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Multi-Step Inertial Regularized Methods for Hierarchical Variational Inequality Problems Involving Generalized Lipschitzian Mappings

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Abstract: In this paper, we construct two multi-step inertial regularized methods for hierarchical inequality problems involving generalized Lipschitzian and hemicontinuous mappings in Hilbert spaces. Then we present two strong convergence theorems and some numerical experiments to show the effectiveness and feasibility of our new iterative methods.

Keywords: hierarchical variational inequality problem; multi-step inertial method; generalized Lipschitzian mapping; strong convergence

MSC: 47H09; 47H10; 47H04

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Citation: Jiang, B.; Wang, Y.; Yao, J.-C. Multi-Step Inertial Regularized Methods for Hierarchical Variational Inequality Problems Involving Generalized Lipschitzian Mappings. *Mathematics* **2021**, *9*, 2103. https://doi.org/10.3390/math9172103

Academic Editor: Christopher Goodrich

Received: 23 July 2021
Accepted: 27 August 2021
Published: 31 August 2021

1. Introduction

Throughout this paper, we let H be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$, respectively, C be a nonempty closed convex subset of H and $A: H \to H$ be a mapping. We recall the variational inequality problem (VIP) is to find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1}$$

We denote the solution set of VIP (1) by $\mathrm{VI}(C, A)$.

The variational inequality problem is one of the most important problems in nonlinear analysis. Now, this problem has been studied by many scholars.

In 2000, Tseng [1] introduced the following method:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \delta Ax_n), \\ x_{n+1} = y_n - \delta(Ay_n - Ax_n), \end{cases} \tag{2}$$

where A is monotone and L-Lipschitzian (see in Section 2, Definition 1), $\mathrm{VI}(C, A) \neq \emptyset$, $\delta \in \left(0, \frac{1}{L}\right)$. This algorithm has a weak convergence result.

In 2011, Censor et al. [2] proposed the subgradient extragradient method for solving VIP (1), as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \delta Ax_n), \\ G_n = \{w \in H : \langle x_n - \delta Ax_n - y_n, w - y_n\rangle \geq 0\}, \\ x_{n+1} = P_{G_n}(x_n - \delta Ay_n). \end{cases} \tag{3}$$

The conditions of A and δ are the same as those in Tseng's method. Then the sequence $\{x_n\}$ converges weakly to some point $z \in \mathrm{VI}(C, A)$ under some conditions.

In recent years, some scholars have paid attention to the following hierarchical variational inequality problems (HVIP):

$$\text{Find } \hat{x} \in \mathrm{VI}(C, A) \text{ such that } \langle F\hat{x}, x - \hat{x} \rangle \geq 0, \ \forall x \in \mathrm{VI}(C, A), \tag{4}$$

where $F : H \to H$ is a strongly monotone and Lipschitzian mapping.

In 2020, Hieu et al. [3,4] proposed regularized subgradient extragradient method (Algorithm 1 RSEGM) and regularized Tseng's extragradient method (Algorithm 2 RTEGM) for solving HVIP (4). Both of these two methods have strong convergence results.

On the other hand, in order to accelerate the convergence, the inertial methods have been studied extensively by many scholars [5–12]. One of the important results is the inertial Mann algorithm which is introduced by Maingé [5] in 2007:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \beta_n)w_n + \beta_n T w_n. \end{cases} \tag{5}$$

Under some conditions, the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2019, Dong et al. [7] proposed the multi-step inertial Krasnosel'skiǐ-Mann algorithm for finding a fixed point of a nonexpansive mapping, as follows:

$$\begin{cases} y_n = x_n + \sum_{k \in \mathcal{S}_n} \alpha_{k,n}(x_{n-k} - x_{n-k-1}), \\ w_n = x_n + \sum_{k \in \mathcal{S}_n} \beta_{k,n}(x_{n-k} - x_{n-k-1}), \\ x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T w_n. \end{cases} \tag{6}$$

This algorithm has a weak convergence result under certain conditions.

In this paper, motivated by the results of [3,4,7], we construct a multi-step inertial regularized subgradient extragradient method and a multi-step inertial regularized Tseng's extragradient method for solving HVIP (4) in a Hilbert space when F is a generalized Lipschitzian and hemicontinuous mapping (see in Section 2, Definitions 2 and 3). Then, we present two strong convergence theorems and give some numerical experiments to show the effectiveness and feasibility of our new iterative methods.

Algorithm 1 Regularized subgradient extragradient method (RSEGM)

Initialization: Given $\lambda_1 > 0$, $\mu \in (0, 1)$. Choose $x_0, x_1 \in H$ arbitrarily and a sequence $\{\beta_n\} \subset (0, +\infty)$ such that

$$\lim_{n \to +\infty} \beta_n = 0, \quad \sum_{n=1}^{+\infty} \beta_n = +\infty, \quad \text{and} \quad \lim_{n \to +\infty} \frac{\beta_{n+1} - \beta_n}{\beta_n^2} = 0.$$

Iterative step: Calculate x_{n+1} for $n \geq 1$ as follows:

Step 1. Compute

$$y_n = P_C[x_n - \lambda_n(Aw_n + \beta_n F x_n)].$$

Step 2. Compute

$$x_{n+1} = P_{G_n}[x_n - \lambda_n(Ay_n + \beta_n F x_n)],$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}\right\}, & \text{if } Ax_n \neq Ay_n, \\ \lambda_n, & \text{otherwise,} \end{cases}$$

where $G_n = \{z \in H : \langle x_n - \lambda_n(Ax_n + \beta_n F x_n) - y_n, z - y_n \rangle \leq 0\}$.

Set $n := n + 1$ and go to Step 1.

Algorithm 2 Regularized Tseng's extragradient method (RTEGM)

Initialization: Given $\lambda_1 > 0$, $\mu \in (0,1)$. Choose $x_0, x_1 \in H$ arbitrarily and a sequence $\{\beta_n\} \subset (0, +\infty)$ such that

$$\lim_{n\to+\infty} \beta_n = 0, \quad \sum_{n=1}^{+\infty} \beta_n = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\beta_{n+1} - \beta_n}{\beta_n^2} = 0.$$

Iterative step: Calculate x_{n+1} for $n \geq 1$ as follows:

Step 1. Compute

$$y_n = P_C[x_n - \lambda_n(Ax_n + \beta_n F x_n)].$$

Step 2. Compute

$$x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n),$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}\right\}, & \text{if } Ax_n \neq Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to Step 1.

2. Preliminaries

In this section, we present some necessary definitions and lemmas which are needed for our main results.

Definition 1 ([13]). *Let $A : H \to H$ be a mapping.*

(i) *A is ζ-strongly monotone ($\zeta > 0$) if*

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|x - y\|^2, \quad \forall x, y \in H.$$

(ii) *A is monotone if*

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

(iii) *A is L-Lipschitzian ($L > 0$) if*

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

Let $\{x_n\} \subset H$ be a sequence. We use $x_n \to z$ and $x_n \rightharpoonup z$ to indicate that $\{x_n\}$ converges strongly and weakly to z, respectively.

Definition 2 ([14]). *Let $A : H \to H$ be a mapping. A is said to be hemicontinuous if $\forall x \in H$, $\forall y \in H$, $\delta_n \to 0$ implies $A(x + \delta_n y) \rightharpoonup Ax$ as $n \to +\infty$.*

It is obvious that a continuous mapping must be hemicontinuous, but the converse is not true.

Lemma 1 ([14]). *If $A : H \to H$ be a hemicontinuous and strongly monotone mapping in VIP (1), then VIP (1) exists a unique solution.*

Lemma 2 ([15]). *If $A : C \to H$ be a monotone and hemicontinuous mapping. Then $\tilde{x} \in \mathrm{VI}(C, A)$ if and only if $\langle Ax, x - \tilde{x}\rangle \geq 0$, $\forall x \in C$.*

Definition 3 ([14]). *Let $A : H \to H$ be a mapping. A is said to be L-generalized Lipschitzian ($L > 0$) if*

$$\|Ax - Ay\| \leq L(\|x - y\| + 1), \quad \forall x, y \in H.$$

Remark 1. *We can easily see that a Lipschitzian mapping must be a generalized Lipschitzian mapping, but the converse is not true. A generalized Lipschitzian mapping may even not be hemicontinuous. For example, consider the sign function $f : \mathbb{R} \to \mathbb{R}$, defined by*

$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Then f is generalized Lipschitzian but not hemicontinuous [14]. A continuous generalized Lipschitzian mapping may not be Lipschitzian. For example, let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x - 1, & x < -1, \\ x - \sqrt{1 - (x+1)^2}, & -1 \leq x < 0, \\ x + \sqrt{1 - (x-1)^2}, & 0 \leq x \leq 1, \\ x + 1, & x > 1. \end{cases}$$

We can see that g is continuous and generalized Lipschitzian but not Lipschitzian. In the past, many scholars studied Lipschitzian mappings, but in this paper, we pay attention to generalized Lipschitzian. Therefore, the research in this paper is meaningful.

