mathematics

MDPI

Article

Warped Product Submanifolds in Locally Golden Riemannian Manifolds with a Slant Factor

Cristina E. Hretcanu [1,*,†] and Adara M. Blaga [2,†]

[1] Faculty of Food Engineering, Stefan cel Mare University of Suceava, 720229 Suceava, Romania
[2] Faculty of Mathematics and Computer Science, West University of Timisoara, 300223 Timisoara, Romania; adarablaga@yahoo.com
* Correspondence: criselenab@yahoo.com
† These authors contributed equally to this work.

Abstract: In the present paper, we study some properties of warped product pointwise semi-slant and hemi-slant submanifolds in Golden Riemannian manifolds, and we construct examples in Euclidean spaces. Additionally, we study some properties of proper warped product pointwise semi-slant (and, respectively, hemi-slant) submanifolds in a locally Golden Riemannian manifold.

Keywords: Golden Riemannian structure; warped product submanifold; pointwise slant; semi-slant; hemi-slant; bi-slant submanifold

MSC: 53B20; 53B25; 53C42; 53C15

Citation: Hretcanu, C.E.; Blaga, A.M. Warped Product Submanifolds in Locally Golden Riemannian Manifolds with a Slant Factor. *Mathematics* **2021**, *9*, 2125. https://doi.org/10.3390/math9172125

Academic Editor: Marian Ioan Munteanu

Received: 26 July 2021
Accepted: 27 August 2021
Published: 2 September 2021

1. Introduction

B.-Y. Chen studied CR-submanifolds of a Kähler manifold, which are warped products of holomorphic and totally real submanifolds, respectively [1–3]. In addition, in his new book [4], he presents a multitude of properties for warped product manifolds and submanifolds, such as: warped product of Riemannian and Kähler manifolds, warped product submanifolds of Kähler manifolds (with the particular cases: warped product CR-submanifolds, warped product semi-slant or hemi-slant submanifolds of Kähler manifolds), CR-warped products in complex space forms and so on.

Metallic Riemannian manifolds and their submanifolds were defined and studied by C. E. Hretcanu, M. Crasmareanu and A. M. Blaga in [5,6], as a generalization of Golden Riemannian manifolds studied in [7–9]. The authors of the present paper studied some properties of invariant, anti-invariant and slant submanifolds [10], semi-slant submanifolds [11] and, respectively, hemi-slant submanifolds [12] in metallic and Golden Riemannian manifolds and they obtained integrability conditions for the distributions involved in these types of submanifolds. Moreover, properties of metallic and Golden warped product Riemannian manifolds were presented in the two previous works of the authors [13,14]. Lately, the study of submanifolds in metallic Riemannian manifolds has been continued by many authors [15–17], which introduced the notion of a lightlike submanifold of a metallic semi-Riemannian manifold.

In the present paper, we study warped product pointwise semi-slant and hemi-slant submanifolds in locally Golden Riemannian manifolds. In Section 2, we recall the main properties of Golden Riemannian manifolds and of their submanifolds, and we prove some immediate consequences of the Gauss and Weingarten equations for an isometrically immersed submanifold in a Golden Riemannian manifold. In Section 3, we give some properties of pointwise slant submanifolds in Golden Riemannian manifolds. In Section 4, we study some properties of pointwise bi-slant submanifolds in Golden Riemannian manifolds. In Section 5, we discuss warped product pointwise bi-slant submanifolds in Golden Riemannian manifolds, and in Section 6, we find some properties of pointwise semi-slant and hemi-slant submanifolds in locally metallic Riemannian manifolds. We also provide examples of pointwise slant and pointwise bi-slant submanifolds, of warped product semi-slant, hemi-slant and pointwise bi-slant submanifolds in Golden Riemannian manifolds.

2. Preliminaries

The Golden number $\phi = \frac{1+\sqrt{5}}{2}$ is the positive solution of the equation

$$x^2 - x - 1 = 0. \tag{1}$$

It is a member of the metallic numbers family introduced by Spinadel [18], given by the positive solution $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ of the equation $x^2 - px - q = 0$, where p and q are positive integer values.

The *Golden structure* J is a particular case of polynomial structure on a manifold [19,20], which satisfies

$$J^2 = J + I, \tag{2}$$

where I is the identity operator on $\Gamma(T\overline{M})$.

