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Article

A Remark on the Change of Variable Theorem for the Riemann Integral

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Abstract: In 1961, Kestelman first proved the change in the variable theorem for the Riemann integral in its modern form. In 1970, Preiss and Uher supplemented his result with the inverse statement. Later, in a number of papers (Sarkhel, Výborný, Puoso, Tandra, and Torchinsky), the alternative proofs of these theorems were given within the same formulations. In this note, we show that one of the restrictions (namely, the boundedness of the function f on its entire domain) can be omitted while the change of variable formula still holds.

Keywords: real analysis; Riemann integral; change of variable

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1. Introduction

Throughout this paper, we denote $[a,b]$ as the closed interval connecting the points $a,b \in \mathbb{R}$, and denote $R[a,b]$ as the class of all Riemann-integrable real functions on $[a,b]$. In 1961, Kestelman (see [1]) first proved the following fundamental theorem for the Riemann integral.

Theorem 1. *Suppose that $g \in R[\alpha,\beta]$, $c \in \mathbb{R}$,*

$$G(t) := \int_{\alpha}^{t} g(y)dy + c \tag{1}$$

and $f \in R\big(G([\alpha,\beta])\big)$. Then, $(f \circ G)g \in R[\alpha,\beta]$ and the following ***change of variable formula*** *holds:*

$$\int_{G(\alpha)}^{G(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(G(t)\big)g(t)dt \tag{2}$$

In 1970, Preiss and Uher (see [2]) supplemented this result with the following statement.

Theorem 2. *Suppose that $g \in R[\alpha,\beta]$, G is defined by (1), f is bounded on $[c,d] := G([\alpha,\beta])$ and $(f \circ G)g \in R[\alpha,\beta]$. Then $f \in R[c,d] \subset R[G(\alpha),G(\beta)]$ and the change of variable Formula (2) holds.*

Later, in a number of papers (see [3–6]), the alternative Proofs of Theorems 1 and 2 were given within the same formulations. The main goal of this note is to abandon the requirement of boundedness of the function f on $[c,d] := G([\alpha,\beta])$ in Theorem 2. At the same time, the condition for the boundedness of the function f on $[G(\alpha), G(\beta)]$ is essential for the existence of the integral on the left-hand side of (2) and does not follow from other conditions of the theorem, which are shown by the example at the end of [3]. Let us now proceed to formulating the main result.

2. The Main Result

Theorem 3. *Suppose that $g \in R[\alpha, \beta]$, G is defined by (1), f is bounded on $I := [G(\alpha), G(\beta)]$ and $(f \circ G)g \in R[\alpha, \beta]$. Then, $f \in R(I)$ and the change of variable Formula (2) holds.*

For the proof of Theorem 3, we need the following lemma.

Lemma 1. *If $g, gh \in R[\alpha, \beta]$, then $g|h| \in R[\alpha, \beta]$.*

Proof. By Lebesgue's criterion, the functions g and gh are both continuous a.e. on $[\alpha, \beta]$. Let $x_0 \in [\alpha, \beta]$ be the point of their mutual continuity. If h is continuous at x_0, then $g|h|$ is continuous at x_0. If h is discontinuous at x_0, then the equality $g(x_0) = 0$ must hold because otherwise, h must be continuous at x_0 as a quotient of continuous functions gh and g. Then, we have the following:

$$g(x)h(x) \to g(x_0)h(x_0) = 0,$$

and therefore,

$$g(x)|h(x)| = g(x)h(x)\operatorname{sgn}\big(h(x)\big) \to 0 = g(x_0)|h(x_0)|$$

as $x \to x_0$, which means the continuity of $g|h|$ at x_0, and thus, its continuity a.e. on $[\alpha, \beta]$. Thus, $g|h| \in R[\alpha, \beta]$ by Lebesgue's criterion. □

Proof of Theorem 3. By the hypothesis of the theorem, there is $M_1 > 0$ such that $|f(x)| \leq M_1$ for all $x \in I$. For all $n \in \mathbb{N}$, let $c_n := M_1 + n$ and define for all $x \in [c, d] := G([\alpha, \beta])$ the following function:

$$f_n(x) := \begin{cases} f(x), \text{ if } |f(x)| \leq c_n; \\ c_n, \text{ if } f(x) > c_n; \\ -c_n, \text{ if } f(x) < -c_n. \end{cases}$$