Recall the metric projection operator P_C from H onto C, defined as follows:

$$P_C x = \arg\min_{y \in C} \|x - y\|, \quad x \in H.$$

Lemma 3 ([16,17]). *Given $x \in H$ and $q \in C$, we have*

(i) $q = P_C x$ *if and only if*
$$\langle x - q, q - y \rangle \geq 0, \quad \forall y \in C;$$

(ii) P_C *is firmly nonexpansive, i.e.,*
$$\langle P_C y - P_C z, y - z \rangle \geq \|P_C y - P_C z\|^2, \quad \forall y, z \in H;$$

(iii) $\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C x\|^2, \quad \forall y \in C.$

The following lemma is important to prove the strong convergence.

Lemma 4 ([18,19]). *Let $\{\zeta_n\}$ be a sequence of nonnegative real numbers satisfying*

$$\zeta_{n+1} \leq (1 - \gamma_n)\zeta_n + \gamma_n \xi_n + \sigma_n, \quad \forall n \in \mathbb{N},$$

where $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\sigma_n\}$ satisfy the following conditions:

(i) $\{\gamma_n\} \subset (0, 1)$ *with* $\sum_{n=1}^{+\infty} \gamma_n = +\infty$;

(ii) $\limsup_{n \to +\infty} \xi_n \leq 0$;

(iii) $\sigma_n \geq 0$ *with* $\sum_{n=1}^{+\infty} \sigma_n < +\infty$.

Then $\lim_{n \to +\infty} \zeta_n = 0$.

Now, we focus on HVIP (4) when A and F satisfy the following conditions:
(CD1) A is monotone on C and L-Lipschitzian on H.
(CD2) F is η-strongly monotone, K-generalized Lipschitzian and hemicontinuous on H.
(CD3) $\mathrm{VI}(C, A)$ is nonempty.

From Lemma 1, we know that HVIP (4) exists the unique solution if the conditions (CD1)–(CD3) are satisfied. We denote this solution by $\hat{x}$. Consider the following variational inequality problem:

$$\text{Find } x^* \in C \text{ such that } \langle (A + \beta F)x^*, x - x^* \rangle \geq 0,\ \forall x \in C, \tag{7}$$

where $\beta > 0$, A and F satisfy conditions (CD1)-(CD3). It is obvious that $(A + \beta F) : H \to H$ is strongly monotone and hemicontinuous, so VIP (7) has the unique solution according to Lemma 1. We denote this solution by x_β.

We have the following lemmas.

Lemma 5. $\|x_\beta\| \leq \frac{1}{\eta}\|F\hat{x}\| + \|\hat{x}\|$.

Proof. For each $u \in \mathrm{VI}(C, A)$, since x_β is the solution of VIP (2.1), we have

$$\langle Ax_\beta + \beta Fx_\beta, u - x_\beta \rangle \geq 0$$

and

$$\langle Au, x_\beta - u \rangle \geq 0.$$

Adding the inequalities above, we obtain

$$\langle Ax_\beta - Au + \beta Fx_\beta, u - x_\beta \rangle \geq 0. \tag{8}$$

Since A is monotone on C, we get

$$\langle Au - Ax_\beta, u - x_\beta \rangle \geq 0. \tag{9}$$

Adding (8) and (9), we obtain

$$\langle \beta Fx_\beta, u - x_\beta \rangle \geq 0,$$

which means

$$\langle Fx_\beta, u - x_\beta \rangle \geq 0. \tag{10}$$

From the η-strongly monotonicity of F, we get

$$\langle Fu - Fx_\beta, u - x_\beta \rangle \geq \eta\|x_\beta - u\|^2. \tag{11}$$

Adding (10) and (11), we obtain

$$\langle Fu, u - x_\beta \rangle \geq \eta\|x_\beta - u\|^2. \tag{12}$$

Hence

$$\|x_\beta - u\|^2 \leq \frac{1}{\eta}\langle Fu, u - x_\beta \rangle \leq \frac{1}{\eta}\|Fu\|\|u - x_\beta\|,$$

which implies

$$\|x_\beta - u\| \leq \frac{1}{\eta}\|Fu\|. \tag{13}$$

So we obtain

$$\|x_\beta\| \leq \|x_\beta - u\| + \|u\| \leq \frac{1}{\eta}\|Fu\| + \|u\|. \tag{14}$$

Particularly, this is also true for $u = \hat{x}$. □

Lemma 6. *For all* $\alpha, \beta > 0$, $\|x_\alpha - x_\beta\| \leq \frac{|\beta - \alpha|}{\alpha}M$, *where* M *is a positive constant. More precisely,* $M = \frac{1}{\eta}\left[\left(1 + \frac{K}{\eta}\right)\|F\hat{x}\| + 2K\|\hat{x}\| + K\right]$.

Proof. Since $x_\alpha \in \mathrm{VI}(C, A+\alpha F)$ and $x_\beta \in \mathrm{VI}(C, A+\beta F)$, we have

$$\langle Ax_\alpha + \alpha F x_\alpha, x_\beta - x_\alpha \rangle \geq 0$$

and

$$\langle Ax_\beta + \beta F x_\beta, x_\alpha - x_\beta \rangle \geq 0.$$

Adding the two inequalities above, we obtain

$$\langle Ax_\alpha - Ax_\beta, x_\beta - x_\alpha \rangle + \alpha \langle F x_\alpha, x_\beta - x_\alpha \rangle + \beta \langle F x_\beta, x_\alpha - x_\beta \rangle \geq 0,$$

which, by the monotonicity of A, implies that

$$\alpha \langle F x_\alpha, x_\beta - x_\alpha \rangle + \beta \langle F x_\beta, x_\alpha - x_\beta \rangle \geq 0.$$

Hence

$$(\beta - \alpha) \langle F x_\beta, x_\alpha - x_\beta \rangle \geq \alpha \langle F x_\beta - F x_\alpha, x_\beta - x_\alpha \rangle. \tag{15}$$

From the η-strong monotonicity of F, we obtain

$$\langle F x_\beta - F x_\alpha, x_\beta - x_\alpha \rangle \geq \eta \|x_\alpha - x_\beta\|^2.$$

Substituting the last relation into (15), we have

$$\begin{aligned} & \|x_\alpha - x_\beta\|^2 \\ \leq\ & \frac{1}{\eta} \langle F x_\beta - F x_\alpha, x_\beta - x_\alpha \rangle \\ \leq\ & \frac{1}{\eta} \frac{\beta - \alpha}{\alpha} \langle F x_\beta, x_\alpha - x_\beta \rangle \\ \leq\ & \frac{1}{\eta} \frac{|\beta - \alpha|}{\alpha} \|F x_\beta\| \|x_\alpha - x_\beta\|, \end{aligned}$$

which means

$$\|x_\alpha - x_\beta\| \leq \frac{1}{\eta} \frac{|\beta - \alpha|}{\alpha} \|F x_\beta\|. \tag{16}$$

Since F is generalized Lipschitzian, by Lemma 5, we get

$$\begin{aligned} & \|F x_\beta\| \\ \leq\ & \|F x_\beta - F \hat{x}\| + \|F \hat{x}\| \\ \leq\ & K(\|x_\beta - \hat{x}\| + 1) + \|F \hat{x}\| \\ \leq\ & K\|x_\beta\| + K\|\hat{x}\| + K + \|F \hat{x}\| \\ \leq\ & K\left(\frac{1}{\eta}\|F\hat{x}\| + \|\hat{x}\|\right) + K\|\hat{x}\| + K + \|F\hat{x}\| \\ =\ & \left(1 + \frac{K}{\eta}\right)\|F\hat{x}\| + 2K\|\hat{x}\| + K. \end{aligned} \tag{17}$$

Substituting (17) into (16), we obtain

$$\begin{aligned} & \|x_\alpha - x_\beta\| \\ \leq\ & \frac{1}{\eta} \frac{|\beta - \alpha|}{\alpha} \left[\left(1 + \frac{K}{\eta}\right)\|F\hat{x}\| + 2K\|\hat{x}\| + K\right] \\ =\ & \frac{|\beta - \alpha|}{\alpha} M, \end{aligned} \tag{18}$$

where $M = \frac{1}{\eta}\left[\left(1 + \frac{K}{\eta}\right)\|F\hat{x}\| + 2K\|\hat{x}\| + K\right]$. □

Lemma 7. $\lim_{\beta\to 0^+} x_\beta = \hat{x}$.

