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On Robust Saddle-Point Criterion in Optimization Problems with Curvilinear Integral Functionals

Savin Treanță ^{1,*}  and Koushik Das ² ¹ Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania² Department of Mathematics, Taki Government College, Taki 743429, India; koushikdas.maths@gmail.com

* Correspondence: savin.treanta@upb.ro

Abstract: In this paper, we introduce a new class of multi-dimensional robust optimization problems (named (P)) with mixed constraints implying second-order partial differential equations (PDEs) and inequations (PDIs). Moreover, we define an auxiliary (modified) class of robust control problems (named $(P)_{(\bar{b}, \bar{c})}$), which is much easier to study, and provide some characterization results of (P) and $(P)_{(\bar{b}, \bar{c})}$ by using the notions of normal weak robust optimal solution and robust saddle-point associated with a Lagrange functional corresponding to $(P)_{(\bar{b}, \bar{c})}$. For this aim, we consider path-independent curvilinear integral cost functionals and the notion of convexity associated with a curvilinear integral functional generated by a controlled closed (complete integrable) Lagrange 1-form.

Keywords: Lagrange 1-form; second-order Lagrangian; normal weak robust optimal solution; modified objective function method; robust saddle-point



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1. Introduction

As we all know, partial differential equations (PDEs) and partial differential inequations (PDIs) are essential in modeling and investigating many processes in engineering and science. In this respect, many researchers have taken a special interest in their study. We specify, for example, the research works of Mititelu [1], Treanță [2–4], Mititelu and Treanță [5], Olteanu and Treanță [6], Preeti et al. [7], and Jayswal et al. [8] on the study of some optimization problems with ODE, PDE, or isoperimetric constraints. In order to reduce the complexity of the considered optimization problems, some auxiliary optimization problems were formulated to investigate the initial problems more easily (Treanță [9–12]). Nevertheless, since the real-life processes and phenomena often imply uncertainty in initial data, many researchers have turned their attention to optimization issues governed by first- and second-order PDEs, isoperimetric restrictions, stochastic PDEs, uncertain data, or a combination thereof. In this context, we mention the following research papers: Wei et al. [13], Liu and Yuan [14], Jeyakumar et al. [15], Sun et al. [16], Preeti et al. [7], Lu et al. [17], and Treanță [18]. The structure of approximate solutions associated with some autonomous variational problems on large finite intervals was studied by Zaslavski [19]. Furthermore, Geldhauser and Valdinoci [20] investigated an optimization problem with SPDE constraints, with the peculiarity that the control parameter s is the s -th power of the diffusion operator in the state equation. In [21], Babamiyi et al. focused on identifying a distributed parameter in a saddle point problem with application to the elasticity imaging inverse problem. Very recently, Debnath and Qin [22], investigated the robust optimality and duality for minimax fractional programming problems with support functions.

Motivated and inspired by previous research works, in this paper, we introduce and study new classes of robust optimization problems. More exactly, by taking curvilinear integral objective functionals with mixed (equality and inequality) constraints implying data uncertainty and second-order partial derivatives, we introduce the robust control problems under study. Further, by using the concept of convexity associated with curvilinear integral

functionals and the notion of robust saddle-point associated with a Lagrange functional corresponding to the modified robust optimization problem, we formulate and prove some characterization results for the considered classes of control problems. The novelty elements included in the paper, in comparison with other research papers in this field, are provided by the presence of uncertain data both in the objective functional and in the constraint functionals and also by the presence of second-order partial derivatives. Moreover, the proofs associated with the main results are established in an innovative way. Furthermore, since the mathematical framework introduced here is appropriate for various scientific approaches and viewpoints on complex spatial behaviors, the current paper could be seen as a definitive research work for a large community of researchers in engineering and science.

The paper is structured as follows. Section 2 provides the preliminary and necessary mathematical tools, which will be used in the next sections. Section 3 includes the main results of this paper. Under convexity assumption of the cost functional, the first main result establishes a connection between a robust saddle point of the Lagrange functional associated with the associated modified problem $(P)_{(\bar{b}, \bar{c})}$ and a weak robust optimal solution of (P) . By assuming the convexity hypotheses of the constraint functionals, the converse of the first main result is presented in the second main result. In Section 4, we formulate the conclusions and further development.

2. Preliminaries

In this paper, we use the following working hypotheses and notations:

- Consider $\mathbb{R}^p, \mathbb{R}^q, \mathbb{R}^r$ and \mathbb{R}^n as Euclidean spaces of dimension p, q, r and n , respectively;
- Consider $\Theta \subset \mathbb{R}^p$ as a compact domain and the point $t = (t^\alpha) \in \Theta$ as a *multi-parameter of evolution* or *multi-time*;
- Consider $\Gamma \subset \Theta$ as a piecewise smooth curve joining the points t_0 and t_1 in Θ ;
- \mathcal{B} is the space of C^4 -class state functions $b = (b^\tau) : \Theta \rightarrow \mathbb{R}^q$ and $b_\alpha := \frac{\partial b}{\partial t^\alpha}$, $b_{\alpha\beta} := \frac{\partial^2 b}{\partial t^\alpha \partial t^\beta}$ denote the *partial speed* and *partial acceleration*, respectively;
- \mathcal{C} is the space of C^1 -class control functions $c = (c^j) : \Theta \rightarrow \mathbb{R}^r$;
- Consider T as the transpose for a given vector;
- Consider the following convention for inequalities and equalities of any two vectors $x, y \in \mathbb{R}^n$:
 - (i) $x < y \Leftrightarrow x_i < y_i, \forall i = \overline{1, n}$,
 - (ii) $x = y \Leftrightarrow x_i = y_i, \forall i = \overline{1, n}$,
 - (iii) $x \leq y \Leftrightarrow x_i \leq y_i, \forall i = \overline{1, n}$,
 - (iv) $x \leq y \Leftrightarrow x_i \leq y_i, \forall i = \overline{1, n}$ and $x_i < y_i$ for some i .

