we prove a series of integral formulas for a codimension-one foliated sub-Riemannian manifold $(M, \mathcal{D}, \mathcal{F}, g)$ with $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$.

Recall that the \mathcal{F} -divergence of a vector field X tangent to \mathcal{F} on (M, g) is defined by

$$\text{div } X = \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle,$$

where $\{e_i\}$ is a local orthonormal frame of $T\mathcal{F}$. Following [4], define the \mathcal{F} -divergence of the Newton transformation $T_r(A_N)$ by

$$\text{div}_{\mathcal{F}} T_r(A_N) = \sum_{i=1}^n (\nabla_{e_i}^{\mathcal{F}} T_r(A_N)) e_i. \quad (18)$$

Please note that $\text{div}_{\mathcal{F}} T_0(A_N) = 0$.

For any $X \in \mathcal{D}$, define a linear operator $\mathcal{R}_X^P : T\mathcal{F} \rightarrow T\mathcal{F}$ by

$$\mathcal{R}_X^P : V \rightarrow R^P(V, X)N, \quad V \in T\mathcal{F}. \quad (19)$$

The following result generalizes Lemma 2.2 in [4].

Lemma 4. The leafwise divergence of $T_r(A_N)$ for $r > 0$ satisfies the inductive formula

$$\langle \operatorname{div}_{\mathcal{F}} T_r(A_N), X \rangle = -\langle \operatorname{div}_{\mathcal{F}} T_{r-1}(A_N), A_N X \rangle + \operatorname{tr}_{\mathcal{F}}(T_{r-1}(A_N) \mathcal{R}_X^P), \quad (20)$$

where $X \in \Gamma(\mathcal{F})$. Equivalently, for $r > 0$ we have

$$\langle \operatorname{div}_{\mathcal{F}} T_r(A_N), X \rangle = \sum_{j=1}^r (-1)^{j-1} \operatorname{tr}_{\mathcal{F}}(T_{r-j}(A_N) \mathcal{R}_{A_N^{j-1}X}^P). \quad (21)$$

Proof. Using the recursive definition of $T_r(A_N)$, we have

$$\operatorname{div}_{\mathcal{F}} T_r(A_N) = \nabla^{\mathcal{F}} \sigma_r - A_N \operatorname{div}_{\mathcal{F}} T_{r-1}(A_N) - \sum_{i=1}^n (\nabla_{e_i}^{\mathcal{F}} A_N) T_{r-1}(A_N) e_i.$$

Applying Codazzi type equation (10) and the last formula in Lemma 3, we obtain

$$\begin{aligned} \sum_{i=1}^n \langle (\nabla_{e_i}^{\mathcal{F}} A_N) T_{r-1}(A_N) e_i, X \rangle &= \sum_{i=1}^n \langle T_{r-1}(A_N) e_i, (\nabla_{e_i}^{\mathcal{F}} A_N) X \rangle \\ &= \sum_{i=1}^n \langle T_{r-1}(A_N) e_i, (\nabla_X^{\mathcal{F}} A_N) e_i - R^P(e_i, X) N \rangle \\ &= \operatorname{tr}_{\mathcal{F}}(T_{r-1}(A_N) (\nabla_X^{\mathcal{F}} A_N)) - \sum_{i=1}^n \langle R^P(e_i, X) N, T_{r-1}(A_N) e_i \rangle \\ &= X(\sigma_r) - \operatorname{tr}_{\mathcal{F}}(T_{r-1}(A_N) \mathcal{R}_X^P). \end{aligned}$$

Here, we used $R^P(e_i, X)N = \mathcal{R}_X^P(e_i)$, see (19). Hence, the inductive Formula (20) holds. Finally, (21) follows directly from the above. \square

Remark 1. (a) If $P = \operatorname{id}_{TM}$, i.e., \mathcal{F} is a codimension-one foliation of M , then $R^P = R$, and using the symmetry $\langle R(X, Y)U, V \rangle = \langle R(U, V)X, Y \rangle$, we simplify Equation (20) to the form

$$\operatorname{div}_{\mathcal{F}} T_r(A_N) = -A_N \operatorname{div}_{\mathcal{F}} T_{r-1}(A_N) + \sum_{i=1}^n (R(N, T_{r-1}(A_N) e_i) e_i)^{\top},$$

(see Lemma 2.2 in [4]), where $^{\top}$ denotes the orthogonal projection on the vector bundle $T\mathcal{F}$.