If $(\overline{M}, \overline{g})$ is a Riemannian manifold endowed with a Golden structure J such that the Riemannian metric $\overline{g}$ is J-compatible, i.e.,

$$\overline{g}(JX, Y) = \overline{g}(X, JY), \tag{3}$$

for any X, $Y \in \Gamma(T\overline{M})$, then $(\overline{M}, \overline{g}, J)$ is called a *Golden Riemannian manifold* [7].

In this case, $\overline{g}$ verifies

$$\overline{g}(JX, JY) = \overline{g}(J^2X, Y) = \overline{g}(JX, Y) + \overline{g}(X, Y), \tag{4}$$

for any X, $Y \in \Gamma(T\overline{M})$.

Let M be an isometrically immersed submanifold in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. The tangent space $T_x\overline{M}$ of $\overline{M}$ in a point $x \in M$ can be decomposed into the direct sum $T_x\overline{M} = T_xM \oplus T_x^{\perp}M$, for any $x \in M$, where $T_x^{\perp}M$ is the normal space of M in x. Let i_* be the differential of the immersion $i : M \to \overline{M}$. Then, the induced Riemannian metric g on M is given by $g(X, Y) = \overline{g}(i_*X, i_*Y)$, for any X, $Y \in \Gamma(TM)$. In the rest of the paper, we shall denote by X the vector field i_*X for any $X \in \Gamma(TM)$.

For any $X \in \Gamma(TM)$, let $TX := (JX)^T$ and $NX := (JX)^{\perp}$ be the tangential and normal components, respectively, of JX, and for any $V \in \Gamma(T^{\perp}M)$, let $tV := (JV)^T$ and $nV := (JV)^{\perp}$ be the tangential and normal components, respectively, of JV. Then, we have

$$JX = TX + NX, \quad JV = tV + nV, \tag{5}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The maps T and n are $\overline{g}$-symmetric [10]:

$$\overline{g}(TX, Y) = \overline{g}(X, TY), \quad \overline{g}(nU, V) = \overline{g}(U, nV), \tag{6}$$

$$\overline{g}(NX, V) = \overline{g}(X, tV), \tag{7}$$

for any X, $Y \in \Gamma(TM)$ and U, $V \in \Gamma(T^{\perp}M)$. Moreover, from [12] for $p = q = 1$ in the metallic structure, we obtain

$$T^2X = TX + X - tNX, \quad NX = NTX + nNX, \tag{8}$$

$$n^2V = nV + V - NtV, \quad tV = TtV + tnV, \tag{9}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and on its submanifold (M, g), respectively. The Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{10}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the shape operator, which satisfy

$$\overline{g}(h(X, Y), V) = \overline{g}(A_V X, Y). \tag{11}$$

For any $X, Y \in \Gamma(TM)$, the covariant derivatives of T and N are given by

$$(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \quad (\overline{\nabla}_X N)Y = \nabla^{\perp}_X NY - N(\nabla_X Y). \tag{12}$$

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, the covariant derivatives of t and n are given by

$$(\nabla_X t)V = \nabla_X tV - t(\nabla^{\perp}_X V), \quad (\overline{\nabla}_X n)V = \nabla^{\perp}_X nV - n(\nabla^{\perp}_X V). \tag{13}$$

From (2), we obtain

$$\overline{g}((\overline{\nabla}_X J)Y, Z) = \overline{g}(Y, (\overline{\nabla}_X J)Z), \tag{14}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$, which implies [21]

$$\overline{g}((\nabla_X T)Y, Z) = \overline{g}(Y, (\nabla_X T)Z), \quad \overline{g}((\overline{\nabla}_X N)Y, V) = \overline{g}((\nabla_X t)V, Y), \tag{15}$$

for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The analogue concept of locally product manifold is considered in the context of Golden geometry, having the name of *locally Golden manifold* [14]. Thus, we say that the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is *locally Golden* if J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$, i.e., $\overline{\nabla} J = 0$.

Remark 1. *Any almost product structure F on $\overline{M}$ induces two Golden structures on $\overline{M}$ [9]:*

$$J = \pm \frac{2\phi - 1}{2} F + \frac{1}{2} I, \tag{16}$$

where ϕ is the Golden number.

In addition, for an almost product structure F, the decompositions into the tangential and normal components of FX and FV are given by

$$FX = fX + \omega X, \quad FV = BV + CV, \tag{17}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $fX := (FX)^T$, $\omega X := (FX)^{\perp}$, $BV := (FV)^T$ and $CV := (FV)^{\perp}$.