In the following, we consider $g = (g_1, \dots, g_m) = (g_l) : J^2(\Theta, \mathbb{R}^q) \times \mathcal{C} \times \mathcal{U}_l \rightarrow \mathbb{R}^m$, $l = \overline{1, m}$, $f_\kappa : J^2(\Theta, \mathbb{R}^q) \times \mathcal{C} \times \mathcal{W}_\kappa \rightarrow \mathbb{R}$, $\kappa = \overline{1, p}$, $h = (h_1, \dots, h_n) = (h_\zeta) : J^2(\Theta, \mathbb{R}^q) \times \mathcal{C} \times \mathcal{V}_\zeta \rightarrow \mathbb{R}^n$, $\zeta = \overline{1, n}$, are C^3 -class functionals. Furthermore, let us assume that $w = (w_\kappa), u = (u_l)$ and $v = (v_\zeta)$ are the uncertain parameters for some convex compact subsets $\mathcal{W} = (\mathcal{W}_\kappa) \subset \mathbb{R}^p, \mathcal{U} = (\mathcal{U}_l) \subset \mathbb{R}^m$ and $\mathcal{V} = (\mathcal{V}_\zeta) \subset \mathbb{R}^n$, respectively. Denote by $J^2(\Theta, \mathbb{R}^q)$ the second-order jet bundle associated with Θ and \mathbb{R}^q . Furthermore, assume that the previous multi-time-controlled second-order Lagrangians f_κ determine a controlled closed (complete integrable) Lagrange 1-form (see summation over the repeated indices, Einstein summation):

$$f_\kappa(t, b(t), b_\sigma(t), b_{\alpha\beta}(t), c(t), w) dt^\kappa,$$

which generates the following controlled path-independent curvilinear integral functional:

$$\int_{\Gamma} f_{\kappa}(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), w) dt^{\kappa}.$$

The second-order PDE and PDI constrained variational control problem with uncertainty in the objective and constraint functionals is defined as follows:

$$(P) \quad \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} f_{\kappa}(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), w) dt^{\kappa}$$

subject to

$$g(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), u) \leq 0, \quad t \in \Theta$$

$$h(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), v) = 0, \quad t \in \Theta$$

$$b(t_0) = b_0, \quad b(t_1) = b_1, \quad b_{\sigma}(t_0) = b_{\sigma 0}, \quad b_{\sigma}(t_1) = b_{\sigma 1}.$$

The associated robust counterpart of the aforementioned variational control problem (P) is defined as:

$$(RP) \quad \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} \max_{w \in \mathcal{W}} f_{\kappa}(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), w) dt^{\kappa}$$

subject to

$$g(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), u) \leq 0, \quad t \in \Theta, \quad \forall u \in \mathcal{U}$$

$$h(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), v) = 0, \quad t \in \Theta, \quad \forall v \in \mathcal{V}$$

$$b(t_0) = b_0, \quad b(t_1) = b_1, \quad b_{\sigma}(t_0) = b_{\sigma 0}, \quad b_{\sigma}(t_1) = b_{\sigma 1}.$$

Further, denote by

$$\mathcal{D} = \{ (b, c) \in \mathcal{B} \times \mathcal{C} : g(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), u) \leq 0, \\ h(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), v) = 0, \quad b(t_0) = b_0, \quad b(t_1) = b_1, \\ b_{\sigma}(t_0) = b_{\sigma 0}, \quad b_{\sigma}(t_1) = b_{\sigma 1}, \quad t \in \Theta, \quad u \in \mathcal{U}, \quad v \in \mathcal{V} \}$$

the feasible solution set in (RP), and we call it the *robust feasible solution set* of (P).

To simplify the presentation, we use the following notation:

$$\pi = (t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t)).$$

The associated first-order partial derivatives of f_{κ} , $\kappa = \overline{1, p}$, are defined as

$$\frac{\partial f_{\kappa}}{\partial b} = \left(\frac{\partial f_{\kappa}}{\partial b^1}, \dots, \frac{\partial f_{\kappa}}{\partial b^q} \right), \quad \frac{\partial f_{\kappa}}{\partial c} = \left(\frac{\partial f_{\kappa}}{\partial c^1}, \dots, \frac{\partial f_{\kappa}}{\partial c^r} \right).$$

In the same manner, we have $g_b := \frac{\partial g}{\partial b}$ and $g_c := \frac{\partial g}{\partial c}$ by using matrices with m rows and $h_b := \frac{\partial h}{\partial b}$ and $h_c := \frac{\partial h}{\partial c}$ by using matrices with n rows.

Further, in accordance to Treanță [3], we define the notion of a weak robust optimal solution of the considered class of constrained variational control problems. This notion will be used to establish the associated robust necessary conditions of optimality and the main results derived in the paper.

Definition 1. A pair $(\bar{b}, \bar{c}) \in \mathcal{D}$ is said to be a weak robust optimal solution to (P) if there does not exist another point $(b, c) \in \mathcal{D}$ such that

$$\int_{\Gamma} \max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) dt^{\kappa} < \int_{\Gamma} \max_{w \in \mathcal{W}} f_{\kappa}(\bar{\pi}, w) dt^{\kappa}.$$

Next, we shall use the Saunders’s multi-index notation (Saunders [23], Treanță [3,24]) to formulate the concept of convexity and the robust necessary optimality conditions for (P).