Proof. From Lemma 5, we know that $\{x_\beta\}$ is bounded. So there exists a sequence $\{\beta_m\} \subset (0,\infty)$ such that $\beta_m \to 0$ and $x_{\beta_m} \rightharpoonup \bar{x}$ as $m \to +\infty$ by the reflexivity of H. Since x_β is the solution of VIP (7), we have

$$\langle Ax_\beta + \beta F x_\beta, x - x_\beta\rangle \geq 0, \quad \forall x \in C,$$

which, by the monotonicity of A, implies that

$$\langle Ax + \beta F x_\beta, x - x_\beta\rangle \geq 0, \quad \forall x \in C.$$

Replacing β with β_m in the last relation, we get

$$\langle Ax + \beta_m F x_{\beta_m}, x - x_{\beta_m}\rangle \geq 0, \quad \forall x \in C. \tag{19}$$

Since $\{x_{\beta_m}\}$ is bounded and F is generalized Lipschitzian, $\{Fx_{\beta_m}\}$ is also bounded. Taking limit in (19), we obtain

$$\langle Ax, x - \bar{x}\rangle \geq 0, \quad \forall x \in C. \tag{20}$$

It follows from Lemma 2 that $\bar{x} \in \mathrm{VI}(C, A)$. From (12), we deduce

$$\langle Fu, u - x_{\beta_m}\rangle \geq 0, \quad \forall u \in \mathrm{VI}(C, A). \tag{21}$$

Taking limit in (21), we obtain

$$\langle Fu, u - \bar{x}\rangle \geq 0, \quad \forall u \in \mathrm{VI}(C, A). \tag{22}$$

It means that $\bar{x}$ is a solution of HVIP (6). Since HVIP (6) has the unique solution $\hat{x}$, we conclude $\bar{x} = \hat{x}$. Thus, $x_\beta \rightharpoonup \hat{x}$ as $\beta \to 0^+$. Replacing u with $\hat{x}$ in (12), we get

$$\|x_\beta - \hat{x}\|^2 \leq \frac{1}{\eta}\langle F\hat{x}, \hat{x} - x_\beta\rangle. \tag{23}$$

It follows from the fact $x_\beta \rightharpoonup \hat{x}$ that $\lim_{\beta\to 0^+} x_\beta = \hat{x}$. □

3. Multi-Step Inertial RSEGM

In this section, we propose a new multi-step inertial method for solving HVIP (4) based on Algorithm 1 (RSEGM). Under certain conditions, it has a strong convergence result.

We need the following lemma to analyze the convergence of $\{x_n\}$ generated by Algorithm 3.

Lemma 8 ([20])**.** *The sequence $\{\lambda_n\}$ generated by Algorithm 3 is non-increasing and*

$$\lim_{n\to+\infty} \lambda_n = \lambda > 0.$$

More precisely, we have $\lambda \geq \min\{\lambda_1, \frac{\mu}{L}\} > 0$.

Theorem 1. *Under the conditions (CD1)-(CD3), the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to $\hat{x} \in \mathrm{VI}(C, A)$, where $\hat{x}$ is the unique solution of HVIP (4).*

Algorithm 3 Multi-step inertial RSEGM (MIRSEGM)

Initialization: Given $\lambda_1 > 0$, $\mu \in (0,1)$. Choose $x_0, x_1 \in H$ arbitrarily and a sequence $\{\beta_n\} \subset (0,+\infty)$ such that

$$\lim_{n\to+\infty} \beta_n = 0, \quad \sum_{n=1}^{+\infty} \beta_n = +\infty, \quad \sum_{n=1}^{+\infty} \beta_n^2 < +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{\beta_{n+1} - \beta_n}{\beta_n^2} = 0.$$

For each $i = 1, 2, \cdots, N$ (where N is a chosen positive integer), choose a sequence $\{\sigma_{i,n}\} \subset (0,+\infty)$ satisfying

$$\lim_{n\to+\infty} \frac{\sigma_{i,n}}{\beta_n} = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \sigma_{i,n} < +\infty.$$

Iterative step: Calculate x_{n+1} for $n \geq 1$ as follows:

Step 1. Compute

$$w_n = x_n + \sum_{i=1}^{\min\{N,n\}} \alpha_{i,n}(x_{n-i+1} - x_{n-i}),$$

where $0 \leq \alpha_{i,n} \leq \alpha_i$ for some $\alpha_i \in H$ with

$$\alpha_{i,n} = \begin{cases} \min\Big\{\alpha_i, \frac{\sigma_{i,n}}{\|x_{n-i+1} - x_{n-i}\|}\Big\}, & \text{if } x_{n-i+1} \neq x_{n-i}, \\ \alpha_i, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$y_n = P_C[w_n - \lambda_n(Aw_n + \beta_n F w_n)].$$

Step 3. Compute

$$x_{n+1} = P_{T_n}[w_n - \lambda_n(Ay_n + \beta_n F w_n)],$$

and

$$\lambda_{n+1} = \begin{cases} \min\Big\{\lambda_n, \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}\Big\}, & \text{if } Aw_n \neq Ay_n, \\ \lambda_n, & \text{otherwise,} \end{cases}$$

where $T_n = \{z \in H : \langle w_n - \lambda_n(Aw_n + \beta_n F w_n) - y_n, z - y_n\rangle \leq 0\}$.

Set $n := n + 1$ and go to Step 1.

Proof. From Lemma 2, we know that for each $n \in \mathbb{N}$, there exists the unique element $x_{\beta_n} \in C$ such that

$$\langle (A + \beta_n F)x_{\beta_n}, x - x_{\beta_n}\rangle \geq 0, \quad \forall x \in C.$$

From Lemma 7, we know that $x_{\beta_n} \to \hat{x}$, so it is only to be shown that $x_n - x_{\beta_n} \to 0$. According to Lemma 3, we have

$$\begin{aligned}
& \|x_{n+1} - x_{\beta_n}\|^2 \\
\le\ & \|w_n - \lambda_n(Ay_n + \beta_n F w_n) - x_{\beta_n}\|^2 - \|w_n - \lambda_n(Ay_n + \beta_n F w_n) - x_{n+1}\|^2 \\
=\ & \|(w_n - x_{\beta_n}) - \lambda_n(Ay_n + \beta_n F w_n)\|^2 - \|(w_n - x_{n+1}) - \lambda_n(Ay_n + \beta_n F w_n)\|^2 \\
=\ & \|w_n - x_{\beta_n}\|^2 - \|x_{n+1} - w_n\|^2 + 2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - x_{n+1}\rangle \\
=\ & \|w_n - x_{\beta_n}\|^2 - \|x_{n+1} - w_n\|^2 + 2\lambda_n\langle Aw_n + \beta_n F w_n, y_n - x_{n+1}\rangle \\
& +2\lambda_n\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle + 2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n\rangle \\
=\ & \|w_n - x_{\beta_n}\|^2 - \|x_{n+1} - w_n\|^2 + 2\langle w_n - y_n, y_n - x_{n+1}\rangle \\
& +2\langle w_n - \lambda_n(Aw_n + \beta_n F w_n) - y_n, x_{n+1} - y_n\rangle \\
& +2\lambda_n\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle + 2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n\rangle. \qquad (24)
\end{aligned}$$

Since $x_{n+1} \in T_n$, it follows from the definition of T_n that

$$\langle w_n - \lambda_n(Aw_n + \beta_n F w_n) - y_n, x_{n+1} - y_n\rangle \le 0. \qquad (25)$$

It is easy to see that

$$2\langle w_n - y_n, y_n - x_{n+1}\rangle = \|x_{n+1} - w_n\|^2 - \|w_n - y_n\|^2 - \|x_{n+1} - y_n\|^2. \qquad (26)$$

