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The Reducibility Concept in General Hyperrings

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Abstract: By using three equivalence relations, we characterize the behaviour of the elements in a hypercompositional structure. With respect to a hyperoperation, some elements play specific roles: their hypercomposition with all the elements of the carrier set gives the same result; they belong to the same hypercomposition of elements; or they have both properties, being essentially indistinguishable. These equivalences were first defined for hypergroups, and here we extend and study them for general hyperrings—that is, structures endowed with two hyperoperations. We first present their general properties, we define the concept of reducibility, and then we focus on particular classes of hyperrings: the hyperrings of formal series, the hyperrings with P-hyperoperations, complete hyperrings, and (H,R)-hyperrings. Our main aim is to find conditions under which these hyperrings are reduced or not.

Keywords: general hyperring; reducibility; fundamental relation; equivalence

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1. Introduction

Algebraic hypercompositional structures, i.e., structures where the result of the synthesis of two elements is a subset of the carrier set, are natural generalizations of the classical algebraic structures, and thus many properties of groups, rings, fields, modules, vector spaces, etc., are extended to hypergroups, hyperrings, hyperfields, hypermodules, vector hyperspaces, etc., more or less in a canonical way. The powerful Hypercompositional Algebra, i.e., the theory of algebraic hypercompositional structures, is given by concepts that do not exist in classical Algebra, and *reducibility* is one of them.

In 1990, James Jantosciak had the idea to describe the behaviour of the elements of a hypergroup with respect to the hyperoperation by defining three equivalence relations, that emphasize the interchangeable role of the elements with respect to the hyperoperation. If two elements in a hypergroup always belong to the same hyperproducts and their hypercomposition with all the elements of the carrier set is the same, then they are called *essentially indistinguishable* [1]. A hypergroup is *reduced* if the equivalence class of each element is a singleton with respect to the essentially indistinguishable relation.

In addition, Jantosciak noticed also that factorizing the hypergroup by this equivalence one obtains a reduced hypergroup, called the *reduced form* of the initial hypergroup. Therefore, he proposed to divide into two parts the study of the hypergroups: the study of the reduced hypergroups and the study of the hypergroups having the same reduced form [1]. Due to this important property, he named as *fundamental* the three equivalences used in the definition of the concept of reducibility.

Inspired by this pioneer paper and the further results obtained by researchers on the reducibility of various types of hypergroups [2–5], we extend here this property to hyperrings. These are algebraic structures containing an additive and a multiplicative part connected by the distributivity law, where at least one of them is a hypercompositional structure. The first type of hyperring was introduced by Krasner [6] as a hypercompositional structure whose additive part is a canonical hypergroup, and the multiplicative one is a semigroup.

Currently, this structure is known as *Krasner hyperring* and considered as an additive hyperring, in order to emphasize that the addition is a hyperoperation. If one considers the multiplication to be a hyperoperation, while the addition stays an operation, the notion of *multiplicative hyperring* was introduced in 1982 by Rota [7], where the additive part is an abelian group and the multiplicative one is a semihypergroup. If both the addition and the multiplication are hyperoperations, then we talk about *general hyperrings*.

There are several types of general hyperrings: one studied by Corsini [8] in 1975 in connection with feebly hypermodules; one defined in 1973 by Mittas [9,10] and called superring, having as additive part a canonical hypergroup; another one studied in 1989 by Spartalis [11], where the additive part is a hypergroup and the multiplicative one is a semihypergroup. Expository and survey articles on this topic have been published by Nakassis [12] in 1988 and recently by Massouros [13,14].

The aim of this manuscript is to define and study the concept of reducibility in the class of hyperrings. We will do this in a very natural way, by extending the three fundamental relations defined by J. Jantosciak to both addition and multiplication. It is clear that it makes sense to do this only in a general hyperring, where the carrier set is endowed with two hyperoperations, because these fundamental equivalences are equivalent with the equality relation when they are considered with respect to an operation.

Thus, the study of the reducibility in a Krasner hyperring or in a multiplicative hyperring is not relevant since it reduces to the study of the reducibility of a hypergroup. This study, covered in Section 4, was conducted first in a general way and then for particular classes of general hyperrings, as the hyperring of formal series, or hyperrings with P-hyperoperations. Particular attention is given to the complete hyperrings and (H,R)-hyperrings. The paper ends with some conclusive remarks and ideas for future work.

2. Preliminaries on Hypergroups and Hyperrings

For a non-empty set H, we denote, by $\mathcal{P}^*(H)$, the family of all non-empty subsets of H. A binary hyperoperation, also called a hyperproduct, is an application $\circ : H\times H \to \mathcal{P}^*(H)$ and the pair $(H,\circ)$ is called a hypergrupoid. It is important to stress that, in a hypergrupoid, the hyperproduct $x\circ y$ between two arbitrary elements x and y in H is a non-empty subset of H. This is a property that we cannot find in classical algebraic structures, such as groupoids and semigroups.

The hyperoperation is extended to non-empty subsets of H as $A\circ B = \bigcup\limits_{a\in A, b\in B} a\circ b$. If the hyperoperation is associative, then the hypercompositional structure $(H,\circ)$ is a semihypergroup, which becomes a hypergroup when the reproducibility property also holds: $x\circ H = H\circ x = H$ for all $x\in H$.

The link between groups and hypergroups is established by the fundamental relation β defined on a semihypergroup $(H,\circ)$ as follows: $\beta = \cup_{n\geq 1}\beta_n$ where β_1 is the diagonal relation on H and for any $n>1$, and $x,y\in H$, $x\beta_n y \Leftrightarrow \exists a_1,a_2,\ldots,a_n\in H$ such that $\{x,y\}\subseteq \prod\limits_{i=1}^{n} a_i = a_1\circ a_2\circ\cdots\circ a_n$. It is clear that β is a reflexive and symmetrical relation, but generally not transitive. That is why we take its transitive closure β^*, which is an equivalence relation. Recall that, for hypergroups, we have $\beta=\beta^*$ [15,16], and the quotient $(H/\beta^*,\otimes)$ is a group with the operation $\beta^*(x)\otimes\beta^*(y) = \beta^*(z)$ for all $x,y\in H$ and $z\in x\circ y$.

Considering now the canonical projection $\varphi_H : H\to H/\beta^*$, which is a good homomorphism, i.e., $\varphi_H(x\circ y) = \varphi_H(x)\otimes\varphi_H(y)$, we may define the heart (or core) of a hypergroup H as the set $\omega_H = \{x\in H \mid \varphi_H(x) = 1\}$, where 1 is the identity of the group H/β^*. This set plays an important role for the structure of a hypergroup, because, if we know it, then we can determine the complete closure of a subset of H.