Definition 2. A curvilinear integral functional

$$F(b, c, \bar{w}) = \int_{\Gamma} f_{\kappa}(t, b(t), b_{\sigma}(t), b_{\alpha\beta}(t), c(t), \bar{w}) dt^{\kappa} = \int_{\Gamma} f_{\kappa}(\pi, \bar{w}) dt^{\kappa}$$

is said to be convex at $(\bar{b}, \bar{c}) \in \mathcal{B} \times \mathcal{C}$ if the following inequality

$$\begin{aligned} F(b, c, \bar{w}) - F(\bar{b}, \bar{c}, \bar{w}) &\geq \int_{\Gamma} \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) [b(t) - \bar{b}(t)] dt^{\kappa} \\ &+ \int_{\Gamma} \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) [b_{\sigma}(t) - \bar{b}_{\sigma}(t)] dt^{\kappa} \\ &+ \frac{1}{n(\alpha, \beta)} \int_{\Gamma} \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) [b_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)] dt^{\kappa} \\ &+ \int_{\Gamma} \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) [c(t) - \bar{c}(t)] dt^{\kappa} \end{aligned}$$

holds for all $(b, c) \in \mathcal{B} \times \mathcal{C}$.

According to Treanță [24], we formulate the robust necessary optimality conditions for (P).

Theorem 1. If $(\bar{b}, \bar{c}) \in \mathcal{D}$ is a weak robust optimal solution to (P) and $\max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) = f_{\kappa}(\pi, \bar{w})$, $\kappa = \overline{1, p}$, then there exist the scalar $\bar{\mu} \in \mathbb{R}$, the piecewise smooth functions $\bar{v} = (\bar{v}_l(t)) \in \mathbb{R}_+^m$, $\bar{\gamma} = (\bar{\gamma}_{\zeta}(t)) \in \mathbb{R}^n$, and the uncertainty parameters $\bar{u} \in \mathcal{U}$ and $\bar{\vartheta} \in \mathcal{V}$ such that the following conditions

$$\bar{\mu} \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) + \bar{v}^T g_b(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_b(\bar{\pi}, \bar{\vartheta}) \tag{1}$$

$$-D_{\sigma} \left[\bar{\mu} \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) + \bar{v}^T g_{b_{\sigma}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\sigma}}(\bar{\pi}, \bar{\vartheta}) \right]$$

$$+ \frac{1}{n(\alpha, \beta)} D_{\alpha\beta}^2 \left[\bar{\mu} \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) + \bar{v}^T g_{b_{\alpha\beta}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\alpha\beta}}(\bar{\pi}, \bar{\vartheta}) \right] = 0, \quad \kappa = \overline{1, p}$$

$$\bar{\mu} \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) + \bar{v}^T g_c(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_c(\bar{\pi}, \bar{\vartheta}) = 0, \quad \kappa = \overline{1, p} \tag{2}$$

$$\bar{v}^T g(\bar{\pi}, \bar{u}) = 0, \quad \bar{v} \geq 0, \tag{3}$$

$$\bar{\mu} \geq 0 \tag{4}$$

hold for all $t \in \Theta$, except at discontinuities.

Remark 1. The robust necessary optimality conditions of (P) are given by the conditions (1)–(4).

Definition 3. A pair $(\bar{b}, \bar{c}) \in \mathcal{D}$ is said to be a normal weak robust optimal solution to (P) if $\bar{\mu} > 0$ in Theorem 1. We can consider $\bar{\mu} = 1$ without loss of generality.

Next, we use the modified objective function method to reduce the complexity of (P). In this direction, let (\bar{b}, \bar{c}) be an arbitrary given robust feasible solution to (P). The modified multi-dimensional variational control problem associated with the original optimization problem (P) is defined as:

$$(P)_{(\bar{b}, \bar{c})} \quad \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} \left\{ \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w)(b(t) - \bar{b}(t)) + \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w)(b_{\sigma}(t) - \bar{b}_{\sigma}(t)) \right. \\ \left. + \frac{1}{n(\alpha, \beta)} \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w)(b_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) + \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w)(c(t) - \bar{c}(t)) \right\} dt^{\kappa}$$

subject to

$$g(\pi, u) \leq 0, \quad t \in \Theta$$

$$h(\pi, v) = 0, \quad t \in \Theta$$

$$b(t_0) = b_0, \quad b(t_1) = b_1, \quad b_{\sigma}(t_0) = b_{\sigma 0}, \quad b_{\sigma}(t_1) = b_{\sigma 1},$$

where the functionals g, f_{κ} and h are given as in (P).

The associated robust counterpart of the modified multi-dimensional variational control problem (P)_(\bar{b}, \bar{c}) is defined as:

$$(RP)_{(\bar{b}, \bar{c})} \quad \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} \max_{w \in \mathcal{W}} \left\{ \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w)(b(t) - \bar{b}(t)) + \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w)(b_{\sigma}(t) - \bar{b}_{\sigma}(t)) \right. \\ \left. + \frac{1}{n(\alpha, \beta)} \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w)(b_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) + \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w)(c(t) - \bar{c}(t)) \right\} dt^{\kappa}$$

subject to

$$g(\pi, u) \leq 0, \quad t \in \Theta, \quad \forall u \in \mathcal{U}$$

$$h(\pi, v) = 0, \quad t \in \Theta, \quad \forall v \in \mathcal{V}$$

$$b(t_0) = b_0, \quad b(t_1) = b_1, \quad b_{\sigma}(t_0) = b_{\sigma 0}, \quad b_{\sigma}(t_1) = b_{\sigma 1}.$$

Remark 2. The robust feasible solution set of the problem (P)_(\bar{b}, \bar{c}) is the same as in (P). Consequently, it is also denoted by \mathcal{D} .