Moreover, the maps f and C are $\overline{g}$-symmetric [22]:

$$\overline{g}(fX, Y) = \overline{g}(X, fY), \quad \overline{g}(CU, V) = \overline{g}(U, CV), \tag{18}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

Remark 2 ([11]). *If M is a submanifold in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$ and J is the Golden structure induced by F on $\overline{M}$, then for any $X \in \Gamma(TM)$, we have*

$$TX = \frac{1}{2} X \pm \frac{2\phi - 1}{2} fX, \quad NX = \pm \frac{2\phi - 1}{2} \omega X. \tag{19}$$

3. Pointwise Slant Submanifolds in Golden Riemannian Manifolds

We shall state the notion of pointwise slant submanifold in a Golden Riemannian manifold, following Chen's definition [23,24] of pointwise slant submanifold of an almost Hermitian manifold.

Definition 1. *A submanifold M of a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is called pointwise slant if, at each point $x \in M$, the angle $\theta_x(X)$ between JX and T_xM (called the Wirtinger angle) is independent of the choice of the nonzero tangent vector $X \in T_xM \setminus \{0\}$, but it depends on $x \in M$. The Wirtinger angle is a real-valued function θ (called the Wirtinger function), verifying*

$$\cos\theta_x = \frac{\overline{g}(JX, TX)}{\|JX\| \cdot \|TX\|} = \frac{\|TX\|}{\|JX\|}, \tag{20}$$

for any $x \in M$ and $X \in T_xM \setminus \{0\}$.

A pointwise slant submanifold of a Golden Riemannian manifold is called *slant submanifold* if its Wirtinger function θ is globally constant.

In a similar manner as in [23], we obtain

Proposition 1. *If M is an isometrically immersed submanifold in the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$, then M is a pointwise slant submanifold if and only if*

$$T^2 = (\cos^2\theta_x)(T + I), \tag{21}$$

for some real-valued function $x \mapsto \theta_x$, for $x \in M$.

From (8) and (21), we have

Proposition 2. *Let M be an isometrically immersed submanifold in the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. If M is a pointwise slant submanifold with the Wirtinger angle θ_x, then*

$$\overline{g}(NX, NY) = (\sin^2\theta_x)[\overline{g}(TX, Y) + \overline{g}(X, Y)], \tag{22}$$

$$tNX = (\sin^2\theta_x)(TX + X), \tag{23}$$

for any $X, Y \in T_xM \setminus \{0\}$ and any $x \in M$.

From (21), by a direct computation, we obtain

Proposition 3. *Let M be an isometrically immersed submanifold in the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. If M is a pointwise slant submanifold with the Wirtinger angle θ_x, then*

$$(\nabla_X T^2)Y = (\cos^2\theta_x)(\nabla_X T)Y - \sin(2\theta_x)X(\theta_x)(TY + Y), \tag{24}$$

for any $X, Y \in T_xM \setminus \{0\}$ and any $x \in M$.

4. Pointwise Bi-Slant Submanifolds in Golden Riemannian Manifolds

In this section, we introduce the notion of pointwise bi-slant submanifold in the Golden Riemannian context.

Definition 2. *Let M be an immersed submanifold in the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$. We say that M is a pointwise bi-slant submanifold of $\overline{M}$ if there exists a pair of orthogonal distributions D_1 and D_2 on M such that*

(i) $TM = D_1 \oplus D_2$;

(ii) $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$;

(iii) The distributions D_1 and D_2 are pointwise slant, with slant functions θ_{1x} and θ_{2x}, for $x \in M$.

The pair $\{\theta_1, \theta_2\}$ of slant functions is called the bi-slant function.

A pointwise bi-slant submanifold M is called proper if its bi-slant functions $\theta_1, \theta_2 \neq 0; \frac{\pi}{2}$ and both θ_1 and θ_2 are not constant on M.

In particular, if $\theta_1 = 0$ and $\theta_2 \neq 0; \frac{\pi}{2}$, then M is called *a pointwise semi-slant submanifold*; if $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0; \frac{\pi}{2}$, then M is called *a pointwise hemi-slant submanifold*.