More exactly, if A is a non-empty subset of H, it is called a *complete part* [17] of H if for any natural number n and any elements $a_1,a_2,\ldots,a_n$ in H, the following implication holds: $A \cap \prod_{i=1}^{n} a_i \neq \emptyset \Rightarrow \prod_{i=1}^{n} a_i \subseteq A$. The intersection of all complete parts of H containing the subset A is called the *complete closure* of A in H, and it is denoted by $C(A)$. Moreover, $C(A) = \omega_H \circ A = A \circ \omega_H$. The complete closure of a set helps us to define a particular type of hypergroups, called *complete hypergroups*.

We say that a hypergroup $(H, \circ)$ is complete if $x \circ y = C(x \circ y)$ for all $x, y \in H$. Moreover, if $(H, \circ)$ is a complete hypergroup, then $x \circ y = C(\{a\}) = \beta(a)$ for every $x, y \in H$ and $a \in x \circ y$. In practice, this definition is substituted with the representation theorem, which we recall here below.

Theorem 1 ([18]). *A hypergroup $(H, \circ)$ is complete if and only if it can be partitioned as $H = \bigcup_{g \in G} A_g$, where G and the subsets A_g of H satisfy the following conditions:*

(1) *$(G, \cdot)$ is a group.*
(2) *For all $g_1 \neq g_2 \in G$, there is $A_{g_1} \cap A_{g_2} = \emptyset$.*
(3) *If $(a, b) \in A_{g_1} \times A_{g_2}$, and then $a \circ b = A_{g_1 g_2}$.*

It is clear that any group is a complete hypergroup; however, this case is not interesting for our study. This is why we will consider only proper complete hypergroups, i.e., complete hypergroups that are not groups. The heart ω_H of a complete hypergroup $(H, \circ)$ has an interesting property: it coincides with the set of identities of H. The complete hypergroups have been studied for their general properties [19], or in connection with their fuzzy grade [20], for their commutativity degree [21], or in relation with their size [22].

General hyperrings are algebraic structures equipped with two hyperoperations, i.e., hyperaddition and hypermultiplication that satisfy the distributivity condition. Here, we will recall the definitions of some particular types of general hyperrings, which will be considered further on in the paper.

Definition 1 ([23]). *A hypercompositional structure $(R, \oplus, \odot)$ is called a hyperringoid if*

1. *$(R, \oplus)$ is a hypergroup.*
2. *$(R, \odot)$ is a semigroup.*
3. *The operation "$\odot$" distributes on both sides over the hyperoperation "$\oplus$."*

This algebraic hypercompositional structure was first introduced by Massouros [24] in a study on languages and automata. If we request that both addition and multiplication are hyperoperations, then the hyperringoid becomes a general hyperring.

Definition 2 ([25]). *A triple $(R, \oplus, \odot)$ is a general hyperring if:*

1. *$(R, \oplus)$ is a hypergroup.*
2. *$(R, \odot)$ is a semihypergroup.*
3. *The multiplication is distributive with respect to the addition, i.e., for all $a, b, c \in R$ $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ and $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$.*

The H_v-structures were introduced by Vougiouklis during the 4th AHA Congress in 1990 [26] as hypercompositional structures with weak associative hyperoperations.

Definition 3. *The hyperstructure $(H, \cdot)$ is an H_v-semigroup if $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset$ for all $x, y, z \in H$. If also the eproducibility property is valid, i.e., $a \cdot H = H \cdot a = H$, then $(H, \cdot)$ is called an H_v-group.*

Definition 4. *A multi-valued system $(R, \oplus, \odot)$ is an H_v-ring if:*

1. *$(R, \oplus)$ is an H_v-group.*
2. *$(R, \odot)$ is an H_v-semigroup.*

3. *The multiplication weakly distributes with respect to the addition, i.e., for all* $a, b, c \in R$ $(a \odot (b \oplus c)) \cap ((a \odot b) \oplus (a \odot c)) \neq \emptyset$ *and* $((a \oplus b) \odot c) \cap ((a \odot c) \oplus (b \odot c)) \neq \emptyset$.

It is important to recall here one of the main properties of hypercompositional structures: the quotient of a group with respect to any of its subgroups is a hypergroup, while the quotient of a group by any equivalence relation gives birth to an H_v-group [14]. A recently published overview of the theory of weak-hyperstructures is covered in [26,27].

In the following, we will recall the construction of two types of hyperrings, which we will study in the next section. The first one leads to an H_v-ring obtained from a ring. This structure was principally studied by Spartalis and Vougiouklis [28,29], in connection with homomorphisms and numeration.

Let $(R, +, \cdot)$ be a ring and P_1 and P_2 be non-empty subsets of R. The hyperoperations defined by $xP_1^*y = x + y + P_1$ and $xP_2^*y = x \cdot y \cdot P_2$ for all $x, y \in R$ are called *P-hyperoperations* [30].

Theorem 2 ([29])**.** *Let* $(R, +, \cdot)$ *be a ring,* $Z(R)$ *be the center of the multiplicative semigroup* $(R, \cdot)$ *and* P_1, P_2 *be non-empty subsets of* R*. If* $0 \in P_1$ *and* $Z(R) \cap P_2 \neq \emptyset$*, then* (R, P_1^*, P_2^*) *is an* H_v*-ring.*

This kind of H_v-ring is called an *H_v-ring with P-hyperoperations*.

We end this section by recalling the construction of the hyperring of the formal series [31,32]. Based on this, we studied the structure of the set of polynomials over a hyperring.

Let $(R, +, \cdot)$ be a general commutative hyperring. A formal series with coefficients in R is an infinite sequence $(a_0, a_1, a_2, \ldots, a_n, \ldots)$ of elements a_i in R. The set of all such series is denoted by $R[[x]]$. We say that two series $(a_0, a_1, a_2, \ldots, a_n, \ldots)$ and $(b_0, b_1, b_2, \ldots, b_n, \ldots)$ are equal if and only if $a_i = b_i$ for all indices i.

Let define on $R[[x]]$ the addition by

$$(a_0, a_1, \ldots, a_n, \ldots) \oplus (b_0, b_1, \ldots, b_n, \ldots) = \{(c_0, c_1, \ldots, c_n, \ldots), c_k \in a_k + b_k\}$$

and the multiplication by

$$(a_0, a_1, \ldots, a_n, \ldots) \odot (b_o, b_1, \ldots, b_n, \ldots) = \{(c_0, c_1, \ldots, c_n, \ldots), c_k \in \sum_{i+j=k} a_i \cdot b_j\}.$$

The structure $(R[[x]], \oplus, \odot)$ is a general hyperring. We recall that the set of the polynomials $R[x]$ with coefficients in R is a superring with the same hyperoperations $\oplus$ and $\odot$ defined above [33]. This means that $(R[x], \oplus)$ is a canonical hypergroup, $(R[x], \odot)$ is a semihypergroup such that 0 is a bilaterally absorbing element and the multiplication is weakly distributive on the left side with respect to the addition, i.e., $f \odot (g \oplus h) \subseteq f \odot g \oplus f \odot h$, for $f, g, h \in R[x]$.