Definition 4. A pair $(\hat{b}, \hat{c}) \in \mathcal{D}$ is said to be a weak robust optimal solution to (P)_(\bar{b}, \bar{c}) if there does not exist another point $(b, c) \in \mathcal{D}$ such that

$$\int_{\Gamma} \max_{w \in \mathcal{W}} \left[\frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w)(b - \bar{b}) + \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w)(b_{\sigma} - \bar{b}_{\sigma}) \right. \\ \left. + \frac{1}{n(\alpha, \beta)} \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w)(b_{\alpha\beta} - \bar{b}_{\alpha\beta}) + \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w)(c - \bar{c}) \right] dt^{\kappa} \\ < \int_{\Gamma} \max_{w \in \mathcal{W}} \left[\frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w)(\hat{b} - \bar{b}) + \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w)(\hat{b}_{\sigma} - \bar{b}_{\sigma}) \right. \\ \left. + \frac{1}{n(\alpha, \beta)} \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w)(\hat{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) + \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w)(\hat{c} - \bar{c}) \right] dt^{\kappa}.$$

3. Saddle-Point Optimality Criterion

In this section, under some convexity assumptions, we establish some connections between a weak robust optimal solution of (P) and a robust saddle-point associated with a Lagrange functional (Lagrangian) corresponding to the modified multi-dimensional variational control problem (P)_(\bar{b}, \bar{c}). In this regard, in accordance with Treanță [9,11,12] and Preeti et al. [7], we formulate the next definitions.

Definition 5. The Lagrange functional $L((b, c), \nu, \gamma, w, u, v) : \mathcal{B} \times \mathcal{C} \times \mathbb{R}_+^m \times \mathbb{R}^n \times \mathcal{W} \times \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{R}$ associated with the modified variational control problem (P)_(\bar{b}, \bar{c}) is defined as

$$\begin{aligned}
 L((b, c), v, \gamma, w, u, v) = & \int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} \left[(b(t) - \bar{b}(t)) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (b_{\sigma}(t) - \bar{b}_{\sigma}(t)) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \right. \\
 & + \frac{1}{n(\alpha, \beta)} (b_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (c(t) - \bar{c}(t)) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \left. \left. \right] \right. \\
 & \left. + v^T(t)g(\pi, u) + \gamma^T(t)h(\pi, v) \right\} dt^{\kappa}.
 \end{aligned}$$

Definition 6. A point $((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}) \in \mathcal{D} \times \mathbb{R}_+^m \times \mathbb{R}^n \times \mathcal{W} \times \mathcal{U} \times \mathcal{V}$ is said to be a robust saddle-point for the Lagrange functional $L((b, c), v, \gamma, w, u, v)$ associated with the modified multi-dimensional variational control problem $(P)_{(\bar{b}, \bar{c})}$ if the following relations are fulfilled:

- (i) $L((\bar{b}, \bar{c}), v, \gamma, w, u, v) \leq L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}), \forall v \in \mathbb{R}_+^m, \forall \gamma \in \mathbb{R}^n, \forall (u, v) \in \mathcal{U} \times \mathcal{V}$,
- (ii) $L((b, c), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}) \geq L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}), \forall (b, c) \in \mathcal{B} \times \mathcal{C}$.

Now, taking into account the above definitions, we establish the following two main results of this paper.

Theorem 2. Let (\bar{b}, \bar{c}) be a robust feasible solution to (P) . Assume that $\max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) = f_{\kappa}(\pi, \bar{w}), \kappa = \overline{1, p}$, and the objective functional $\int_{\Gamma} f_{\kappa}(\pi, \bar{w}) dt^{\kappa}$ is convex at (\bar{b}, \bar{c}) . If the point $((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v})$ is a robust saddle-point for the Lagrange functional $L((b, c), v, \gamma, w, u, v)$ associated with the modified multi-dimensional variational control problem $(P)_{(\bar{b}, \bar{c})}$, then (\bar{b}, \bar{c}) is a weak robust optimal solution to (P) .

Proof. By reductio ad absurdum, let us assume that (\bar{b}, \bar{c}) is not a weak robust optimal solution to (P) . Therefore, by using the convexity property of the objective functional $\int_{\Gamma} f_{\kappa}(\pi, \bar{w}) dt^{\kappa}$, we get

$$\begin{aligned}
 & \int_{\Gamma} \max_{w \in \mathcal{W}} \left[(\tilde{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\tilde{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \\
 & \left. + \frac{1}{n(\alpha, \beta)} (\tilde{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\tilde{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] dt^{\kappa} \\
 & < \int_{\Gamma} \max_{w \in \mathcal{W}} \left[(\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \\
 & \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] dt^{\kappa},
 \end{aligned} \tag{5}$$

for some $(\tilde{b}, \tilde{c}) \in \mathcal{D}$.