If M is a *pointwise bi-slant submanifold* of $\overline{M}$, then the distributions D_1 and D_2 on M verify $T(D_1) \subseteq D_1$ and $T(D_2) \subseteq D_2$.

Now, we provide an example of a pointwise bi-slant submanifold in a Golden Riemannian manifold.

Example 1. *Let $\mathbb{R}^6$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$. Let $i : M \to \mathbb{R}^6$ be the immersion given by*

$$i(u, v) := (\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v, \sin v, \cos v),$$

where $M := \{(u, v) \mid u, v \in (0, \frac{\pi}{2})\}$.

We can find a local orthogonal frame on TM given by

$$Z_1 = -\sin u \cos v \frac{\partial}{\partial x_1} - \sin u \sin v \frac{\partial}{\partial x_2} + \cos u \cos v \frac{\partial}{\partial x_3} + \cos u \sin v \frac{\partial}{\partial x_4}$$

$$Z_2 = -\cos u \sin v \frac{\partial}{\partial x_1} + \cos u \cos v \frac{\partial}{\partial x_2} - \sin u \sin v \frac{\partial}{\partial x_3} + \sin u \cos v \frac{\partial}{\partial x_4} + \cos v \frac{\partial}{\partial x_5} - \sin v \frac{\partial}{\partial x_6}.$$

We define the Golden structure $J : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$J(X_1, X_2, X_3, X_4, X_5, X_6) := (\phi X_1, \overline{\phi} X_2, \phi X_3, \overline{\phi} X_4, \phi X_5, \overline{\phi} X_6),$$

where $\phi := \frac{1+\sqrt{5}}{2}$ is the Golden number and $\overline{\phi} = 1 - \phi$.

We remark that J verifies $J^2 = J + I$ and $\langle JX, Y \rangle = \langle X, JY \rangle$, for any $X, Y \in \mathbb{R}^6$. Additionally, we have

$$JZ_1 = -\phi \sin u \cos v \frac{\partial}{\partial x_1} - \overline{\phi} \sin u \sin v \frac{\partial}{\partial x_2} + \phi \cos u \cos v \frac{\partial}{\partial x_3} + \overline{\phi} \cos u \sin v \frac{\partial}{\partial x_4},$$

$$JZ_2 = -\phi \cos u \sin v \frac{\partial}{\partial x_1} + \overline{\phi} \cos u \cos v \frac{\partial}{\partial x_2} - \phi \sin u \sin v \frac{\partial}{\partial x_3} + {} + \overline{\phi} \sin u \cos v \frac{\partial}{\partial x_4} + \phi \cos v \frac{\partial}{\partial x_5} - \overline{\phi} \sin v \frac{\partial}{\partial x_6}.$$

We remark that $\langle JZ_1, Z_2 \rangle = \langle JZ_2, Z_1 \rangle = 0$, $\langle JZ_1, Z_1 \rangle = \phi \cos^2 v + \overline{\phi} \sin^2 v$ and $\langle JZ_2, Z_2 \rangle = 1$.

On the other hand, we get

$$\|Z_1\|^2 = 1, \ \ \|Z_2\|^2 = 2,$$

$$\|JZ_1\|^2 = \phi^2 \cos^2 v + \overline{\phi}^2 \sin^2 v = \phi \cos^2 v + \overline{\phi} \sin^2 v + 1, \ \ \|JZ_2\|^2 = \phi^2 + \overline{\phi}^2 = 3.$$

We denote by $D_1 := span\{Z_1\}$ the pointwise slant distribution with the slant angle θ_1, where $\cos \theta_1 = \frac{f(u,v)}{\sqrt{f(u,v)+1}}$, for $f(u, v) := \phi \cos^2 v + \overline{\phi} \sin^2 v$ a real function on M. In addition, we denote by $D_2 := span\{Z_2\}$ the slant distribution with the slant angle θ_2, where $\cos \theta_2 = \frac{1}{\sqrt{6}}$.

The distributions D_1 and D_2 satisfy the conditions from Definition 2.

If M_1 and M_2 are the integral manifolds of the distributions D_1 and D_2, respectively, then $M := M_1 \times_{\sqrt{2}} M_2$ with the Riemannian metric tensor

$$g := du^2 + 2dv^2$$

is a pointwise bi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$.