From the given definition for all $n \in \mathbb{N}$, we obtain the boundedness of f_n as well as the following equality:

$$f_n\big|_I = f\big|_I. \tag{3}$$

Additionally, for every $n \in \mathbb{N}$ for all $x \in [c, d]$, we obtain the following:

$$|f_n(x)| \leq |f(x)|, \tag{4}$$

and for all $x \in [c, d]$, we have the following:

$$f_n(x) \to f(x) \tag{5}$$

as $n \to \infty$. Next, we show that $(f_n \circ G)g \in R[\alpha, \beta]$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have the following explicit formula:

$$f_n = \min\{\max\{f, -c_n\}, c_n\} = \frac{1}{4}\big(f - c_n - |f - c_n| + \big|3c_n + f - |f - c_n|\big|\big),$$

from which, for $h := f \circ G$, we obtain the following equality:

$$(f_n \circ G)g = \frac{1}{4}\big(h - c_n - |h - c_n| + \big|3c_n + h - |h - c_n|\big|\big)g. \tag{6}$$

Since by the hypothesis of the theorem $g, gh \in R[\alpha, \beta]$, then by Lemma 1, we have $g|h - c_n| \in R[\alpha, \beta]$, and thus, $g\big|3c_n + h - |h - c_n|\big| \in R[\alpha, \beta]$ by the same lemma. Finally, (6) implies that $(f_n \circ G)g \in R[\alpha, \beta]$ for all $n \in \mathbb{N}$.

Since the function $(f \circ G)g$ is integrable (and, thus, bounded), there exists $M_2 > 0$ such that for all $n \in N$, $t \in [\alpha, \beta]$ holds the inequality as follows:

$$\left| f_n\big(G(t)\big)g(t) \right| \overset{(4)}{\le} \left| f\big(G(t)\big)g(t) \right| \le M_2,$$

Additionally, for all $t \in [\alpha, \beta]$ as $n \to \infty$, we have the following:

$$f_n\big(G(t)\big)g(t) \overset{(5)}{\to} f\big(G(t)\big)g(t).$$

By virtue of (3), using Theorem 2 and Arzela's bounded convergence theorem for the Riemann integral (see [7]), as $n \to \infty$ we obtain the following:

$$\int_{G(\alpha)}^{G(\beta)} f(x)dx \overset{(3)}{=} \int_{G(\alpha)}^{G(\beta)} f_n(x)dx \overset{\text{Th. } 2}{=} \int_{\alpha}^{\beta} f_n\big(G(t)\big)g(t)dt \to \int_{\alpha}^{\beta} f\big(G(t)\big)g(t)dt,$$

which completes the verification of (2) and the proof of the theorem. □

3. Some applications

The following example illustrates Theorem 3 in use: let $\alpha := -1$, $\beta := 2$, $g(t) := 2t$, $G(t) := t^2$ and

$$f(x) := \begin{cases} \dfrac{1}{\sqrt{x}} \text{ if } x > 0; \\ 0 \text{ if } x = 0. \end{cases}$$

Clearly, f is unbounded on $G([-1, 2]) = [0, 4]$, but there exists

$$\int_1^4 \frac{dx}{\sqrt{x}} = \int_{G(\alpha)}^{G(\beta)} f(x)dx \overset{\text{Th. } 3}{=} \int_{\alpha}^{\beta} f\big(G(t)\big)g(t)dt = \int_{-1}^{2} 2\,\text{sgn}(t)dt = 2.$$

To illustrate some other applications of our result, we obtain as a consequence the theorem on the change of a variable in an improper integral (in one direction) under quite general conditions.

Corollary 1 (of Theorem 3)**.** *Suppose that $a < b$, $\alpha < \beta$, f is bounded on $[a, c]$ for all $c \in (a, b)$, $g \in R[\alpha, \gamma]$ for all $\gamma \in (\alpha, \beta)$,*

$$G(t) := \int_{\alpha}^{t} g(y)dy + a \xrightarrow{t \to \beta-} b-$$

and

$$\lim_{z \to \beta-} \int_{\alpha}^{z} f\big(G(t)\big)g(t)dt = I.$$

Then, the following holds:

$$\lim_{x \to b-} \int_{a}^{x} f(s)ds = I.$$

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