(b) Let the distribution $T\mathcal{F}$ be P -curvature invariant, i.e.,

$$R^P(X, Y)V \in T\mathcal{F}, \quad X, Y, V \in T\mathcal{F}. \quad (22)$$

By (9), Equation (21) implies that $\operatorname{div}_{\mathcal{F}} T_r(A_N) = 0$ for every $r \geq 0$. Condition (22) is obviously satisfied, if the distribution $T\mathcal{F}$ is auto-parallel, i.e., $\nabla_X Y \in \Gamma(T\mathcal{F})$ for all $X, Y \in \Gamma(T\mathcal{F})$. A sufficient condition for (22) is the constancy of the sectional P -curvature, i.e., the following equality with some real constant c :

$$R^P(X, Y)V = c(\langle Y, V \rangle X - \langle X, V \rangle Y), \quad X, Y, V \in \mathcal{D}. \quad (23)$$

The following result generalizes Proposition 3.3 in [4].

Theorem 1. For any compact leaf L of \mathcal{F} we have

$$\begin{aligned} \int_L ((r+2) \sigma_{r+2} + N(\sigma_{r+1}) - \sigma_1 \sigma_{r+1} - \operatorname{tr}_{\mathcal{F}}(T_r(A_N) \mathcal{R}_N^P) \\ - \langle T_r(A_N) \mathcal{Z}, \mathcal{Z} \rangle - \langle \operatorname{div}_{\mathcal{F}} T_r(A_N), \mathcal{Z} \rangle) \, d\operatorname{vol}_L = 0, \quad r \in [0, n-2], \end{aligned} \quad (24)$$

where the underlined term is given by (21) with $X = \mathcal{Z}$.

Proof. Using (17) and (18), we compute the divergence of the vector field $T_r(A_N)\mathcal{Z}$ as follows:

$$\begin{aligned}\operatorname{div}_{\mathcal{F}} T_r(A_N)\mathcal{Z} &= \sum_{i=1}^n \langle \nabla_{e_i}(T_r(A_N)\mathcal{Z}), e_i \rangle \\ &= \langle \operatorname{div}_{\mathcal{F}} T_r(A_N), \mathcal{Z} \rangle + \sum_{i=1}^n \langle \nabla_{e_i} \mathcal{Z}, T_r(A_N)e_i \rangle.\end{aligned}$$

Using (14) and (19), we compute $\sum_{i=1}^n \langle \nabla_{e_i} \mathcal{Z}, T_r(A_N)e_i \rangle$ as

$$\begin{aligned}\sum_{i=1}^n (\langle A_N^2 e_i + R^P(e_i, N)N - (\nabla_N^{\mathcal{F}} A_N)e_i, T_r(A_N)e_i \rangle + \langle \mathcal{Z}, e_i \rangle \langle \mathcal{Z}, T_r(A_N)e_i \rangle) \\ = -\operatorname{tr}_{\mathcal{F}}(T_r(A_N)(\nabla_N^{\mathcal{F}} A_N - A_N^2 - \mathcal{R}_N^P)) + \langle T_r(A_N)\mathcal{Z}, \mathcal{Z} \rangle.\end{aligned}$$

By Lemma 3, we can write

$$\operatorname{tr}_{\mathcal{F}}(T_r(A_N)(\nabla_N^{\mathcal{F}} A_N - A_N^2 - \mathcal{R}_N^P)) = N(\sigma_{r+1}) - \sigma_1 \sigma_{r+1} + (r+2)\sigma_{r+2} - \operatorname{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P).$$