Substituting (25) and (26) into (24), we obtain

$$\begin{aligned}
& \|x_{n+1} - x_{\beta_n}\|^2 \\
\le\ & \|w_n - x_{\beta_n}\|^2 - \|w_n - y_n\|^2 - \|x_{n+1} - y_n\|^2 + 2\lambda_n\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle \\
& +2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n\rangle. \qquad (27)
\end{aligned}$$

From the computation of $\{\lambda_n\}$, we deduce

$$\begin{aligned}
& 2\lambda_n\langle Aw_n - Ay_n, x_{n+1} - y_n\rangle \\
\le\ & 2\lambda_n\|Aw_n - Ay_n\|\|x_{n+1} - y_n\| \\
\le\ & 2\mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|\|x_{n+1} - y_n\| \\
\le\ & \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2. \qquad (28)
\end{aligned}$$

Combining (27) and (28), we have

$$\begin{aligned}
& \|x_{n+1} - x_{\beta_n}\|^2 \\
\le\ & \|w_n - x_{\beta_n}\|^2 - \left(1 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 \\
& +2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n\rangle. \qquad (29)
\end{aligned}$$

According to the monotonicity of A, we get

$$\begin{aligned}
& 2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n\rangle \\
=\ & 2\lambda_n\langle Ay_n - Ax_{\beta_n}, x_{\beta_n} - y_n\rangle + 2\lambda_n\langle Ax_{\beta_n} + \beta_n F x_{\beta_n}, x_{\beta_n} - y_n\rangle \\
& +2\lambda_n\beta_n\langle Fw_n - Fx_{\beta_n}, x_{\beta_n} - y_n\rangle \\
\le\ & 2\lambda_n\langle Ax_{\beta_n} + \beta_n F x_{\beta_n}, x_{\beta_n} - y_n\rangle + 2\lambda_n\beta_n\langle Fw_n - Fx_{\beta_n}, x_{\beta_n} - y_n\rangle. \qquad (30)
\end{aligned}$$

Since $x_{\beta_n} \in \mathrm{VI}(C, A + \beta_n F)$ and $y_n \in C$, we have

$$\langle Ax_{\beta_n} + \beta_n F x_{\beta_n}, x_{\beta_n} - y_n \rangle \leq 0,$$

which, by relation (30), yields that

$$2\lambda_n \langle Ay_n + \beta_n Fw_n, x_{\beta_n} - y_n \rangle \leq 2\lambda_n \beta_n \langle Fw_n - Fx_{\beta_n}, x_{\beta_n} - y_n \rangle. \quad (31)$$

Since F is η-strongly monotone, we find

$$\begin{aligned} & 2\lambda_n \beta_n \langle Fw_n - Fx_{\beta_n}, x_{\beta_n} - y_n \rangle \\ = \; & 2\lambda_n \beta_n \langle Fw_n - Fx_{\beta_n}, x_{\beta_n} - w_n \rangle + 2\lambda_n \beta_n \langle Fw_n - Fx_{\beta_n}, w_n - y_n \rangle \\ \leq \; & -2\lambda_n \beta_n \eta \|w_n - x_{\beta_n}\|^2 + 2\lambda_n \beta_n \langle Fw_n - Fx_{\beta_n}, w_n - y_n \rangle. \end{aligned} \quad (32)$$

Let ϵ_1, ϵ_2 and ϵ_3 be there positive real numbers such that

$$2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3 > 0.$$

From Lemma 8, we know that $\mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \to \mu^2 \in (0, 1)$. Since $\beta_n \to 0$ and $\frac{\sigma_{i,n}}{\beta_n} \to 0$ for each $i = 1, 2, \cdots, N$, there exist $\epsilon_4 > 0$ and $n_0 \geq N$ such that

$$1 - \epsilon_4 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} - \frac{\lambda_n \beta_n K}{\epsilon_1} > 0, \quad \forall n \geq n_0,$$

$$\sum_{i=1}^{N} \sigma_{i,n} \leq \epsilon_3 \lambda_n \beta_n, \quad \forall n \geq n_0.$$

Since F is K-generalized Lipschitzian, we deduce

$$\begin{aligned} & 2\lambda_n \beta_n \langle Fw_n - Fx_{\beta_n}, w_n - y_n \rangle \\ \leq \; & 2\lambda_n \beta_n \|Fw_n - Fx_{\beta_n}\| \|w_n - y_n\| \\ \leq \; & 2\lambda_n \beta_n K(\|w_n - x_{\beta_n}\| + 1) \|w_n - y_n\| \\ = \; & 2\lambda_n \beta_n K \|w_n - x_{\beta_n}\| \|w_n - y_n\| + 2\lambda_n \beta_n K \|w_n - y_n\| \\ \leq \; & \epsilon_1 \lambda_n \beta_n K \|w_n - x_{\beta_n}\|^2 + \frac{\lambda_n \beta_n K}{\epsilon_1} \|w_n - y_n\|^2 + \epsilon_4 \|w_n - y_n\|^2 \\ & + \frac{\lambda_n^2 \beta_n^2 K^2}{\epsilon_4} \\ \leq \; & \epsilon_1 \lambda_n \beta_n K \|w_n - x_{\beta_n}\|^2 + \left(\frac{\lambda_n \beta_n K}{\epsilon_1} + \epsilon_4 \right) \|w_n - y_n\|^2 + \frac{\lambda_1^2 \beta_n^2 K^2}{\epsilon_4}. \end{aligned} \quad (33)$$

Combing (31)–(33), we obtain

$$\begin{aligned} & 2\lambda_n \langle Ay_n + \beta_n Fw_n, x_{\beta_n} - y_n \rangle \\ \leq \; & -(2\eta - K\epsilon_1) \lambda_n \beta_n \|w_n - x_{\beta_n}\|^2 + \left(\frac{\lambda_n \beta_n K}{\epsilon_1} + \epsilon_4 \right) \|w_n - y_n\|^2 \\ & + \frac{\lambda_1^2 \beta_n^2 K^2}{\epsilon_4}. \end{aligned} \quad (34)$$

Substituting (34) into (29), for all $n \geq n_0$, we conclude

$$\begin{aligned} & \|x_{n+1} - x_{\beta_n}\|^2 \\ \leq & [1 - (2\eta - K\epsilon_1)\lambda_n\beta_n]\|w_n - x_{\beta_n}\|^2 \\ & -\left(1 - \epsilon_4 - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - \frac{\lambda_n\beta_n K}{\epsilon_1}\right)\|w_n - y_n\|^2 + \frac{\lambda_1^2\beta_n^2K^2}{\epsilon_4} \\ \leq & [1 - (2\eta - K\epsilon_1)\lambda_n\beta_n]\|w_n - x_{\beta_n}\|^2 + \frac{\lambda_1^2\beta_n^2K^2}{\epsilon_4}. \end{aligned} \tag{35}$$

By Lemma 6, for all $n \geq n_0$, we have

$$\begin{aligned} & \|x_{n+1} - x_{\beta_{n+1}}\|^2 \\ = & \|x_{n+1} - x_{\beta_n}\|^2 + \|x_{\beta_{n+1}} - x_{\beta_n}\|^2 + 2\langle x_{n+1} - x_{\beta_n}, x_{\beta_n} - x_{\beta_{n+1}}\rangle \\ \leq & \|x_{n+1} - x_{\beta_n}\|^2 + \|x_{\beta_{n+1}} - x_{\beta_n}\|^2 + 2\|x_{n+1} - x_{\beta_n}\|\|x_{\beta_{n+1}} - x_{\beta_n}\| \\ \leq & \|x_{n+1} - x_{\beta_n}\|^2 + \|x_{\beta_{n+1}} - x_{\beta_n}\|^2 + \epsilon_2\lambda_n\beta_n\|x_{n+1} - x_{\beta_n}\|^2 \\ & + \frac{1}{\epsilon_2\lambda_n\beta_n}\|x_{\beta_{n+1}} - x_{\beta_n}\|^2 \\ = & (1 + \epsilon_2\lambda_n\beta_n)\|x_{n+1} - x_{\beta_n}\|^2 + \frac{1 + \epsilon_2\lambda_n\beta_n}{\epsilon_2\lambda_n\beta_n}\|x_{\beta_{n+1}} - x_{\beta_n}\|^2 \\ \leq & (1 + \epsilon_2\lambda_n\beta_n)\|x_{n+1} - x_{\beta_n}\|^2 + \frac{1 + \epsilon_2\lambda_n\beta_n}{\epsilon_2\lambda_n\beta_n}\left(\frac{\beta_{n+1} - \beta_n}{\beta_n}\right)^2 M_1^2 \\ = & (1 + \epsilon_2\lambda_n\beta_n)\|x_{n+1} - x_{\beta_n}\|^2 + \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{\epsilon_2\lambda_n}\frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^3}, \end{aligned} \tag{36}$$