3. Short Review of the Reducibility in Hypergroups

In this section, we briefly recall the notion of the reducibility of hypergroups. We start with the three fundamental relations introduced by Jantosciak [1] on an arbitrary hypergroup.

Definition 5 ([1])**.** *Two elements* x, y *in a hypergroup* $(H, \circ)$ *are called:*

1. *operationally equivalent or by short o-equivalent, and we write* $x \sim_o y$*, if* $x \circ a = y \circ a$*, and* $a \circ x = a \circ y$*, for any* $a \in H$*;*
2. *inseparable or by short i-equivalent, and we write* $x \sim_i y$*, if, for all* $a, b \in H$*,* $x \in a \circ b \Longleftrightarrow y \in a \circ b$*; and*
3. *essentially indistinguishable or by short e-equivalent, and we write* $x \sim_e y$*, if they are operationally equivalent and inseparable.*

Definition 6 ([1])**.** *A hypergroup H is called reduced if, for any $x \in H$, the equivalence class of x with respect to the essentially indistinguishable relation $\sim_e$ a singleton.*

Proposition 1 ([5])**.** *A total hypergroup is not reduced.*

Theorem 3 ([5])**.** *Any proper complete hypergroup is not reduced.*

Proposition 2. *Let ϕ be a good surjective homomorphism from the hypergroup $(R,+)$ to the hypergroup $(T,\oplus)$. If two elements are essentially indistinguishable with respect to the hyperoperation $+$, then their images are essentially indistinguishable with respect to the hyperoperation $\oplus$.*

Proof. Let x and y be elements from R such that $x+a = y+a$, where $a \in R$. This gives $\{\phi(l) | l \in x+a\} = \{\phi(k) | k \in y+a\}$, and thus $\phi(x+a) = \phi(y+a)$. From here, $\phi(x) \oplus \phi(a) = \phi(y) \oplus \phi(a)$. Denote $\phi(a) = b$ and $\phi(x) = x_1, \phi(y) = y_1$. Thus, $x_1 \oplus b = y_1 \oplus b$. If the equality $x+a = y+a$ holds for every $a \in H$, then the last equality holds for all $b \in T$ since $\{\phi(a) | a \in R\} = T$. Assuming $a+x = a+y$ for all $a \in R$, similarly, we obtain $\phi(a) \oplus \phi(x) = \phi(a) \oplus \phi(y)$ for all $a \in R$. Hence, if $x \sim_o^+ y$ then $\phi(x) \sim_o^{\oplus} \phi(y)$.

Let $x \sim_i^+ y$, i.e., $x \in a+b$ if and only if $y \in a+b$ for all $a, b \in R$. From this equivalence, we find that $\phi(x) \in \{\phi(l) | l \in a+b\}$ if and only if $\phi(y) \in \{\phi(k) | k \in a+b\}$, and thus $\phi(x) \in \phi(a+b)$ if and only if $\phi(y) \in \phi(a+b)$. Since ϕ is homomorphism, $\phi(x) \in \phi(a) \oplus \phi(b)$ if and only if $\phi(y) \in \phi(a) \oplus \phi(b)$. Let $\phi(x) = x_1, \phi(y) = y_1$ and $\phi(a) = a_1, \phi(b) = b_1$. Since the mapping is surjective $a_1 \oplus b_1$ covers whole set T. Hence, $x_1 \in a_1 \oplus b_1$ is equivalent to $y_1 \in a_1 \oplus b_1$, for all $a_1, b_1 \in T$. Here, $x \sim_i^+ y$ implies $\phi(x) \sim_i^{\oplus} \phi(y)$. The definition of the essential indistinguishability relation, together with the above implications, concludes the proof of our claim. □

4. Reducibility in Hyperrings

In a semigroup, the equivalences $\sim_o$ and $\sim_i$ coincide with the diagonal relation, i.e., $x \sim_o y \iff x \sim_i y \iff x = y$. Thus, in a Krasner hyperring or in a multiplicative hyperring (when the referential set is equipped with a hyperoperation and an operation), these two equivalences are not significant. Therefore, in this section, our first aim is to study relationships between these equivalences in a general hyperring $(R, \oplus, \odot)$, where addition and multiplication are both hyperoperations.

For any element $x \in R$, we denote, by $\hat{x}_r^{\oplus}$ and $\hat{x}_r^{\odot}$, the equivalence classes of x with respect to the hyperoperations $\oplus$ and $\odot$, respectively, where $r \in \{o, i, e\}$ denotes the type of the equivalence that we consider in Definition 7. In the following, by hyperring, we mean a general hyperring.

Definition 7. *We say that two elements x and y in a hyperring $(R, \oplus, \odot)$ are operationally equivalent, inseparable or essentially indistinguishable if they have the same property with respect to both hyperoperations, i.e.,*

1. *$x \sim_o y$ if $x \oplus a = y \oplus a$, $a \oplus x = a \oplus y$ and $a \odot x = a \odot y$, $x \odot a = y \odot a$, for all $a \in R$.*
2. *$x \sim_i y$ if $x \in a \oplus b \iff y \in a \oplus b$, for all $a, b \in R$ and $x \in c \odot d \iff y \in c \odot d$, for all $c, d \in R$.*
3. *$x \sim_e y$ if $x \sim_o y$ and $x \sim_i y$.*

Definition 8. *A hyperring R is called reduced if the equivalence class of each element $x \in R$ with respect to the essentially indistinguishable relation $\sim_e$ is a singleton, i.e., $\hat{x}_e = \{x\}$ for any $x \in R$.*

The equivalence class of any element x in R with respect to the essentially indistinguishability relation $\sim_e$ is obtained as $\hat{x}_e = \hat{x}_e^{\oplus} \cap \hat{x}_e^{\odot} = (\hat{x}_o^{\oplus} \cap \hat{x}_i^{\oplus}) \cap (\hat{x}_o^{\odot} \cap \hat{x}_i^{\odot})$. It is important to stress on the following property. If at least one of the hypergroupoids $(R, \oplus)$ or $(R, \odot)$ is reduced, then the hyperring $(R, \oplus, \odot)$ is reduced, too. Reciprocally, if $(R, \oplus, \odot)$ is reduced, then the hypergroupoids $(R, \oplus)$ and $(R, \odot)$ can be reduced or not, as one can see in the following examples.