From the feasibility of (\tilde{b}, \tilde{c}) to the problem (P) and $\bar{v} \in \mathbb{R}_+^m$, we get

$$\int_{\Gamma} \bar{v}^T g(\bar{\pi}, \bar{u}) dt^{\kappa} \leq 0. \tag{6}$$

On the other hand, since $((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v})$ is a robust saddle-point for the Lagrange functional $L((b, c), v, \gamma, w, u, v)$ associated with the modified multi-dimensional variational control problem $(P)_{(\bar{b}, \bar{c})}$, by using Definition 6 (i), we have

$$L((\bar{b}, \bar{c}), v, \gamma, w, u, v) \leq L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}), \forall v \in \mathbb{R}_+^m, \forall \gamma \in \mathbb{R}^n, \forall u \in \mathcal{U}, \forall v_s \in \mathcal{V},$$

which, using of the definition of Lagrange functional, can be rewritten as

$$\int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} \left[(\bar{b}(t) - \bar{b}(t)) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma}(t) - \bar{b}_{\sigma}(t)) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \right.$$

$$\begin{aligned}
 & + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c}(t) - \bar{c}(t)) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \\
 & \quad + v^T(t)g(\bar{\pi}, u) + \gamma^T(t)h(\bar{\pi}, v) \} dt^{\kappa} \\
 & \leq \int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} [(\bar{b}(t) - \bar{b}(t)) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma}(t) - \bar{b}_{\sigma}(t)) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \\
 & \quad \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c}(t) - \bar{c}(t)) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] \\
 & \quad \left. + \bar{v}^T(t)g(\bar{\pi}, \bar{u}) + \bar{\gamma}^T(t)h(\bar{\pi}, \bar{v}) \right\} dt^{\kappa}.
 \end{aligned}$$

Since $\max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) = f_{\kappa}(\pi, \bar{w})$, $\kappa = \overline{1, p}$, it follows that

$$\begin{aligned}
 & \int_{\Gamma} \left\{ (\bar{b}(t) - \bar{b}(t)) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) + (\bar{b}_{\sigma}(t) - \bar{b}_{\sigma}(t)) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) \right. \\
 & \quad \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) + (\bar{c}(t) - \bar{c}(t)) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) \right. \\
 & \quad \left. + v^T(t)g(\bar{\pi}, u) + \gamma^T(t)h(\bar{\pi}, v) \right\} dt^{\kappa} \\
 & \leq \int_{\Gamma} \left\{ (\bar{b}(t) - \bar{b}(t)) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) + (\bar{b}_{\sigma}(t) - \bar{b}_{\sigma}(t)) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) \right. \\
 & \quad \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta}(t) - \bar{b}_{\alpha\beta}(t)) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) + (\bar{c}(t) - \bar{c}(t)) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) \right. \\
 & \quad \left. + \bar{v}^T(t)g(\bar{\pi}, \bar{u}) + \bar{\gamma}^T(t)h(\bar{\pi}, \bar{v}) \right\} dt^{\kappa}.
 \end{aligned}$$

If we set $\nu = 0$ and $\gamma = 0$ in the above inequality, we obtain

$$\int_{\Gamma} \bar{v}^T g(\bar{\pi}, \bar{u}) dt^{\kappa} \geq 0. \tag{7}$$

From (6) and (7), it follows that

$$\int_{\Gamma} \bar{v}^T g(\bar{\pi}, \bar{u}) dt^{\kappa} \leq \int_{\Gamma} v^T g(\bar{\pi}, u) dt^{\kappa},$$

which, along with the inequality (5), gives

$$\begin{aligned}
 & \int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} [(\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \\
 & \quad \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] + \bar{v}^T g(\bar{\pi}, \bar{u}) \} dt^{\kappa} \\
 & < \int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} [(\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \\
 & \quad \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] + v^T g(\bar{\pi}, u) \} dt^{\kappa},
 \end{aligned}$$

equivalently with

$$\begin{aligned}
 & \int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} [(\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \\
 & \quad \left. + \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] + \bar{v}^T g(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h(\bar{\pi}, \bar{v}) \} dt^{\kappa}
 \end{aligned}$$

$$\begin{aligned}
 &< \int_{\Gamma} \left\{ \max_{w \in \mathcal{W}} \left[(\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, w) + (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, w) \right. \right. \\
 &+ \left. \left. \frac{1}{n(\alpha, \beta)} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, w) + (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, w) \right] + \bar{v}^T g(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h(\bar{\pi}, \bar{v}) \right\} dt^{\kappa},
 \end{aligned}$$

or

$$L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}) < L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v}), \quad (\bar{b}, \bar{c}) \in \mathcal{B} \times \mathcal{C},$$

which contradicts Definition 6, and the proof is completed. \square

Theorem 3. Let (\bar{b}, \bar{c}) be a normal weak robust optimal solution to (P). Assume that $\max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) = f_{\kappa}(\pi, \bar{w})$, $\kappa = \overline{1, p}$, and the constraint functionals

$$\int_{\Gamma} \bar{v}^T g(\pi, \bar{u}) dt^{\kappa}, \quad \int_{\Gamma} \bar{\gamma}^T h(\pi, \bar{v}) dt^{\kappa}$$

are convex at (\bar{b}, \bar{c}) . Then, $((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v})$ is a robust saddle-point for the Lagrange functional $L((b, c), v, \gamma, w, u, v)$ associated with the modified variational control problem (P) $_{(\bar{b}, \bar{c})}$.

Proof. Since the relations (1)–(4), with $\bar{\mu} = 1$, are satisfied for all $t \in \Theta$, except at discontinuities, the conditions (1) and (2) yield