where $M_1 = \frac{1}{\eta}\left[\left(1 + \frac{K}{\eta}\right)\|F\hat{x}\| + 2K\|\hat{x}\| + K\right]$ is a positive constant. Since $\beta_n \to 0$, we know that $\{\beta_n\}$ is bounded. Hence $\left\{\frac{\lambda_1^2K^2}{\epsilon_4}(1 + \epsilon_2\lambda_1\beta_n)\right\}$ is bounded. Substituting (35) into (36), for all $n \geq n_0$, we deduce

$$\begin{aligned} & \|x_{n+1} - x_{\beta_{n+1}}\|^2 \\ \leq & (1 + \epsilon_2\lambda_n\beta_n)[1 - (2\eta - K\epsilon_1)\lambda_n\beta_n]\|w_n - x_{\beta_n}\|^2 \\ & + \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{\epsilon_2\lambda_n}\frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^3} + (1 + \epsilon_2\lambda_n\beta_n)\frac{\lambda_1^2\beta_n^2K^2}{\epsilon_4} \\ \leq & [1 - (2\eta - K\epsilon_1 - \epsilon_2)\lambda_n\beta_n - (2\eta - K\epsilon_1)\epsilon_2\lambda_n^2\beta_n^2]\|w_n - x_{\beta_n}\|^2 \\ & + \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{\epsilon_2\lambda_n}\frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^3} + (1 + \epsilon_2\lambda_1\beta_n)\frac{\lambda_1^2\beta_n^2K^2}{\epsilon_4} \\ \leq & [1 - (2\eta - K\epsilon_1 - \epsilon_2)\lambda_n\beta_n]\|w_n - x_{\beta_n}\|^2 \\ & + \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{\epsilon_2\lambda_n}\frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^3} + M_2\beta_n^2, \end{aligned} \tag{37}$$

where $M_2 = \sup_{n\in\mathbb{N}}\left\{\frac{\lambda_1^2K^2}{\epsilon_4}(1 + \epsilon_2\lambda_1\beta_n)\right\}$ is a positive constant. Notice, for all $n \geq n_0$,

$$\begin{aligned}
&\|w_n - x_{\beta_n}\|^2 \\
= \;& \|(x_n - x_{\beta_n}) + \sum_{i=1}^{N} \alpha_{i,n}(x_{n-i+1} - x_{n-i})\|^2 \\
\leq \;& \left(\|x_n - x_{\beta_n}\| + \sum_{i=1}^{N} \alpha_{i,n}\|x_{n-i+1} - x_{n-i}\| \right)^2 \\
= \;& \|x_n - x_{\beta_n}\|^2 + \sum_{i=1}^{N} \alpha_{i,n}^2 \|x_{n-i+1} - x_{n-i}\|^2 \\
& +2\|x_n - x_{\beta_n}\| \sum_{i=1}^{N} \alpha_{i,n}\|x_{n-i+1} - x_{n-i}\| \\
& +2 \sum_{1 \leq i < j \leq N} \alpha_{i,n}\alpha_{j,n}\|x_{n-i+1} - x_{n-i}\|\|x_{n-j+1} - x_{n-j}\| \\
\leq \;& \|x_n - x_{\beta_n}\|^2 + \sum_{i=1}^{N} \sigma_{i,n}^2 + \|x_n - x_{\beta_n}\|^2 \sum_{i=1}^{N} \sigma_{i,n} + \sum_{i=1}^{N} \sigma_{i,n} \\
& +2 \sum_{1 \leq i < j \leq N} \sigma_{i,n}\sigma_{j,n} \\
= \;& \left(1 + \sum_{i=1}^{N} \sigma_{i,n}\right) \|x_n - x_{\beta_n}\|^2 + \bar{\sigma}_n \\
\leq \;& (1 + \epsilon_3 \lambda_n \beta_n)\|x_n - x_{\beta_n}\|^2 + \bar{\sigma}_n,
\end{aligned} \tag{38}$$

where $\bar{\sigma}_n = \sum_{i=1}^{N} \sigma_{i,n}^2 + \sum_{i=1}^{N} \sigma_{i,n} + 2\sum_{1 \leq i < j \leq N} \sigma_{i,n}\sigma_{j,n}$. From the condition of $\{\sigma_{i,n}\}$, we can obviously see that $\sum_{n=1}^{+\infty} \bar{\sigma}_n < +\infty$. Substituting (38) into (37), for all $n \geq n_0$, we conclude

$$\begin{aligned}
&\|x_{n+1} - x_{\beta_{n+1}}\|^2 \\
\leq \;& [1 - (2\eta - K\epsilon_1 - \epsilon_2)\lambda_n\beta_n](1 + \epsilon_3\lambda_n\beta_n)\|x_n - x_{\beta_n}\|^2 \\
& + \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{\epsilon_2\lambda_n} \frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^3} + M_2\beta_n^2 + \bar{\sigma}_n \\
= \;& [1 - (2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_n\beta_n - (2\eta - K\epsilon_1 - \epsilon_2)\epsilon_3\lambda_n^2\beta_n^2]\|x_n - x_{\beta_n}\|^2 \\
& + \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{\epsilon_2\lambda_n} \frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^3} + M_2\beta_n^2 + \bar{\sigma}_n \\
\leq \;& [1 - (2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_n\beta_n]\|x_n - x_{\beta_n}\|^2 \\
& +(2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_n\beta_n \frac{M_1^2(1 + \epsilon_2\lambda_n\beta_n)}{(2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3)\epsilon_2\lambda_n^2} \frac{(\beta_{n+1} - \beta_n)^2}{\beta_n^4} \\
& +M_2\beta_n^2 + \bar{\sigma}_n \\
\leq \;& (1 - \bar{\beta}_n)\|x_n - x_{\beta_n}\|^2 + \bar{\beta}_n M' \left(\frac{\beta_{n+1} - \beta_n}{\beta_n^2} \right)^2 + M_2\beta_n^2 + \bar{\sigma}_n \\
= \;& (1 - \bar{\beta}_n)\|x_n - x_{\beta_n}\|^2 + \bar{\beta}_n\delta_n + M_2\beta_n^2 + \bar{\sigma}_n,
\end{aligned} \tag{39}$$

where $\bar{\beta}_n = (2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_n\beta_n$, $M' = \sup_{n\in\mathbb{N}}\left\{ \frac{M_1^2(1+\epsilon_2\lambda_n\beta_n)}{(2\eta - K\epsilon_1 - \epsilon_2 - \epsilon_3)\epsilon_2\lambda_n^2} \right\}$ is a positive constant and $\delta_n = M'\left(\frac{\beta_{n+1} - \beta_n}{\beta_n^2}\right)^2$. By the conditions of $\{\lambda_n\}$ and $\{\beta_n\}$, we know that $\sum_{n=1}^{+\infty}(M_2\beta_n^2 + \bar{\sigma}_n) < +\infty$, $\bar{\beta}_n \to 0$, $\sum_{n=1}^{+\infty} \bar{\beta}_n = +\infty$ and $\delta_n \to 0$. It follows from Lemma 4 that $x_n - x_{\beta_n} \to 0$ as $n \to +\infty$. □

Remark 2.

(i) *In Algorithm 3, it is not neccecery to know η and K.*

(ii) *In Algorithm 3, $\{\beta_n\}$ can be taken as $\beta_n = n^{-p}$, where $\frac{1}{2} < p < 1$.*

(iii) *If F is strongly monotone and Lipschitz-continuous, then the condition $\sum_{n=1}^{+\infty} \beta_n^2 < +\infty$ can be removed.*

(iv) *Let $f : H \to H$ be a contractive mapping. It is obvious that $I - f$ is strongly monotone and Lipschitz continuous. If $F = I - f$ in Algorithm 3, then $\{x_n\}$ converges strongly to $\hat{x}$, where $\hat{x}$ is the unique fixed point of $P_{\mathrm{VI}(C,A)} f$. Furthermore, if $F = I$ in Algorithm 3, then $\{x_n\}$ converges strongly to $x^\dagger$, where $x^\dagger$ is the mini-norm element in $\mathrm{VI}(C, A)$, i.e., $x^\dagger = P_{\mathrm{VI}(C,A)} 0$.*

4. Multi-Step Inertial RTEGM

In this section, we propose a new method for solving HVIP (6) based on Algorithm 2 (RTEGM). Under certain conditions, it has a strong convergence result.