Example 1. *Let $(R, \oplus, \odot)$ be a hyperring defined by the following Cayley tables:*

$\oplus$	e	a
e	R	R
a	R	R

$\odot$	e	a
e	e	R
a	R	a

Since $(R, \oplus)$ is a total hypergroup, based on Proposition 1, it is not reduced. Here, $\hat{a}_e^{\oplus} = \hat{e}_e^{\oplus} = \{e, a\}$. However, it is easy to check that the hypergroup $(R, \odot)$ is a reduced hypergroup, and $\hat{a}_e^{\odot} = \{a\}, \hat{e}_e^{\odot} = \{e\}$. All together, it gives that $\hat{e}_e = \{e\}$ and $\hat{a}_e = \{a\}$ which shows that $(R, \oplus, \odot)$ is a reduced hyperring.

Example 2. *Let the hyperring $(R, \oplus, \odot)$ be defined by the following Cayley tables:*

$\oplus$	x	y	z
x	x, y	x, y	R
y	x, y	x, y	R
z	R	R	R

$\odot$	x	y	z
x	R	R	R
y	R	y, z	y, z
z	R	y, z	y, z

It is elementary to check that the algebraic hyperstructure $(R, \oplus, \odot)$ is a general hyperring. Since the rows corresponding to x and y are equal in $(R, \oplus)$ and both x, y appear in the same hyperproducts $a \oplus b$, it follows that $x \sim_e^{\oplus} y$, which implies that $(R, \oplus)$ is not reduced. Similarly, $(R, \odot)$ is not a reduced hypergroup since $y \sim_e^{\odot} z$. But, $\hat{x}_e = \hat{x}_e^{\oplus} \cap \hat{x}_e^{\odot} = \{x, y\} \cap x = \{x\}$. Similarly, $\hat{y}_e = \{y\}$, and $\hat{z}_e = \{z\}$, which proves that $(R, \oplus, \odot)$ is a reduced hyperring.

4.1. Some Properties of the Reducibility in Hyperrings

In the following, subsections, we suppose that the ring $(R, +, \cdot)$ has no zero divisors.

First, we will present some relationships between the operationally equivalence (inseparability) with respect to the first hyperoperation of the hyperring and the operationally equivalence (inseparability) with respect to the second hyperoperation of the considered hyperring.

Proposition 3. *Let $(R, \oplus, \odot)$ be a general hyperring, where the hypergroup $(R, \oplus)$ contains a scalar identity. Then, the essentially indistinguishability with respect to the hyperoperation "$\oplus$" implies the essentially indistinguishability with respect to the hyperoperation "$\odot$", i.e., $x \sim_e^{\oplus} y \Rightarrow x \sim_e^{\odot} y$, for all $x, y \in R$.*

Proof. We denote by 0 the scalar identity in $(R, \oplus)$. Let x and y be two elements in R such that $x \sim_o^{\oplus} y$, i.e., $x \oplus a = y \oplus a$ and $a \oplus x = a \oplus y$, for all $a \in R$. This means that, for any $u \in R$ such that $u \in x \oplus a$, it holds $u \in y \oplus a$. Let u in $a \odot x$. Then, since $x = x \oplus 0$, it follows that $u \in a \odot (x \oplus 0)$. Now, using $x \oplus 0 = y \oplus 0$, we get $u \in a \odot (y \oplus 0) = a \odot y$. By symmetry, we can conclude that $a \odot x = a \odot y$, and $x \odot a = y \odot a$, for all $a \in R$. Hence, $x \sim_o^{\odot} y$.

Let us suppose that $x \in a \oplus b$ if and only if $y \in a \oplus b$, for any $a, b \in R$. Let c and d be elements in the hyperring such that $x \in c \odot d$. Thus, $x \in (c \oplus 0) \odot d$. Using the distributibivity, we obtain $x \in c \odot d \oplus 0 \odot d = \{m \oplus n | m \in c \odot d, n \in 0 \odot d\}$. Since x and y appear in the same hyperproducts $a \oplus b$, for any $a, b \in R$, it follows that y also belongs to the same hyperproduct, which gives $y \in c \odot d \oplus 0 \odot d$, i.e., $y \in c \odot d$. This proves the implication $x \sim_i^{\oplus} y \Rightarrow x \sim_i^{\odot} y$. Now the conclusion of the result is clear. □

Corollary 1. *Let $(R, \oplus, \odot)$ be a general hyperring such that $(R, \oplus)$ contains a scalar identity. If $(R, \oplus)$ is not a reduced hypergroup, then the hyperring $(R, \oplus, \odot)$ is not reduced, too.*

Proof. If $(R, \oplus)$ is not a reduced hypergroup, then there exist two distinct elements x and y in R such that $x \sim_e^{\oplus} y$. Based on Proposition 3, it follows that $x \sim_e^{\odot} y$, meaning that the hyperring $(R, \oplus, \odot)$ is not reduced. □

In the second part of this section, we present some particular types of general hyperrings and highlight some of their properties related to the reducibility. We start with some aspects regarding the reducibility of the hyperring of formal series.

Proposition 4. *Let $R[[x]]$ be the hyperring of the formal series with coefficients in the general commutative hyperring $(R,+,\cdot)$. The hyperring $(R,+,\cdot)$ is reduced if and only if the hyperring $(R[[x]],\oplus,\odot)$ is reduced.*

Proof. Let us suppose that the hyperring R is not reduced, i.e., there exist elements a and b such that $a+x=b+x$ and $x+a=x+b$ for all $x\in R$, and also a and b appear in the same hyperproducts $c+d$, where $c,d\in R$. As a direct consequence, the formal series $(a,a,\ldots,a,\ldots)$ and $(b,b,\ldots,b,\ldots)$ are operationally equivalent and inseparable with respect to the hyperoperation $\oplus$. Analogously, the implication holds also if we consider the multiplicative hyperoperation. Hence, if R is not reduced, then the hyperring $(R[[x]],\oplus,\odot)$ is not reduced, too.

Let us prove now that the reducibility in $(R,+,\cdot)$ implies the reducibility in $(R[[x]],\oplus,\odot)$. For that purpose, let us assume that the hyperring $R[[x]]$ is not reduced. Then, there exist two formal series $(a_1,a_2,\ldots,a_n,\ldots)$ and $(b_1,b_2,\ldots,b_n,\ldots)$, which are operationally equivalent with respect to the hyperoperation $\oplus$. This implies that:

$$(a_1,a_2,\ldots,a_n,\ldots)\oplus(x_1,x_2,\ldots,x_n,\ldots)= \tag{1}$$
$$(b_1,b_2,\ldots,b_n,\ldots)\oplus(x_1,x_2,\ldots,x_n,\ldots), \tag{2}$$

and

$$(x_1,x_2,\ldots,x_n,\ldots)\oplus(a_1,a_2,\ldots,a_n,\ldots)= \tag{3}$$
$$(x_1,x_2,\ldots,x_n,\ldots)\oplus(b_1,b_2,\ldots,b_n,\ldots), \tag{4}$$

for any formal series $(x_1,x_2,\ldots,x_n,\ldots)\in R[[x]]$. Using the definition of the hyperaddition in $(R[[x]],\oplus,\odot)$, the previous equalities give that $a_i+x_i=b_i+x_i$ and $x_i+a_i=x_i+b_i$ for any arbitrary $x_i\in R$. Hence, $a_i\sim_o^+ b_i$ for any elements $a_i,b_i\in R$, which are the coordinates of the considered formal series.