$$\begin{aligned}
 &\int_{\Gamma} (b - \bar{b}) \left\{ \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) + \bar{v}^T g_b(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_b(\bar{\pi}, \bar{v}) \right. \\
 &\quad \left. - D_{\sigma} \left[\frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) + \bar{v}^T g_{b_{\sigma}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\sigma}}(\bar{\pi}, \bar{v}) \right] \right. \\
 &+ \left. \frac{1}{n(\alpha, \beta)} D_{\alpha\beta}^2 \left[\frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) + \bar{v}^T g_{b_{\alpha\beta}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\alpha\beta}}(\bar{\pi}, \bar{v}) \right] \right\} dt^{\kappa} \\
 &+ \int_{\Gamma} (c - \bar{c}) \left\{ \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) + \bar{v}^T g_c(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_c(\bar{\pi}, \bar{v}) \right\} dt^{\kappa} \\
 &= \int_{\Gamma} \left[(b - \bar{b}) \left\{ \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) + \bar{v}^T g_b(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_b(\bar{\pi}, \bar{v}) \right\} \right. \\
 &\quad \left. + (b_{\sigma} - \bar{b}_{\sigma}) \left\{ \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) + \bar{v}^T g_{b_{\sigma}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\sigma}}(\bar{\pi}, \bar{v}) \right\} \right. \\
 &+ \left. \frac{1}{n(\alpha, \beta)} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) \left\{ \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) + \bar{v}^T g_{b_{\alpha\beta}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\alpha\beta}}(\bar{\pi}, \bar{v}) \right\} \right] dt^{\kappa} \\
 &+ \int_{\Gamma} (c - \bar{c}) \left\{ \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) + \bar{v}^T g_c(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_c(\bar{\pi}, \bar{v}) \right\} dt^{\kappa} = 0,
 \end{aligned} \tag{8}$$

where we used the formula of integration by parts, the result “A total divergence is equal to a total derivative” (see Treanță [4]) and the boundary conditions formulated in the considered problem.

Further, taking into account the assumption of convexity for the following multiple integral functionals $\int_{\Gamma} \bar{v}^T g(\pi, \bar{u}) dt^{\kappa}$, $\int_{\Gamma} \bar{\gamma}^T h(\pi, \bar{v}) dt^{\kappa}$ at (\bar{b}, \bar{u}) , we obtain

$$\begin{aligned}
 &\int_{\Gamma} \left\{ \bar{v}^T g(\pi, \bar{u}) - \bar{v}^T g(\bar{\pi}, \bar{u}) \right\} dt^{\kappa} \geq \int_{\Gamma} (b - \bar{b}) \bar{v}^T g_b(\bar{\pi}, \bar{u}) dt^{\kappa} \\
 &+ \int_{\Gamma} (b_{\sigma} - \bar{b}_{\sigma}) \bar{v}^T g_{b_{\sigma}}(\bar{\pi}, \bar{u}) dt^{\kappa} + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) \bar{v}^T g_{b_{\alpha\beta}}(\bar{\pi}, \bar{u}) dt^{\kappa} \\
 &+ \int_{\Gamma} (c - \bar{c}) \bar{v}^T g_c(\bar{\pi}, \bar{u}) dt^{\kappa},
 \end{aligned}$$

$$\int_{\Gamma} \left\{ \bar{\gamma}^T h(\pi, \bar{v}) - \bar{\gamma}^T h(\bar{\pi}, \bar{v}) \right\} dt^{\kappa} \geq \int_{\Gamma} (b - \bar{b}) \bar{\gamma}^T h_b(\bar{\pi}, \bar{v}) dt^{\kappa} \\ + \int_{\Gamma} (b_{\sigma} - \bar{b}_{\sigma}) \bar{\gamma}^T h_{b_{\sigma}}(\bar{\pi}, \bar{v}) dt^{\kappa} + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) \bar{\gamma}^T h_{b_{\alpha\beta}}(\bar{\pi}, \bar{v}) dt^{\kappa} \\ + \int_{\Gamma} (c - \bar{c}) \bar{\gamma}^T h_c(\bar{\pi}, \bar{v}_{s.}) dt^{\kappa},$$

implying

$$\int_{\Gamma} \left\{ \bar{v}^T g(\pi, \bar{u}) + \bar{\gamma}^T h(\pi, \bar{v}) \right\} dt^{\kappa} - \int_{\Gamma} \left\{ \bar{v}^T g(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h(\bar{\pi}, \bar{v}) \right\} dt^{\kappa} \\ \geq \int_{\Gamma} (b - \bar{b}) \left[\bar{v}^T g_b(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_b(\bar{\pi}, \bar{v}) \right] dt^{\kappa} \\ + \int_{\Gamma} (b_{\sigma} - \bar{b}_{\sigma}) \left[\bar{v}^T g_{b_{\sigma}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\sigma}}(\bar{\pi}, \bar{v}) \right] dt^{\kappa} \\ + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) \left[\bar{v}^T g_{b_{\alpha\beta}}(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_{b_{\alpha\beta}}(\bar{\pi}, \bar{v}) \right] dt^{\kappa} \\ + \int_{\Gamma} (c - \bar{c}) \left[\bar{v}^T g_c(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h_c(\bar{\pi}, \bar{v}_{s.}) \right] dt^{\kappa} \\ = - \int_{\Gamma} (b - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) dt^{\kappa} - \int_{\Gamma} (b_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ - \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) dt^{\kappa} - \int_{\Gamma} (c - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) dt^{\kappa},$$

by considering (8). The previous inequality can be formulated as follows

$$\int_{\Gamma} (b - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (b_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (b_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (c - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ + \int_{\Gamma} \left\{ \bar{v}^T g(\pi, \bar{u}) + \bar{\gamma}^T h(\pi, \bar{v}) \right\} dt^{\kappa} \\ \geq \int_{\Gamma} (\bar{b} - b) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (\bar{b}_{\sigma} - b_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (\bar{b}_{\alpha\beta} - b_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (\bar{c} - c) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ + \int_{\Gamma} \left\{ \bar{v}^T g(\bar{\pi}, \bar{u}) + \bar{\gamma}^T h(\bar{\pi}, \bar{v}) \right\} dt^{\kappa},$$

which involves the inequality

$$L((b, c), \bar{v}, \bar{\gamma}, w, \bar{u}, \bar{v}) \geq L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, w, \bar{u}, \bar{v}), \quad \forall (b, c) \in \mathcal{B} \times \mathcal{C}. \tag{9}$$