Theorem 2. *Under the conditions (CD1)-(CD3), the sequence $\{x_n\}$ generated by Algorithm 4 converges strongly to $\hat{x} \in \mathrm{VI}(C, A)$, where $\hat{x}$ is the unique solution of HVIP (4).*

Proof. From Lemma 2, we know that for each $n \in \mathbb{N}$, there exists the unique element $x_{\beta_n} \in C$ such that

$$\langle (A + \beta_n F) x_{\beta_n}, x - x_{\beta_n} \rangle \geq 0, \quad \forall x \in C.$$

By Lemma 7, we need to prove that $x_n - x_{\beta_n} \to 0$. From the expression of x_{n+1}, we have

$$\begin{aligned} & \|x_{n+1} - x_{\beta_n}\|^2 \\ = & \|(y_n - x_{\beta_n}) - \lambda_n (Ay_n - Aw_n)\|^2 \\ = & \|y_n - x_{\beta_n}\|^2 + \lambda_n^2 \|Ay_n - Aw_n\|^2 - 2\lambda_n \langle Ay_n - Aw_n, y_n - x_{\beta_n} \rangle \\ = & \|y_n - x_{\beta_n}\|^2 + \lambda_n^2 \|Ay_n - Aw_n\|^2 + 2\lambda_n \langle Aw_n + \beta_n F w_n, y_n - x_{\beta_n} \rangle \\ & -2\lambda_n \langle Ay_n + \beta_n F w_n, y_n - x_{\beta_n} \rangle \\ = & \|y_n - x_{\beta_n}\|^2 + \lambda_n^2 \|Ay_n - Aw_n\|^2 + 2\langle w_n - y_n, y_n - x_{\beta_n} \rangle \\ & +2\langle w_n - \lambda_n (Aw_n + \beta_n F w_n) - y_n, x_{\beta_n} - y_n \rangle \\ & -2\lambda_n \langle Ay_n + \beta_n F w_n, y_n - x_{\beta_n} \rangle. \end{aligned} \tag{40}$$

By Lemma 3 and the expression of y_n, we get

$$\langle w_n - \lambda_n (Aw_n + \beta_n F w_n) - y_n, p - y_n \rangle \leq 0, \quad \forall p \in C.$$

Now

$$\langle w_n - \lambda_n (Aw_n + \beta_n F w_n) - y_n, x_{\beta_n} - y_n \rangle \leq 0, \tag{41}$$

which is due to the fact that $x_{\beta_n} \in C$. It is easy to see that

$$2\langle w_n - y_n, y_n - x_{\beta_n} \rangle = \|w_n - x_{\beta_n}\|^2 - \|w_n - y_n\|^2 - \|y_n - x_{\beta_n}\|^2. \tag{42}$$

Substituting (41) and (42) into (40), we obtain

$$\begin{aligned} & \|x_{n+1} - x_{\beta_n}\|^2 \\ \leq & \|w_n - x_{\beta_n}\|^2 - \|w_n - y_n\|^2 + \lambda_n^2 \|Ay_n - Aw_n\|^2 \\ & +2\lambda_n \langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n \rangle. \end{aligned} \tag{43}$$

From the computation of $\{\lambda_n\}$, we deduce

$$\lambda_n^2\|Ay_n - Aw_n\|^2 \leq \mu^2 \frac{\lambda_n}{\lambda_{n+1}}\|y_n - w_n\|^2.$$

Substituting the last inequality into (43), we have

$$\begin{aligned} & \|x_{n+1} - x_{\beta_n}\|^2 \\ \leq\ & \|w_n - x_{\beta_n}\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2 \\ & +2\lambda_n\langle Ay_n + \beta_n F w_n, x_{\beta_n} - y_n\rangle. \end{aligned} \tag{44}$$

The rest proof is the same as that in Theorem 1. □

Algorithm 4 Multi-step inertial RTEGM (MIRTEGM)

Initialization: Given $\lambda_1 > 0$, $\mu \in (0,1)$. Choose $x_0, x_1 \in H$ arbitrarily and a sequence $\{\beta_n\} \subset (0,+\infty)$ such that

$$\lim_{n\to+\infty} \beta_n = 0, \quad \sum_{n=1}^{+\infty} \beta_n = +\infty, \quad \sum_{n=1}^{+\infty} \beta_n^2 < +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\beta_{n+1} - \beta_n}{\beta_n^2} = 0.$$

For each $i = 1, 2, \cdots, N$ (where N is a chosen positive integer), choose a sequence $\{\sigma_{i,n}\} \subset (0,+\infty)$ satisfying

$$\lim_{n\to+\infty} \frac{\sigma_{i,n}}{\beta_n} = 0 \quad \text{and} \quad \sum_{n=1}^{+\infty} \sigma_{i,n} < +\infty.$$

Iterative step: Calculate x_{n+1} for $n \geq 1$ as follows:

Step 1. Compute

$$w_n = x_n + \sum_{i=1}^{\min\{N,n\}} \alpha_{i,n}(x_{n-i+1} - x_{n-i}),$$

where $0 \leq \alpha_{i,n} \leq \alpha_i$ for some $\alpha_i \in H$ with

$$\alpha_{i,n} = \begin{cases} \min\left\{\alpha_i, \frac{\sigma_{i,n}}{\|x_{n-i+1} - x_{n-i}\|}\right\}, & \text{if } x_{n-i+1} \neq x_{n-i}, \\ \alpha_i, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$y_n = P_C[w_n - \lambda_n(Aw_n + \beta_n F w_n)].$$

Step 3. Compute

$$x_{n+1} = y_n - \lambda_n(Ay_n - Aw_n),$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}\right\}, & \text{if } Aw_n \neq Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to Step 1.

5. Numerical Experiments

In this section, we give two numerical examples to illustrate the effectiveness and feasibility of our algorithms and compare with Algorithm 1 (RSEGM) and Algorithm 2 (RTEGM). We denote Algorithm 3 for $N = 1$, $N = 2$, $N = 3$ by IRSEGM, 2-MIRSEGM and 3-MIRSEGM, respectively, denote Algorithm 4 for $N = 1$, $N = 2$, $N = 3$ by IRTEGM, 2-MIRTEGM and 3-MIRTEGM, respectively. We write all the programmes in Matlab 9.0 and performed on PC Desktop Intel(R) Core(TM) i5-1035G1 CPU @ 1.00 GHz 1.19 GHz, RAM 16.0 GB.

Example 1. *Let $H = \mathbb{R}$ and $C = [-2, 5]$. Let A be a function defined as*

$$Ax := x + \sin x,$$

for each $x \in \mathbb{R}$. It is easy to see that A is monotone and Lipschitz continuous. Let $F = I$.

Choose $x_0 = 1$, $\alpha_i = 0.1$ and $\sigma_{i,n} = n^{-2}$ for IRSEGM, 2-MIRSEGM, 3-MIRSEGM, IRTEGM, 2-MIRTEGM and 3-MIRTEGM. Choose $\mu = 0.6$ and $\beta_n = n^{-3/4}$ for each algorithm. It is obvious that $\mathrm{VI}(C, A) = \{0\}$ and hence $x^* = 0$ is the unique solution of HVIP (4). We use $\|x_n - x^*\| \leq 10^{-6}$ for stopping criterion. We show the numerical results in Tables 1 and 2. From these tables, we can easily see that the number of iterations of our algorithms is 10–40% less than RSEGM and RTEGM. Convergence of our algorithms is also much faster than RSEGM and RTEGM in term of elapsed time.