Assuming now that the series $(a_1,a_2,\ldots,a_n,\ldots)$ and $(b_1,b_2,\ldots,b_n,\ldots)$ are inseparable with respect to the hyperoperation $\oplus$, it easily follows that a_i and b_i appear in the same hyperproducts $c+d$, where $c,d\in R$, so they are inseparable with respect to the hyperproduct "$+$" on R. Similarly, we can prove that the essentially indistinguishability with respect to the hypermultiplication "$\odot$" implies essentially indistinguishability with respect to the hyperoperation "$\cdot$". We finally find that $(R,+,\cdot)$ is not reduced, which concludes the proof. □

The next part of this subsection is dedicated to the study of reducibility of the hyperrings with P-hyperoperations.

Proposition 5. *Let $(R,+,\cdot)$ be a commutative principal ideal domain with two units, i.e., 1 and -1. If $P_1=nR$, with $n\in R$, and $P_2=R$, then the structure (R,P_1^*,P_2^*) is a commutative H_v-ring with P-hyperoperations, which is a non-reduced hyperring.*

Proof. Any principal ideal contains 0; therefore, $0\in P_1$. As the ring R is commutative, it coincides with its center $Z(R)$, and therefore the set $P_2=R$ has a non-empty intersection with $Z(R)$, and thus the conditions of Theorem 2 are satisfied, proving that the hyperstructure (R,P_1^*,P_2^*) is a commutative H_v-ring.

Let x and y be distinct elements such that $xP_1^*a=yP_1^*a$ for all a in R, meaning that $x+a+P_1=y+a+P_1$, i.e., $x+a+nR=y+a+nR$, for the fixed element $n\in R$ and any $a\in R$. Since the principal ideal nR is a subgroup, then the equality holds whenever $x-y\in nR$. Therefore, the elements x and y are operationally equivalent with respect to the hyperoperation P_1^* if and only if $x-y\in nR$.

Let x and y be two elements such that $x - y \in nR$. Let us suppose that $x \in aP_1^*b$, where $a, b \in R$. The element x belongs to $a + b + nR$, i.e., $x = a + b + n \cdot s$, with $s \in R$. Since $x = y + n \cdot k$, with $k \in R$, it follows that $y + n \cdot k = a + b + n \cdot s$, meaning that $y \in a + b + nR$. Hence, $y \in aP_1^*b$. Similarly, we can prove the other implication. Thus, $x \sim_i^{P_1^*} y$. Conversely, if $x \sim_i^{P_1^*} y$, then it is clear that $x - y \in nR = P_1$. Hence, for any two distinct elements $x, y \in R$, $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$.

Now, suppose that x and y are operational equivalent with respect to the hyperoperation P_2^*. Thus $xP_2^*a = yP_2^*a$, i.e., $x \cdot a \cdot P_2 = y \cdot a \cdot P_2$, for any $a \in R$. Using the property that two principal ideals are equal when their generators are associated, we obtain that there exists a unit u such that $ya = uxa$, and similarly, there exists a unit v such that $xa = vya$. Both together imply that $ya = uvya$, with $uv = 1$. Since the ring R contains only two units, we have exactly two possibilities. If both units u and v are the multiplicative identity 1, then we obtain that $xa - ya = 0$, i.e., $(x - y)a = 0$, which implies that $x = y$. The second case is when $u = v = -1$ and we obtain $ya = -xa$, for any $a \in R$, thus $y = -x$.

Regarding the inseparability with respect to the hyperoperation P_2^*, we easily see that for any $x \in R$, there is $x \sim_i^{P_2^*} (-x)$ and, moreover, $x \sim_e^{P_2^*} (-x)$.

Based on these two results, it follows clearly that $x \sim_e (-x)$, for any $x \in P_1$ which says that the H_v-ring (R, P_1^*, P_2^*) is not reduced. □

Example 3. *An example of an H_v-ring with P-hyperoperations satisfying Proposition 5 can be obtained taking $R = \mathbb{Z}$, the ring of integers.*

In the following, we will construct other examples of H_v-rings with P-hyperoperations and study their reducibility.

Example 4. *Let $\mathbb{Z}$ be the ring of integers and set $P_1 = n\mathbb{Z}$ with $n \in \mathbb{Z}$ and $P_2 = \mathbb{Z}^+$, the set of positive integers. Then, the hyperstructure $(\mathbb{Z}, P_1^*, P_2^*)$ is a commutative H_v-ring with P-hyperoperations, which is reduced.*

It is easy to see that the conditions of the Theorem 2 are all fulfilled, which implies that the hyperstructure $(\mathbb{Z}, P_1^*, P_2^*)$ is an H_v-ring. Similarly, as in Example 3, we conclude that $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$, i.e., $x - y = ns$ for some $s \in \mathbb{Z}$.

Let us suppose that $xP_2^*a = yP_2^*a$, i.e., $x \cdot a \cdot \mathbb{Z}^+ = y \cdot a \cdot \mathbb{Z}^+$, for any $a \in \mathbb{Z}$. Choosing $a = 1$, it follows that $\{xk \mid k \in \mathbb{Z}^+\} = \{yk \mid k \in \mathbb{Z}^+\}$. The equality is satisfied only in the case when $x = y$. Thus, the H_v-ring $(\mathbb{Z}, P_1^*, P_2^*)$ is reduced.

Example 5. *Let (R, P_1^*, P_2^*) be a commutative H_v- ring with $P-$ hyperoperations such that $(R, \cdot)$ is a group and let P_1 be a subgroup of $(R, +)$ and $P_2 = R$. Then, the H_v-ring (R, P_1^*, P_2^*) is not reduced.*

It is easy to check that the hyperstructure (R, P_1^*, P_2^*) is an H_v-ring with P-hyperoperations. Let us prove its non-reducibility. Indeed, following the procedure explained in Proposition 5, we conclude that $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$. Hence, for any two distinct elements $x, y \in R$, such that $x - y \in P_1$, there is $\hat{x}_e^{P_1^*} = \hat{y}_e^{P_1^*} \supseteq \{x, y\}$. Taking $P_2 = R$ we easily get that $xP_2^*a = yP_2^*a$, for all $a \in R$, and if x belongs to aP_2^*b, obviously also y belongs to it. Therefore, for an arbitrary element x in R, there is $\hat{x}_e^{P_2^*} = R$.

Combining the two results, we get $x \sim_e y$, whenever $x - y \in P_1$, meaning that the considered H_v-ring is not reduced.