Finally, we obtain

$$\begin{aligned}\sum_{i=1}^n \langle \nabla_{e_i} \mathcal{Z}, T_r(A_N)e_i \rangle &= -N(\sigma_{r+1}) + \sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2} \\ &\quad + \langle T_r(A_N)\mathcal{Z}, \mathcal{Z} \rangle + \operatorname{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P).\end{aligned}$$

This proves the following:

$$\begin{aligned}\operatorname{div}_{\mathcal{F}} T_r(A_N)\mathcal{Z} &= \frac{\langle \operatorname{div}_{\mathcal{F}} T_r(A_N), \mathcal{Z} \rangle + \operatorname{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P) + \langle T_r(A_N)\mathcal{Z}, \mathcal{Z} \rangle}{-(r+2)\sigma_{r+2} - N(\sigma_{r+1}) + \sigma_1 \sigma_{r+1}}.\end{aligned}\quad (25)$$

Applying the Divergence Theorem to (25) along any compact leaf, we obtain (24). \square

We can view $\operatorname{tr}_{\mathcal{F}} \mathcal{R}_X^P = \sum_{i=1}^n \langle R^P(e_i, X)N, e_i \rangle$ as the Ricci P -curvature $\operatorname{Ric}_{X,N}^P$. Then

$$\operatorname{Ric}_{N,N}^P = \operatorname{tr}_{\mathcal{F}} \mathcal{R}_N^P \quad (26)$$

is the Ricci P -curvature in the N -direction.

Corollary 1. Let $\tau_1 = \text{const.}$ and $\operatorname{Ric}_{N,N}^P > 0$. Then \mathcal{F} has no compact leaves.

Proof. Let L be a compact leaf. By (24) with $r = 0$, using $2\sigma_2 = \tau_1^2 - \tau_2$, we obtain

$$\int_L (\tau_2 - N(\tau_1) + \operatorname{tr}_{\mathcal{F}} \mathcal{R}_N^P + \langle \mathcal{Z}, \mathcal{Z} \rangle) d\operatorname{vol}_L = 0.$$

By conditions and (26), we obtain $N(\tau_1) = 0$ and $\operatorname{tr}_{\mathcal{F}} \mathcal{R}_N^P > 0$. This yields a contradiction. \square

For any vector field X in $T\mathcal{F}$, we have

$$\operatorname{div} X = \operatorname{div}_{\mathcal{F}} X - \langle X, \mathcal{Z} \rangle - \langle X, H^\perp \rangle, \quad (27)$$

where H^\perp is the mean curvature vector field of the distribution \mathcal{D}^\perp (orthogonal to \mathcal{D}) in (M, g) . Recall that a distribution on a Riemannian manifold is called *harmonic* if its mean curvature vector field vanishes. There are topological restrictions for the existence of a Riemannian metric on closed manifold, for which a given distribution becomes harmonic, see [14].

The following statement generalizes (1).

Theorem 2. For a closed sub-Riemannian manifold (M, \mathcal{D}, g) with $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$ and a harmonic distribution \mathcal{D}^\perp , the following integral formula is valid:

$$\int_M \sigma_1 \, d\text{vol}_g = 0. \quad (28)$$

Proof. Recall that $\sigma_1 = \text{tr } A_N$, see (6), and observe that

$$\text{div}_{\mathcal{F}} N = \sum_{i=1}^n \langle \nabla_{e_i} N, e_i \rangle = -\sigma_1.$$

Thus, using the assumption $H^\perp = 0$, we obtain

$$\text{div } N = \text{div}_{\mathcal{F}} N - \langle N, H^\perp \rangle = -\sigma_1. \quad (29)$$

Applying the Divergence Theorem, yields (28). \square

We supplement (28) with a series of integral formulas with total r -th mean curvatures for $r \in [0, n-2]$, which generalize (2).

Theorem 3. For a closed sub-Riemannian manifold (M, g) with $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$ and a harmonic distribution \mathcal{D}^\perp , the following integral formula is valid for $r \in [0, n-2]$:

$$\int_M ((r+2)\sigma_{r+2} - \text{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P) - \langle \text{div}_{\mathcal{F}} T_r(A_N), \mathcal{Z} \rangle) \, d\text{vol}_g = 0, \quad (30)$$

where the underlined term is given by (21) with $X = \mathcal{Z}$.