Table 1. Numerical results of RSEGM, IRSEGM, 2-MIRSEGM and 3-MIRSEGM as regards Example 1.

x_1	λ_1	RSEGM		IRSEGM		2-MIRSEGM		3-MIRSEGM	
		Iter.	**Time [s]**	**Iter.**	**Time [s]**	**Iter.**	**Time [s]**	**Iter.**	**Time [s]**
1	0.5	49	0.6557	44	0.6287	34	0.5419	28	0.4817
	0.1	76	0.9035	69	0.8155	57	0.7259	42	0.6009
	0.05	143	1.3715	128	1.3037	111	1.1908	87	0.9745
2	0.5	51	0.6835	46	0.6415	35	0.5482	28	0.4879
	0.1	80	0.9425	73	0.8266	60	0.7397	37	0.5490
	0.05	151	1.4822	136	1.3876	119	1.2319	97	1.0492

Table 2. Numerical results of RTEGM, IRTEGM, 2-MIRTEGM and 3-MIRTEGM as regards Example 1.

x_1	λ_1	RTEGM		IRTEGM		2-MIRTEGM		3-MIRTEGM	
		Iter.	**Time [s]**	**Iter.**	**Time [s]**	**Iter.**	**Time [s]**	**Iter.**	**Time [s]**
1	0.5	49	0.6543	44	0.6119	34	0.5208	34	0.5178
	0.1	76	0.8435	69	0.7972	57	0.7043	38	0.5724
	0.05	143	1.4172	123	1.2759	111	1.1249	88	0.9873
2	0.5	51	0.6771	46	0.6325	35	0.5372	34	0.5230
	0.1	80	0.9263	73	0.8278	61	0.7532	43	0.6107
	0.05	151	1.4739	137	1.3962	119	1.2103	98	1.0871

Example 2. *Let $H = \mathbb{R}^s$. We consider the HpHard problem [20,21]. Let $A : \mathbb{R}^s \to \mathbb{R}^s$ be a mapping defined by*

$$Ax := Mx + q,$$

for each $x \in \mathbb{R}^s$, where

$$M = BB^{\mathrm{T}} + S + D,$$

B is a matrix in $\mathbb{R}^{s\times s}$, S is a skew-symmetric matrix in $\mathbb{R}^{s\times s}$, D is a diagonal matrix in $\mathbb{R}^{s\times s}$ whose diagonal entries are positive, and $q \in \mathbb{R}^s$ is a vector. Thus, M is positive definite. Let C be a set defined by

$$C = \{(x^{(1)}, x^{(2)}, \cdots, x^{(s)})^{\mathrm{T}} \in \mathbb{R}^s : -2 \le x^{(i)} \le 5,\ i = 1, 2, \cdots, s\}.$$

It is clear that A is monotone and Lipschitz continuous. Let $F = I$. It is obvious that $\mathrm{VI}(C, A) = \{(0, 0, \cdots, 0)^{\mathrm{T}}\}$ and hence $x^* = (0, 0, \cdots, 0)^{\mathrm{T}}$ is the unique solution of HVIP (1.6).

For the experiments, all the entries of B and S are generated randomly and uniformly in $(-2,2)$, the diagonal entries of D are generated randomly and uniformly in $(0,2)$, $q = (0, 0, \cdots, 0)^{\mathrm{T}}$. We choose $x_0 = (1, 1, \cdots, 1)^{\mathrm{T}}$, $\alpha_i = 0.1$ and $\sigma_{i,n} = n^{-2}$ for IRSEGM, 2-MIRSEGM, 3-MIRSEGM, IRTEGM, 2-MIRTEGM and 3-MIRTEGM, choose $x_1 = (1, 1, \cdots, 1)^{\mathrm{T}}$, $\mu = 0.6$, $\lambda_1 = 0.01$ and $\beta_n = n^{-3/4}$ for each algorithm. We show the numerical results in Figures 1–6. From these figures, we see that the algorithms we proposed have advantages over RSEGM and RTEGM.

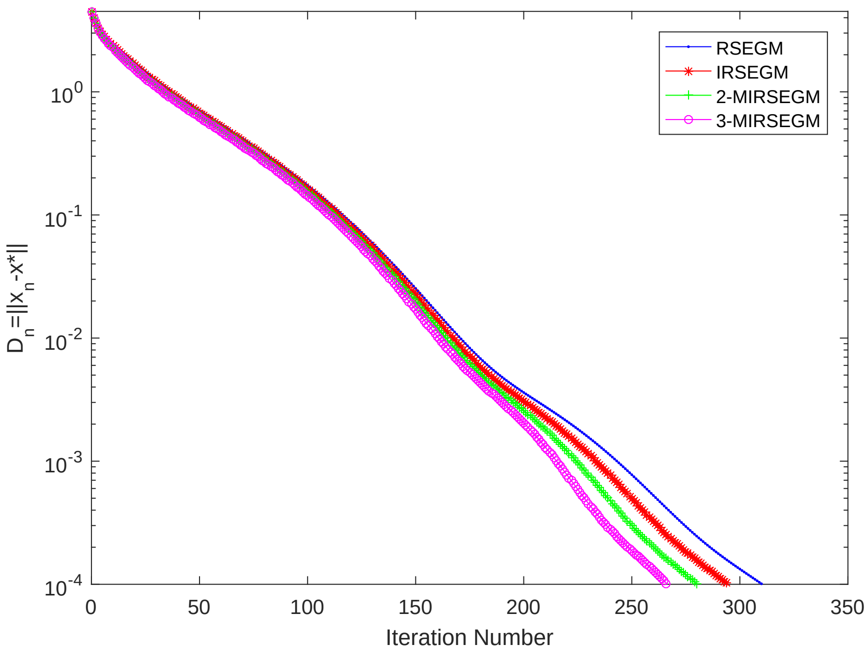

Figure 1. Comparison of RSEGM, IRSEGM, 2-MIRSEGM and 3-MIRSEGM in Example 2 with $s = 20$.

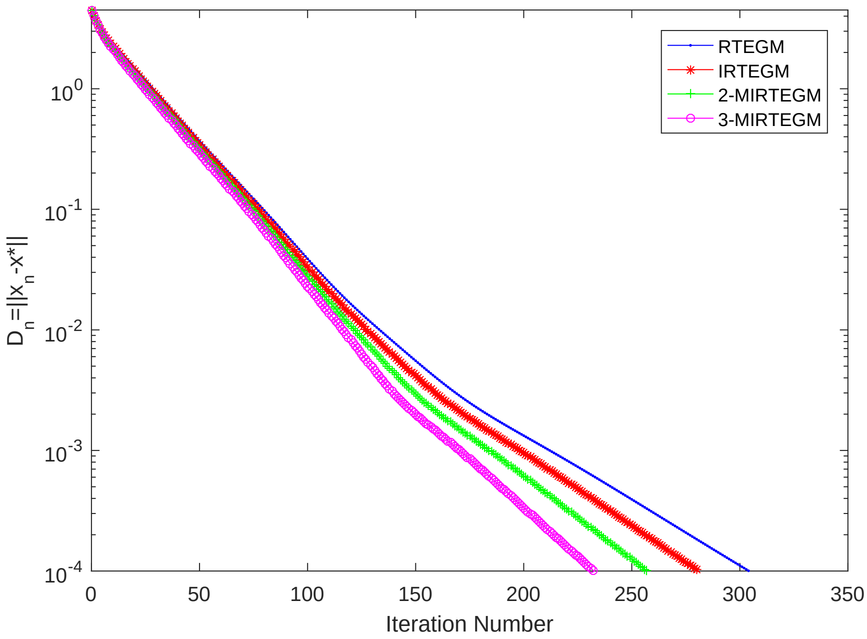

Figure 2. Comparison of RTEGM, IRTEGM, 2-MIRTEGM and 3-MIRTEGM in Example 2 with $s = 20$.