Furthermore, the following inequality is satisfied

$$\int_{\Gamma} v^T g(\bar{\pi}, u) dt^{\kappa} + \int_{\Gamma} \gamma^T h(\bar{\pi}, v) dt^{\kappa} \leq 0$$

for all $(v, \gamma) \in \mathbb{R}_+^m \times \mathbb{R}^n$, $(u, v) \in \mathcal{U} \times \mathcal{V}$ and, using the feasibility of (\bar{b}, \bar{u}) , we obtain

$$\int_{\Gamma} v^T g(\bar{\pi}, u) dt^{\kappa} + \int_{\Gamma} \gamma^T h(\bar{\pi}, v) dt^{\kappa} \\ \leq \int_{\Gamma} \bar{v}^T g(\bar{\pi}, \bar{u}) dt^{\kappa} + \int_{\Gamma} \bar{\gamma}^T h(\bar{\pi}, \bar{v}) dt^{\kappa},$$

or, equivalently,

$$\begin{aligned} & \int_{\Gamma} (\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ & + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ & \quad \int_{\Gamma} v^T g(\bar{\pi}, u) dt^{\kappa} + \int_{\Gamma} \gamma^T h(\bar{\pi}, v) dt^{\kappa} \\ & \leq \int_{\Gamma} (\bar{b} - \bar{b}) \frac{\partial f_{\kappa}}{\partial b}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (\bar{b}_{\sigma} - \bar{b}_{\sigma}) \frac{\partial f_{\kappa}}{\partial b_{\sigma}}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ & + \frac{1}{n(\alpha, \beta)} \int_{\Gamma} (\bar{b}_{\alpha\beta} - \bar{b}_{\alpha\beta}) \frac{\partial f_{\kappa}}{\partial b_{\alpha\beta}}(\bar{\pi}, \bar{w}) dt^{\kappa} + \int_{\Gamma} (\bar{c} - \bar{c}) \frac{\partial f_{\kappa}}{\partial c}(\bar{\pi}, \bar{w}) dt^{\kappa} \\ & \quad \int_{\Gamma} \bar{v}^T g(\bar{\pi}, \bar{u}) dt^{\kappa} + \int_{\Gamma} \bar{\gamma}^T h(\bar{\pi}, \bar{v}) dt^{\kappa}, \end{aligned}$$

involving

$$L((\bar{b}, \bar{c}), v, \gamma, w, u, v) \leq L((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, w, \bar{u}, \bar{v}), \quad \forall v \in \mathbb{R}_+^m, \quad \forall \gamma \in \mathbb{R}^n, \quad \forall (u, v) \in \mathcal{U} \times \mathcal{V}. \quad (10)$$

Consequently, by (9) and (10), we conclude that $((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, w, \bar{u}, \bar{v})$ is a robust saddle-point for the Lagrange functional $L((b, c), v, \gamma, w, u, v)$ associated with the modified multi-dimensional variational control problem $(P)_{(\bar{b}, \bar{c})}$, and the proof is completed. \square

Illustrative application. Let us minimize the mechanical work performed by the variable force $\bar{F}(c^2(t) + w_1, c^2(t) + w_2)$, including the uncertain parameters $w_{\kappa} \in \mathcal{W}_{\kappa} = [0, 1], \kappa = 1, 2$, to move its point of application along the piecewise smooth curve Γ , contained in $\Theta = [0, 3]^2 = [0, 3] \times [0, 3]$ and joining the points $t_0 = (0, 0)$ and $t_1 = (3, 3)$, such that the following constraints

$$\begin{aligned} u_1(b - 2)(b + 2) &\leq 0, \quad t = (t^1, t^2) \in \Theta \\ \frac{\partial b}{\partial t^1} &= 3 - c + v_1, \quad t = (t^1, t^2) \in \Theta \\ \frac{\partial b}{\partial t^2} &= 3 - c + v_2, \quad t = (t^1, t^2) \in \Theta \\ b(0, 0) &= 1, \quad b(3, 3) = 2, \end{aligned}$$

are satisfied, where $v_{\zeta} \in \mathcal{V}_{\zeta} = [1, 2], \zeta = 1, 2$ and $u_1 \in \mathcal{U}_1 = \left[\frac{1}{2}, 1\right]$.

To solve the previous problem, for $m = 1, n = p = 2$, we consider

$$f_{\kappa}(\pi, w) dt^{\kappa} = f_1(\pi, w) dt^1 + f_2(\pi, w) dt^2 = (c^2 + w_1)(t) dt^1 + (c^2 + w_2) dt^2$$

and the constrained robust control problem:

$$(P1) \quad \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} f_{\kappa}(\pi, w) dt^{\kappa}$$

subject to

$$u_1(b - 2)(b + 2) \leq 0, \quad t = (t^1, t^2) \in \Theta \tag{11}$$

$$\frac{\partial b}{\partial t^1} = 3 - c + v_1, \quad t = (t^1, t^2) \in \Theta \tag{12}$$

$$\frac{\partial b}{\partial t^2} = 3 - c + v_2, \quad t = (t^1, t^2) \in \Theta \tag{13}$$

$$b(0, 0) = 1, \quad b(3, 3) = 2. \tag{14}$$

The robust counterpart of (P1) is formulated as follows:

$$\begin{aligned} (RP1) \quad & \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} \max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) dt^{\kappa} \\ & \text{subject to} \\ & u_1(b - 2)(b + 2) \leq 0, \quad \forall u_1 \in \mathcal{U}_1, t = (t^1, t^2) \in \Theta \\ & \frac{\partial b}{\partial t^1} = 3 - c + v_1, \quad \forall v_1 \in \mathcal{V}_1, t = (t^1, t^2) \in \Theta \\ & \frac{\partial b}{\partial t^2} = 3 - c + v_2, \quad \forall v_2 \in \mathcal{V}_2, t = (t^1, t^2) \in \Theta \\ & b(0, 0) = 1, \quad b(3, 3) = 2. \end{aligned}$$