Proof. Using (29), we calculate the divergence of $\sigma_{r+1} \cdot N$ as

$$\text{div}(\sigma_{r+1} \cdot N) = \sigma_{r+1} \text{div } N + N(\sigma_{r+1}) = N(\sigma_{r+1}) - \sigma_1 \sigma_{r+1}.$$

Then, by (27) with $X = T_r(A_N)\mathcal{Z}$, (since \mathcal{D}^\perp is a harmonic distribution), we obtain

$$\text{div}(T_r(A_N)\mathcal{Z} + \sigma_{r+1}N) = \langle \text{div}_{\mathcal{F}} T_r(A_N), \mathcal{Z} \rangle - (r+2)\sigma_{r+2} + \text{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P).$$

Applying the Divergence Theorem, we obtain (30). \square

Example 1. If N is a P -geodesic vector field, i.e., $\mathcal{Z} = 0$, then (30) shortens to the formula

$$\int_M ((r+2)\sigma_{r+2} - \text{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P)) \, d\text{vol}_g = 0.$$

For $r = 0$, (30) gives us the following generalization of (3):

$$\int_M (2\sigma_2 - \text{Ric}_{N,N}^P) \, d\text{vol}_g = 0. \quad (31)$$

For $n = 1$, we obtain $\sigma_2 = 0$, thus, (31) gives zero integral of the “Gaussian P -curvature” of \mathcal{D} . Next, for $r = 1, 2$, (30) reduces to the following:

$$\begin{aligned} \int_M (3\sigma_3 - \text{tr}_{\mathcal{F}}(T_1(A_N)\mathcal{R}_N^P + \mathcal{R}_{\mathcal{Z}}^P)) \, d\text{vol}_g &= 0, \\ \int_M (4\sigma_4 + \langle T_2(A_N)\mathcal{Z}, \mathcal{Z} \rangle - \text{tr}_{\mathcal{F}}(T_2(A_N)\mathcal{R}_N^P + T_1(A_N)\mathcal{R}_{\mathcal{Z}}^P - \mathcal{R}_{A_N\mathcal{Z}}^P)) \, d\text{vol}_g &= 0. \end{aligned}$$

Example 2. Here is an amazing consequence of (28) and (31). Let us consider a closed connected sub-Riemannian manifold $(M, g; \mathcal{D})$, $\dim \mathcal{D} \geq 3$ with a harmonic orthogonal distribution \mathcal{D}^\perp and the condition

$$\text{Ric}^P \geq 2c > 0$$

for some real c . Suppose that $(M, g; \mathcal{D})$ is equipped with a codimension-one foliation \mathcal{F} , i.e., $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$. Then the image of the function $\sigma_2 : M \rightarrow \mathbb{R}$ contains an interval $[0, c + \varepsilon]$ for some $\varepsilon > 0$.

Indeed, by Reeb type formula (28), $\sigma_1(x) = 0$ at some $x \in M$. Then $2\sigma_2(x) = -\tau_2(x) \leq 0$, see (7). By (31), $\sigma_2(\mathcal{F}) \geq c \text{Vol}(M, g)$. Hence, there exists $y \in M$ such that $\sigma_2(y) > c$ (otherwise, $\sigma_2 \leq c$ on M , therefore, $\sigma_2 \equiv c > 0$ on M – a contradiction to $\sigma_2(x) \leq 0$). Put $\varepsilon = \sigma_2(y) - c$. Since M is connected and $\sigma_2 : M \rightarrow \mathbb{R}$ is a continuous function, σ_2 takes all values in the interval $[\sigma_2(x), \sigma_2(y)]$, which obviously includes $[0, c + \varepsilon]$.

Using (31), we obtain the following non-existence results for P -harmonic, i.e., $\sigma_1 = 0$, and P -totally umbilical, i.e., $A_N = (\sigma_1/n) \text{id}_{T\mathcal{F}}$, foliations.