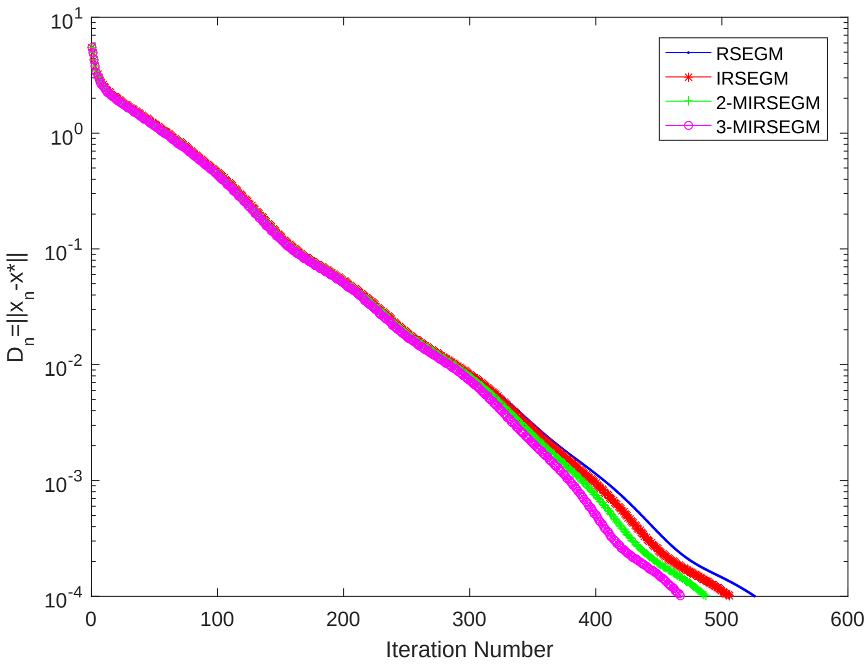

Figure 3. Comparison of RSEGM, IRSEGM, 2-MIRSEGM and 3-MIRSEGM in Example 2 with $s = 30$.

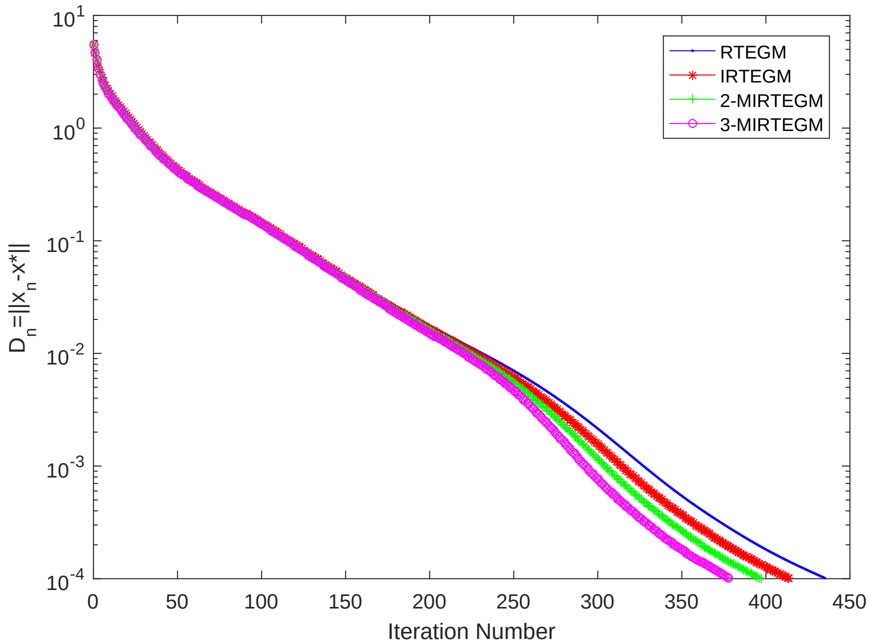

Figure 4. Comparison of RTEGM, IRTEGM, 2-MIRTEGM and 3-MIRTEGM in Example 2 with $s = 30$.

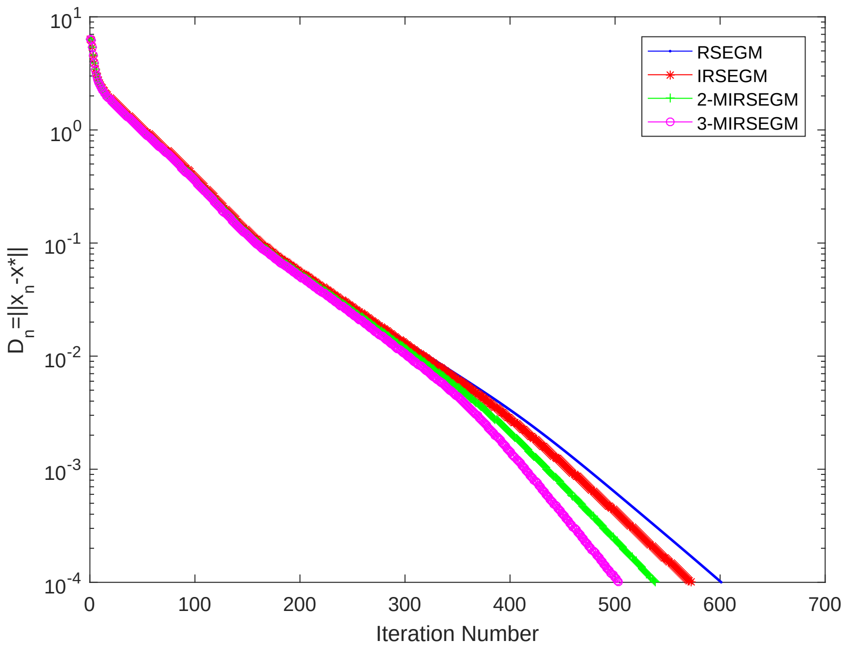

Figure 5. Comparison of RSEGM, IRSEGM, 2-MIRSEGM and 3-MIRSEGM in Example 2 with $s = 40$.

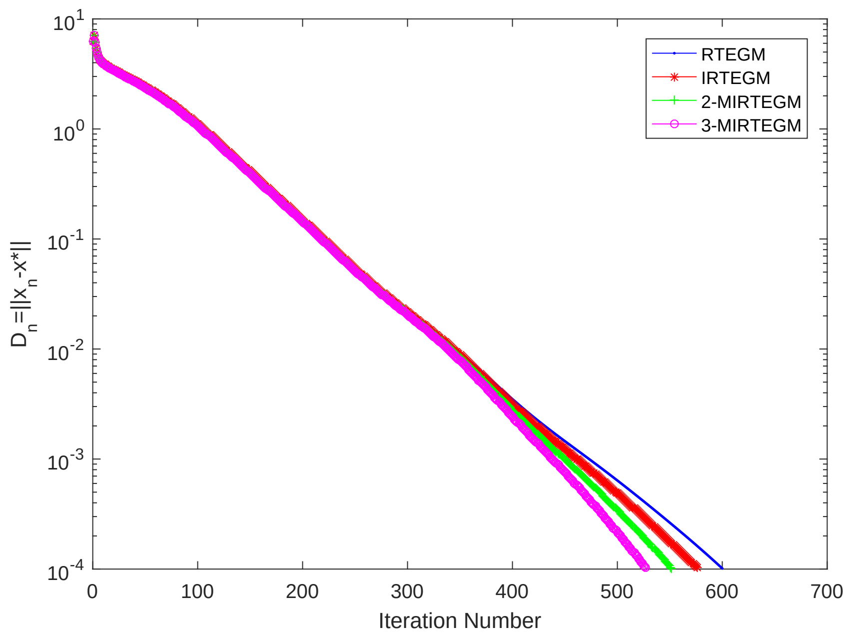

Figure 6. Comparison of RTEGM, IRTEGM, 2-MIRTEGM and 3-MIRTEGM in Example 2 with $s = 40$.

6. Conclusions

In this paper, we constructed a multi-step inertial regularized subgradient extragradient method and a multi-step inertial Tseng's extragradient method for solving HVIP (6) in a Hilbert space when F is a generalized Lipschitzian and hemicontinuous mapping, which are based on the multi-step inertial methods, Algorithm 1 (RSEGM) and Algorithm 2 (RTEGM). We presented two strong convergence theorems. Finally, we gave some numerical experiments to show the effectiveness and feasibility of our new iterative methods. From the numerical results, we can obviously see that our methods have advantages over Algorithms 1 and 2.

Our Algorithms 3 and 4 extend and improve Algorithms 1 and 2 in the following ways:

(i) The inertial method is used in Algorithms 3 and 4.
(ii) The Lipschitzian mapping F is generalized to a generalized Lipschitzian and hemicontinuous mapping.

In other words, if we let $\alpha_i = 0$ and L be a Lipschitzian mapping, then Algorithm 3 (or Algorithm 4) reduces to Algorithm 1 (or Algorithm 2).

Author Contributions: Conceptualization, J.-C.Y.; Data curation, B.J.; Formal analysis, Y.W.; Funding acquisition, Y.W. and J.-C.Y.; Investigation, B.J.; Methodology, B.J. and Y.W.; Resources, J.-C.Y.; Supervision, Y.W. and J.-C.Y.; Writing–original draft, B.J. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China, grant number 11671365.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declare that they have no competing interests.

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