Clearly, the set of all feasible solutions in (RP1) is

$$\begin{aligned} \mathcal{D} = \{ & (b, c) \in \mathcal{S} \times \mathcal{C} : -2 \leq b \leq 2, \frac{\partial b}{\partial t^1} = \frac{\partial b}{\partial t^2}, b(0, 0) = 1, b(3, 3) = 2, \\ & t \in \Theta, u_1 \in \mathcal{U}_1, v_{\zeta} \in \mathcal{V}_{\zeta}, \zeta = 1, 2\}. \end{aligned}$$

Now, we are interested in finding a weak robust optimal solution to the problem (P1). This means that we must find the control function $\bar{c} : \Theta \rightarrow \mathbb{R}$ (that determines the state function $\bar{b} : \Theta \rightarrow \mathbb{R}$), which satisfies the dynamical system (11), (12) and (13) with respect to the boundary conditions (14). Additionally, we assume that the state function is affine.

Let $(\bar{b}, \bar{c}) \in \mathcal{D}$ be a weak robust optimal solution to the problem (P1) and consider $\max_{w \in \mathcal{W}} f_{\kappa}(\pi, w) = f_{\kappa}(\pi, \bar{w}), \kappa = \overline{1, 2}$. Then, according to Theorem 1, there exists the scalar $\bar{\mu} \in \mathbb{R}$, the piecewise smooth functions $\bar{v} = \bar{v}_1(t) \in \mathbb{R}_+, \bar{\gamma} = (\bar{\gamma}_1(t), \bar{\gamma}_2(t)) \in \mathbb{R}^2$, and the uncertainty parameters $\bar{u}_1 \in \mathcal{U}_1$ and $\bar{v}_{\zeta} \in \mathcal{V}_{\zeta}, \zeta = 1, 2$, such that the following conditions

$$2\bar{v}_1\bar{u}_1\bar{b} + \frac{\partial \bar{\gamma}_1}{\partial t^1} + \frac{\partial \bar{\gamma}_2}{\partial t^2} = 0, \tag{15}$$

$$2\bar{\mu}\bar{c} - \bar{\gamma}_1 - \bar{\gamma}_2 = 0, \tag{16}$$

$$\bar{v}_1\bar{u}_1(\bar{b}^2 - 4) = 0, \quad \bar{v}_1 \geq 0, \quad \bar{\mu} \geq 0 \tag{17}$$

hold for all $t \in \Theta$, except at discontinuities.

One can easily verify that the robust necessary optimality conditions (15)–(17) are satisfied at $(\bar{b}, \bar{c}) = \left(\frac{1}{6}(t^1 + t^2) + 1, \frac{29}{6}\right)$, with the Lagrange multipliers $\bar{\mu} = 1, \bar{v}_1 = 0, \bar{\gamma}_1 + \bar{\gamma}_2 = d_1 + d_2$ (with $d_1 + d_2 = \bar{\mu}\left(\frac{17}{3} + \bar{v}_1 + \bar{v}_2\right)$) and the uncertain parameters $\bar{w}_1 = w_2 = \bar{u}_1 = 1, \bar{v}_1 = \bar{v}_2 = 2 \in [1, 2]$. Further, it can also be easily verified that the objective functional $\int_{\Gamma} f_{\kappa}(\pi, \bar{w}) dt^{\kappa}$ is convex at (\bar{b}, \bar{c}) and that $((\bar{b}, \bar{c}), \bar{v}, \bar{\gamma}, \bar{w}, \bar{u}, \bar{v})$ is a robust saddle-point for the Lagrange functional $L((b, c), v, \gamma, w, u, v)$ associated with the modified multi-dimensional variational control problem

$$(P1)_{(\bar{b}, \bar{c})} \quad \min_{(b(\cdot), c(\cdot))} \int_{\Gamma} \left(\frac{29}{3} + w_1\right) \left(c - \frac{29}{6}\right) dt^1 + \left(\frac{29}{3} + w_2\right) \left(c - \frac{29}{6}\right) dt^2$$

subject to

$$u_1(b - 2)(b + 2) \leq 0, \quad t = (t^1, t^2) \in \Theta$$

$$\frac{\partial b}{\partial t^1} = 3 - c + v_1, \quad t = (t^1, t^2) \in \Theta$$

$$\frac{\partial b}{\partial t^2} = 3 - c + v_2, \quad t = (t^1, t^2) \in \Theta$$

$$b(0,0) = 1, \quad b(3,3) = 2.$$

Hence, all the conditions of Theorem 2 are satisfied, which ensures that $(\bar{b}, \bar{c}) = \left(\frac{1}{6}(t^1 + t^2) + 1, \frac{29}{6}\right)$ is a weak robust optimal solution to the problem (P1).

4. Conclusions and Further Development

In this paper, by considering path-independent curvilinear integral cost functionals with mixed (equality and inequality) constraints implying data uncertainty and second-order partial derivatives, we have introduced new classes of robust optimization problems. More precisely, by using the notion of convexity for curvilinear integral functionals, the concept of a normal weak robust optimal solution and the robust saddle-point of a considered Lagrange functional, we have established some characterization results of the problems under study.

As an immediate subsequent development of the results presented in this paper, the author mentions the study of well-posedness for the considered classes of robust control problems.

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