Example 6. *Let (R, P_1^*, P_2^*) be a commutative H_v-ring with P-hyperoperations, such that $(R, +, \cdot)$ is a field and let K be a subfield of R. If $P_1 = P_2 = K$, then the H_v- ring (R, P_1^*, P_2^*) is not reduced.*

Let x and y be arbitrary elements from R. Analogously to Example 5, $x \sim_e^{P_1^*} y$ if and only if $x - y \in P_1$.

Let us suppose that the equality $xP_2^*a = yP_2^*a$ is satisfied for all $a \in R$, i.e., $xaK = yaK$ for any $a \in R$. This is equivalent to $xK = yK$, which is satisfied for any $x, y \in K$.

Merging both conclusions, we get that the hyperring (R, P_1^*, P_2^*) is not reduced, since any two elements x and y in K are essentially indistinguishable.

We conclude this subsection with the study of the reducibility of the hyperrings constructed with Corsini hypergroups. Let us recall first the definition of such a hypergroup.

Definition 9 ([34]). *A hypergroup $(H, \circ)$ is called a Corsini hypergroup, if, for any two elements $x, y \in H$, the following conditions hold:*

1. $x \circ y = x \circ x \cup y \circ y$,
2. $x \in x \circ x$,
3. $y \in x \circ x$ *if and only if* $x \in y \circ y$,
4. *for any* $(a, c) \in H^2$, $c \circ c \circ c \setminus c \circ c \subseteq a \circ a \circ a$.

Proposition 6. *Let $(H, \circ)$ be a Corsini hypergroup and $(H, \star)$ be a B-hypergroup, i.e., $x \star y = \{x, y\}$ for all $x, y \in H$. Then, the hyperring $(H, \star, \circ)$ is a reduced hyperring.*

Proof. Based on Al-Tahan and Davvaz [35], it is known that, if $(H, \circ)$ is a Corsini hypergroup and $(H, \star)$ is a B-hypergroup, then the structure $(H, \star, \circ)$ is a commutative hyperring. Kankaraš has proved in [4] that any $B-$ hypergroup is a reduced hypergroup, which easily gives that the hyperring $(H, \star, \circ)$ is reduced, too. □

Example 7. *Endow the set $R = \{x, y, z\}$ with the hyperoperations $\oplus$ and $\odot$ given by the following tables:*

$\oplus$	x	y	z
x	x, y	x, y	R
y	x, y	x, y	R
z	R	R	z

$\odot$	x	y	z
x	x	x, y	x, z
y	x, y	y	y, z
z	x, z	y, z	z

The hypergroup $(R, \oplus)$ is a Corsini hypergroup [35] and $(R, \odot)$ is a B-hypergroup. Here, $x \oplus a = y \oplus a$ for any $a \in R$. Thus, $x \sim_o^{\oplus} y$. x and y appear in the same hyperproducts, which gives $x \sim_i^{\oplus} y$. Considering the second hyperoperation, it easily follows that $\hat{x}_e^{\odot} = \{x\}$ for any $x \in R$. Hence, $(R, \oplus, \odot)$ is a reduced hyperring.

Remark 1. *If we consider that $(R, \oplus)$ is the hypergroup defined in Example 7 and $(R, \odot)$ is the total hypergroup, then both hypergroups are Corsini hypergroups; hwoever, the hyperring $(R, \oplus, \odot)$ is not reduced since $\hat{x}_e = \hat{y}_e = \{x, y\}$.*

4.2. Reducibility in Complete Hyperrings

The definition of complete hyperrings is based on the definition of complete hypergroups.

Definition 10 ([36]). *Let $(H, \oplus, \odot)$ be a hyperring. If $(H, \oplus)$ is a complete hypergroup, then we say that H is $\oplus$-complete. If $(H, \odot)$ is a complete semihypergroup, then we say that H is $\odot$-complete and if both $(H, \oplus)$ and $(H, \odot)$ are complete, then we say that H is a complete hyperring.*

Following the construction of complete hypergroups, De Salvo [36] proposed a method to obtain complete hyperrings starting with rings. Let us recall here this construction.

Let $(R, +, \cdot)$ be a ring, and $\{A(g)\}_{g \in R}$ be a family of nonempty sets, such that:

1. $\forall g, g' \in R, g \neq g' \Rightarrow A(g) \cap A(g') = \emptyset$
2. $g \notin R \cdot R \Rightarrow |A(g)| = 1$.

Set $H_R = \bigcup_{g \in R} A(g)$ and define on H_R two hyperoperations $\oplus$ and $\odot$ as follows: for any $a, b \in H_R$, there exist $g, g' \in R$ such that $a \in A(g), b \in A(g')$ and define

$$a \oplus b = A(g + g'), a \odot b = A(gg').$$

Lemma 1 ([36]). *Using the previous notations, for all $g, g' \in R$ and any $a \in A(g), b \in A(g')$ we have:*
$a \oplus b = A(g + g') = A(g) \oplus A(g')$,
$a \odot b = A(gg') = A(g) \odot A(g')$.

In [37] Corsini proved that $(H_R, \oplus)$ and $(H_R, \odot)$ are, respectively, a complete commutative hypergroup and a complete semihypergroup.

Remark 2. *All complete hyperrings can be constructed by the above described procedure, since it is known that any complete semihypergroup (hypergroup) can be constructed as the union of disjoint sets $A(g), g \in G$ (see Theorem 1).*

Based on Theorem 3, any complete (semi)hypergroup is not reduced; however, this property does not imply directly the non-reducibility of any complete hyperring. That's why we need to study its reducibility in a different way, as shown in the next result.

Theorem 4. *Any complete hyperring $(H_R, \oplus, \odot)$ is not reduced.*

Proof. Let $(H_R, \oplus, \odot)$ be a complete hyperring. Therefore the hypergroup $(H_R, \oplus)$ and the semihypergroup $(H_R, \odot)$ are both complete, so both are not reduced. It follows that there exist $a \neq b \in H_R$ such that $a \sim_e^{\oplus} b$. Now it is enough to prove that $a \sim_e^{\oplus} b$ implies $a \sim_e^{\odot} b$ for $a, b \in H_R$, because in this case $\hat{a}_e = \hat{a}_e^{\oplus} \cap \hat{a}_e^{\odot} \supseteq \{a, b\}$, which shows that $(H_R, \oplus, \odot)$ is not reduced.