Corollary 2. Let (M, \mathcal{D}, g) be a closed sub-Riemannian manifold with a harmonic distribution \mathcal{D}^\perp .

- (i) If $\text{Ric}^P > 0$, then there are no P -harmonic codimension-one foliations in \mathcal{D} .
- (ii) If $\text{Ric}^P < 0$, then there are no P -totally umbilical codimension-one foliations in \mathcal{D} .

Proof. (i) If \mathcal{F} is a P -harmonic codimension-one foliation in \mathcal{D} , then $2\sigma_2 = -\tau_2 \leq 0$, see (7).

(ii) If \mathcal{F} is a P -totally umbilical codimension-one foliation in \mathcal{D} , then

$$2\sigma_2 = \frac{n-1}{n}(\sigma_1)^2 \geq 0.$$

In both cases, (i) and (ii), we obtain a contradiction to integral formula (31). \square

Obviously, P -harmonic and P -totally umbilical distributions are harmonic and totally umbilical, respectively, but the opposite is not true.

4. Some Consequences

Here, we apply our formulas to sub-Riemannian manifolds with restrictions on the shape operator A_N or the induced curvature tensor R^P .

4.1. Foliations of Constant Sectional P -Curvature

The total r -th mean curvature of a codimension-one (relative to \mathcal{D}) foliation \mathcal{F} is given by

$$\sigma_r(\mathcal{F}, N) = \int_M \sigma_r \, d\text{vol}_g.$$

The following corollary of Theorem 3 generalizes (4), see also Section 4.1 in [4].

Corollary 3. Let (M, \mathcal{D}, g) be a closed sub-Riemannian manifold with $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$ and a harmonic orthogonal distribution \mathcal{D}^\perp . If condition (23) is satisfied, then $\sigma_r(\mathcal{F}, N)$ depend on r, n, c and the volume of (M, g) only, i.e., the following integral formula is valid:

$$\sigma_r(\mathcal{F}, N) = \begin{cases} c^{r/2} \binom{n/2}{r/2} \text{Vol}(M, g), & n \text{ and } r \text{ even,} \\ 0, & \text{either } n \text{ or } r \text{ odd.} \end{cases} \quad (32)$$

Proof. Using Lemma 3, we obtain

$$\text{tr}_{\mathcal{F}}(T_r(A_N)\mathcal{R}_N^P) = \sum_{i=1}^n \langle R^P(e_i, N)N, T_r(A_N)e_i \rangle = c \, \text{tr}_{\mathcal{F}} T_r(A_N) = c(n-r)\sigma_r.$$

From (30) we obtain the equality

$$(r+2)\sigma_{r+2}(\mathcal{F}, N) = c(n-r)\sigma_r(\mathcal{F}, N). \quad (33)$$

Since $\sigma_1(\mathcal{F}, N) = 0$, see (28), by induction we get $\sigma_r(\mathcal{F}, N) = 0$ for any odd r . For even $r = 2s$ and $n = 2l$, using (33) and induction, we obtain $\sigma_{2s}(\mathcal{F}, N) = c^s \binom{l}{s} \text{Vol}(M, g)$. \square

4.2. P -Totally Umbilical Foliations

Here, we obtain similar to (32) integral formulas when (M, \mathcal{D}, g) has the property

$$\text{Ric}_{X,N}^P = C \langle X, N \rangle, \quad X \in \mathcal{D}, \quad (34)$$

for some $C \in \mathbb{R}$ and \mathcal{F} (with $\dim \mathcal{F} > 1$) is a P -totally umbilical foliation. Please note that Einstein manifolds satisfy condition (34) when $\mathcal{D} = TM$.

The following corollary of Theorem 3 generalizes result in ([4] Section 4.2) on codimension-one totally umbilical foliations of Einstein manifolds.