First, we will prove that the operational equivalence relation with respect to the hyperoperation $\oplus$ implies the operational equivalence relation with respect to $\odot$. Let a, b be elements from H_R such that $a \oplus c = b \oplus c$, for all $c \in H_R$. It follows that there exist $g_a, g_b, g_c \in R$ such that $a \in A(g_a), b \in A(g_b)$ and $c \in A(g_c)$. According to Lemma 1, we have $a \oplus c = A(g_a + g_c)$ and $b \oplus c = A(g_b + g_c)$, which leads to the equality $A(g_a + g_c) = A(g_b + g_c)$, and so $g_a + g_c = g_b + g_c$ in the group $(R, +)$. Therefore, $g_a = g_b$, that implies that $g_a \cdot g_c = g_b \cdot g_c$. Therefore, $a \odot c = A(g_a \cdot g_c) = A(g_b \cdot g_c) = A(g_b) \odot A(g_c) = b \odot c$. Similarly, $c \oplus a = c \oplus b$ implies that $c \odot a = c \odot b$. This means that $a \sim_o^{\oplus} b$ implies $a \sim_o^{\odot} b$ for all $a, b \in H_R$.

Next, we will show that the indistinguishability relation with respect to $\oplus$ implies the indistinguishability relation with respect to $\odot$.

Let us suppose $a \sim_i^{\oplus} b$. This means that a and b appear in the same hyperproducts $d \oplus e$, for $d, e \in H_R$. Thus $a \in A(g_d) \oplus A(g_e) \iff b \in A(g_d) \oplus A(g_e)$, with $g_d, g_e \in R$ such that $d \in A(g_d), e \in A(g_e)$. It follows that $a \in A(g_d + g_e) \iff b \in A(g_d + g_e)$, meaning that $a, b \in A(g)$, with $g \in R$. If we consider now $a \in k \odot l$, then $a \in A(g_k \cdot g_l)$, where $k \in A(g_k), l \in A(g_l)$. Since a and b are in the same A_g, it follows that $b \in A(g_k \cdot g_l) = k \odot l$, equivalently, $b \in k \odot l$. Similarly, if $b \in k \odot l$, then $a \in k \odot l$. Hence, $a \sim_i^{\odot} b$. $\square$

Example 8. *Let the hyperring $R = (\{a, b, c, d, e\}, \oplus, \odot)$ be defined as shown in the following tables:*

$\oplus$	a	b	c	d	e
a	a	b, c	b, c	d	e
b	b, c	d	d	e	a
c	b, c	d	d	e	a
d	d	e	e	a	b, c
e	e	a	a	b, c	d

$\odot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b, c	b, c	d	e
c	a	b, c	b, c	d	e
d	a	d	d	a	d
e	a	e	e	d	b, c

The hyperring $(R, \oplus, \odot)$ is a commutative complete hyperring [38]. Since the rows corresponding to the elements b and c are exactly the same in both tables, we conclude that $b \sim_o^{\oplus} c$ and $b \sim_o^{\odot} c$, which further gives $b \sim_o c$, i.e., $\hat{b}_o = \hat{c}_o \supseteq \{b, c\}$. Furthermore, we notice that $\hat{b}_o = \hat{c}_o = \{b, c\}$. In addition, the elements b and c appear together in the same hyperproducts in $(R, \oplus)$, as well as in $(R, \odot)$, whence $b \sim_i c$, and thus $\hat{b}_i = \hat{c}_i = \{b, c\}$. Hence, $\hat{b}_e = \hat{c}_e = \{b, c\}$, which implies that the given hyperring is not reduced.

Remark 3. *Since $(R, \cdot)$ is generally a semigroup, and not a group, it may happen that the operational equivalence relation with respect to the hyperoperation $\odot$ does not imply the operational equivalence relation with respect to the hyperoperation $\oplus$.*

4.3. Reducibility in (H,R)-Hyperrings

(H, R)-hyperrings were introduced by De Salvo in [36], when he generalized the construction of (H,G)-hypergroups described in [39]. In the following, we will present their construction.

Let $(H, \circ, \bullet)$ be a hyperring and $\{A_i\}_{i \in R}$ be a family of nonempty sets such that:

1. $(R, +, \cdot)$ is a ring.
2. $A_{0_R} = H$.
3. For any $i \neq j \in R$, $A_i \cap A_j = \emptyset$.

Set $K = \bigcup_{i \in R} A_i$ and define on K the following hyperoperations:

$$\text{for any} \quad x, y \in H, x \oplus y = x \circ y \tag{5}$$
$$\text{and} \quad x \odot y = H \tag{6}$$

For any $x \in A_i$ and $y \in A_j$ such that $A_i \times A_j \neq H \times H$, define

$$x \oplus y = A_k \quad \text{if} \quad i + j = k, \tag{7}$$
$$x \odot y = A_m \quad \text{if} \quad i \cdot j = m. \tag{8}$$

The structure $(K, \oplus, \odot)$ is a general hyperring, called an (H, R)-*hyperring*. Moreover, if ω is the heart of the hypergroup $(K, \oplus)$, then $\omega = H$ and $H \odot K = K \odot H = H$ [36].

In the following, we will better describe the operational equivalence and the inseparability in an (H, R)-hyperring.

Lemma 2. *Let $(K, \oplus, \odot)$ be an (H,R)-hyperring, where $K = \bigcup_{i \in R} A_i$, with $(R, +, \cdot)$ a ring and $(H, \circ, \bullet)$ a hyperring.*

1. *Two elements x and y in $A_{0_R} = H$ are operationally equivalent with respect to the hyperoperation $\oplus$ if and only if they are operationally equivalent with respect to the hyperoperation $\circ$ on H.*
2. *Two elements x and y in $K \setminus A_{0_R}$ are operationally equivalent with respect to the hyperoperation $\oplus$ if and only if they belong to the same subset $A_i \subset K$.*
3. *Two elements x and y in K are inseparable with respect to the hyperoperation $\oplus$ if and only if they belong to the same subset $A_i \subset K$.*

Proof. 1. Let x, y be in $A_{0_R} = H$ such that $x \oplus a = y \oplus a$, for all $a \in K$. If $a \in A_{i_a}$, with $i_a \neq 0_R$, then the equality always holds. If $a \in A_{0_R}$, then $x \oplus a = y \oplus a$ whenever $x \circ a = y \circ a$, and thus the result is proved.

2. Let x and y be in $K \setminus H$, such that $x \in A_{i_x}$ and $y \in A_{i_y}$, with $i_x, i_y \in R$ and consider $x \oplus a = y \oplus a$, for all $a \in K$. If $a \in A_{0_R}$, then $x \oplus a = A_{i_x}$ and $y \oplus a = A_{i_y}$; therefore x and y are operationally equivalent if and only if $i_x = i_y$. If $a \in K \setminus A_{0_R}$, for example $a \in A_{i_a}$, then $x \oplus a = y \oplus a$ is equivalent with $i_x + i_a = i_y + i_a$, meaning again $i_x = i_y$.

3. Let us consider $x \sim_i^{\oplus} y$, meaning that $x \in a \oplus b$ if and only if $y \in a \oplus b$. If $a, b \in A_{0_R}$, then $a \oplus b = a \circ b$, and therefore $x \sim_i^{\oplus} y$ whenever $x, y \in a \circ b \subset A_{0_R}$. If $a \in A_{i_a}$ and

$b \in A_{i_b}$ with $A_{i_a} \times A_{i_b} \neq H \times H$, then $a \oplus b = A_{i_a + i_b} = A_i$, and therefore $x \sim_i^{\oplus} y$ whenever $x, y \in A_i$, with $i \in R$. Combining the two cases, we find that x and y are inseparable if and only if they are in the same subset A_i. □

Lemma 3. *Let $(K, \oplus, \odot)$ be an (H,R)-hyperring, where $K = \bigcup_{i \in R} A_i$, with $(R, +, \cdot)$ an integral domain and $(H, \circ, \bullet)$ a hyperring. Two elements x and y in K are essentially indistinguishable with respect to the hyperoperation $\odot$ if and only if they belong to the same subset $A_i \subset K$.*

Proof. The proof is similar to the one of Lemma 2. The only difference here is in the case of the relation "$\sim_o$", where the condition regarding R to be an integral domain is fundamental. □

Proposition 7. *Let $(K, \oplus, \odot)$ be an (H,R)-hyperring, where $K = \bigcup_{i \in R} A_i$, with $(R, +, \cdot)$ an integral domain and $(H, \circ, \bullet)$ a hyperring. Then, the hyperring $(K, \oplus, \odot)$ is not reduced if and only if there exists $i \in R, i \neq 0_R$, with $|A_i| \geq 2$, or the hypergroup $(H, \circ)$ is not reduced.*

Proof. Let us suppose that the hyperring $(K, \oplus, \odot)$ is not reduced. Then, there exist two distinct elements x and y in K such that $x \sim_e y$, i.e., $x \sim_e^{\oplus} y$ and $x \sim_e^{\odot} y$. Based on Lemma 2 and and Lemma 3, if x and y belong to the same subset A_i, with $i \neq 0_R$, we conclude that $|A_i| \geq 2$. Otherwise, if all sets $A_i, i \neq 0_R$ are singletons, then $x, y \in A_{0_R} = H$, which implies that $x \sim_o^{\circ} y$ and $x \sim_i^{\circ} y$, i.e., the structure $(H, \circ)$ is not a reduced hypergroup.

Conversely, suppose there exists $i \in R \setminus \{0_R\}$ such that $|A_i| \geq 2$. Then, there exist two elements x and y in the set A_i, implying that $x \sim_e^{\oplus} y$ and $x \sim_e^{\odot} y$. In other words, $x \sim_e y$, meaning that the (H, R)-hyperring $(K, \oplus, \odot)$ is not reduced. Assuming that $(H, \circ)$ is not reduced, let x and y be two elements such that $x \sim_e^{\circ} y$. According with Lemma 2 and and Lemma 3, we further conclude that $x \sim_e^{\oplus} y$. Due to the definition of the hyperoperation $\odot$, for any $x, y \in H$, it easily follows that $x \sim_e^{\odot} y$. Hence, $x \sim_e y$, i.e., $(K, \oplus, \odot)$ is not a reduced hyperring. □

Corollary 2. *If $(H, \circ, \bullet)$ is a not reduced hyperring, then the (H, R)-hyperring $(K, \oplus, \odot)$ is not reduced, too.*

In the following, we will give an example of an (H,R)-hyperring and show its non-reducibility.

Example 9. *Let endow the set $R = \{0, a, b, c\}$ with the following operations*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

It easily follows that $(R, +, \cdot)$ is a ring. Furthermore, let $(H, \circ, \bullet)$ be a hyperring given by the tables

$\circ$	c	d
c	c	c, d
d	c, d	c, d

$\bullet$	c	d
c	c	c, d
d	c	c, d

The structure $(H, \circ, \bullet)$ is a general hyperring [36]. It is easy to check that $(H, \circ)$ is a reduced hypergroup and thus, the hyperring $(H, \circ, \bullet)$ is reduced, too.

We will endow the set $K = \{c, d, a_1, a_2, a_3, a_4, a_5, a_6\}$, where $A_0 = H, A_a = \{a_1, a_2\}, A_b = \{a_3, a_4, a_5\}, A_c = \{a_6\}$, with an (H,R)-hyperstructure, by defining the hyperaddition $x \oplus y = x \circ y$

if both x, y belong to H, otherwise, let $x \oplus y = A_k$, with $x \in A_i, y \in A_j$ and $k = i + j$. We define $x \odot y = H$, where $x, y \in H$ and $x \oplus y = A_k$ with $x \in A_i, y \in A_j$ and $k = i \cdot j$. Then, the structure $(K, \oplus, \odot)$ is an (H,R)-hyperring.

Let us prove that $a_1 \sim_e a_2$, i.e., $a_1 \oplus x = a_2 \oplus x$ for all $x \in K$. Indeed, if $x \in H, a_1 \oplus x = A_{a+0} = a_2 \oplus x$. If $x \in A_a, a_1 \oplus x = A_{a+a} = A_0 = a_2 \oplus x = A_0$. For $x \in A_b, a_1 \oplus x = A_{a+b} = A_c = a_2 \oplus x$. Finally, $a_1 \oplus x = a_2 \oplus x = A_{a+c} = A_b$ for $x \in A_c$. Due to the commutativity of the ring R, $x \oplus a_1 = x \oplus a_2$ for any $x \in K$. Similarly, $a_1 \odot x = a_2 \odot x$ and $x \odot a_1 = x \odot a_2$ for any $x \in K$. Thus, $a_1 \sim_o a_2$.

Since $x \oplus y \subseteq H$ if both $x, y \in H = A_0 = \{c, d\}$, we conclude that the elements a_1 and a_2 do not appear in such hyperproducts. All other hyperproducts $x \oplus y$ are equal to some sets A_k, where $k \in \{a, b, c\}$, with A_a, A_b and A_c being disjoint sets. Hence, a_1 and a_2 appear in the hyperproducts which are equal to A_a, so they always appear together. Analogously, a_1 and a_2 appear in the same hyperproducts $x \odot y$. Hence, $a_1 \sim_i a_2$.

Similarly, one proves that $\hat{a}_{3_e} = \hat{a}_{4_e} = \hat{a}_{5_e} = \{a_3, a_4, a_5\}$. Thereby we conclude that the (H,R)-hyperring $(K, \oplus, \odot)$ is not reduced.

5. Conclusions

In this paper, we defined and studied the reducibility of some particular types of general hyperrings, thus, extending the concept of reducibility in hypergroups. We presented some properties of the fundamental relations in general hyperrings, and we investigated the reducibility for complete and (H, R)-hyperrings, hyperrings of formal series, and hyperrings constructed with Corsini hypergroups. In a future work, our goal is to extend this study of reducibility to the fuzzy case, i.e., to define and investigate the fuzzy reducibility in hyperrings.

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