Corollary 4. *Let (M, \mathcal{D}, g) be a closed sub-Riemannian manifold with $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$, P -totally umbilical foliation \mathcal{F} with $\dim \mathcal{F} > 1$, a harmonic orthogonal distribution \mathcal{D}^\perp , and satisfying (34). Then $\sigma_r(\mathcal{F}, N)$ depends on r, n, C and the volume of (M, g) only, i.e., the following formula holds:*

$$\sigma_r(\mathcal{F}, N) = \begin{cases} (C/n)^{r/2} \binom{n/2}{r/2} \text{Vol}(M, g), & n, r \text{ even,} \\ 0, & r \text{ odd.} \end{cases} \quad (35)$$

Proof. In this case, $T_r(A_N)$ has the form

$$T_r(A_N) = a_r (\sigma_1/n)^r \text{id}_{T\mathcal{F}}, \quad \text{where} \quad a_r = \sum_{i=0}^r (-1)^{r-i} \binom{n}{i}.$$

Since $\text{tr}_{T\mathcal{F}} T_r(A_N) = (n-r)\sigma_r = (n-r) \binom{n}{r} (\sigma_1/n)^r$, then

$$a_r = \frac{n-r}{n} \binom{n}{r}, \quad T_r(A_N) = \frac{n-r}{n} \sigma_r \text{id}_{T\mathcal{F}}.$$

By our assumption (34),

$$\text{Ric}_{Z,N}^P = 0, \quad \text{Ric}_{N,N}^P = C.$$

Thus, for a closed manifold with a P -totally umbilical foliation \mathcal{F} and condition (34), formula (30) becomes similar to (33):

$$\sigma_{r+2}(\mathcal{F}, N) = \frac{C}{n} \cdot \frac{n-r}{r+2} \sigma_r(\mathcal{F}, N).$$

Using induction similarly to Corollary 3, we obtain (35). \square

Remark 2. Our integral formulas provide more conditions for the mean curvature $H = \sigma_1/n$ of \mathcal{F} . In the case of a P -totally umbilical codimension-one foliation in \mathcal{D} , such conditions can be easily derived from (30) using $\sigma_r = \binom{n}{r} H^r$:

$$\int_M H^{r-1} \left(H^3 (n-1)(n-r-1)n - H(n-1)(r+1) \text{Ric}_{N,N}^P - r(r+1) \text{Ric}_{Z,N}^P \right) d\text{vol}_g = 0. \quad (36)$$

Here we used the following identity with binomial coefficients:

$$\sum_{j=1}^r (-1)^{j-1} \frac{n-r+j}{n} \binom{n}{r-j} = \binom{n-2}{r-1}.$$

These integrals contain polynomials depending on H , and one can obtain obstructions for existence of P -totally umbilical foliations. For example, if $r = n - 1$, then (36) reads as

$$\int_M H^{n-2} (H \operatorname{Ric}_{N,N}^P + \operatorname{Ric}_{Z,N}^P) \, d \operatorname{vol}_g = 0;$$

thus, if $\operatorname{Ric}^P > 0$, then any P -totally umbilical codimension-one foliation in \mathcal{D} with $Z = 0$ is P -totally geodesic, i.e., $A_N = 0$.

5. Conclusions

In the article, integral formula (2) and its consequences for foliated space forms are generalized for a Riemannian manifold equipped with a foliation \mathcal{F} and a unit vector field N orthogonal to \mathcal{F} . The results can be applied to foliated sub-Riemannian manifolds. Moreover, our integral formulas can be easily extended for non-integrable distributions and foliations defined outside of a “singularity set” Σ (a finite union of pairwise disjoint closed submanifolds of codimension at least k of a closed manifold M) under additional assumption of convergence of certain integrals. Namely, instead of the Divergence theorem, we apply the following result, see [9]: if $(k-1)(q-1) \geq 1$ and X is a vector field on $M \setminus \Sigma$ such that $\int_M \|X\|^q \, d \operatorname{vol}_g < \infty$, then $\int_M \operatorname{div} X \, d \operatorname{vol}_g = 0$. One can also try to extend our integral formulas for holomorphic foliations of complex (sub-)Riemannian manifolds, see [15] for the case of Riemannian manifolds, i.e., $\mathcal{D} = TM$, and for foliations of metric affine manifolds, see [16]. Finally note that our integral formulas are less cumbersome than the integral formulas in [17].

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