Example 2. *If, in Example 1, we consider that f is a Golden function (i.e., $f^2 = f + 1$), then $\cos\theta_1 = 1$, and we remark that M is a semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$, with the slant angle $\theta = \arccos \frac{1}{\sqrt{6}}$.*

Example 3. *On the other hand, if, in Example 2, we consider $f = 0$ (i.e., $\tan v = \pm\phi$), then $\cos\theta_1 = 0$, and we remark that M is a hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$, with the slant angle $\theta = \arccos \frac{1}{\sqrt{6}}$.*

If we denote by P_i the projections from TM onto D_i for $i \in \{1,2\}$, then $X = P_1X + P_2X$ for any $X \in \Gamma(TM)$. In particular, if $X \in D_i$, then $X = P_iX$, for $i \in \{1,2\}$.

If we denote by $T_i = P_i \circ T$ for $i \in \{1,2\}$, then, from (5), we obtain

$$JX = T_1X + T_2X + NX. \tag{25}$$

In a similar manner as in [24], we obtain

Lemma 1. *Let M be a pointwise bi-slant submanifold of a locally Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with pointwise slant distributions D_1 and D_2 and slant functions θ_1 and θ_2, respectively. Then*

(i) for any $X, Y \in D_1$ and $Z \in D_2$, we have

$$(\sin^2\theta_1 - \sin^2\theta_2)\overline{g}(\nabla_X Y, T_2Z + Z) \tag{26}$$

$$= \overline{g}(\nabla_X Y, T_2Z) + \overline{g}(\nabla_X Z, T_1Y) + (\cos^2\theta_1 + 1)\overline{g}(A_{NZ}Y + A_{NY}Z, X)$$
$$-\overline{g}(A_{NT_1Y}Z + A_{NT_2Z}Y, X) - \overline{g}(A_{NZ}T_1Y + A_{NY}T_2Z, X);$$

(ii) for any $X \in D_1$ and $Z, W \in D_2$, we have

$$(\sin^2\theta_2 - \sin^2\theta_1)\overline{g}(\nabla_Z W, T_1X + X) \tag{27}$$

$$= \overline{g}(\nabla_Z W, T_1X) + \overline{g}(\nabla_Z X, T_2W) + (\cos^2\theta_2 + 1)\overline{g}(A_{NX}W + A_{NW}X, Z)$$
$$-\overline{g}(A_{NT_2W}X + A_{NT_1X}W, Z) - \overline{g}(A_{NW}T_1X + A_{NX}T_2W, Z).$$

Proof. From (2), we have

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z) = \overline{g}(J^2\overline{\nabla}_X Y, Z) - \overline{g}(J\overline{\nabla}_X Y, Z), \tag{28}$$

for any $X, Y \in D_1$ and $Z \in D_2$.

By using (3) and $(\overline{\nabla}_X J)Y = 0$, we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X J^2 Y, Z) - \overline{g}(\overline{\nabla}_X JY, Z). \tag{29}$$

From (25), we obtain $JX = T_1X + NX$, $JY = T_1Y + NY$ and $JZ = T_2Z + NZ$ for any $X, Y \in D_1$ and $Z \in D_2$ and, from here, we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X JT_1Y, Z) + \overline{g}(\overline{\nabla}_X JNY, Z) - \overline{g}(\overline{\nabla}_X(T_1Y + NY), Z)$$

$$= \overline{g}(\overline{\nabla}_X T_1^2 Y, Z) + \overline{g}(\overline{\nabla}_X N T_1 Y, Z) + \overline{g}(\overline{\nabla}_X NY, JZ) - \overline{g}(\overline{\nabla}_X T_1 Y, Z)$$
$$+\overline{g}(A_{NY}X, Z)$$
$$= \overline{g}(\overline{\nabla}_X(\cos^2\theta_1(T_1Y + Y)), Z) - \overline{g}(A_{NT_1Y}X, Z) + \overline{g}(\overline{\nabla}_X NY, T_2Z + NZ)$$
$$+\overline{g}(T_1Y, \overline{\nabla}_X Z) + \overline{g}(A_{NY}X, Z).$$

Thus, we obtain

$$\overline{g}(\nabla_X Y, Z) = \cos^2\theta_1 \overline{g}(\overline{\nabla}_X(T_1Y + Y), Z) - \sin(2\theta_1)X(\theta_1)\overline{g}(T_1Y + Y, Z)$$
$$-\overline{g}(A_{NT_1Y}X, Z) - \overline{g}(A_{NY}X, T_2Z) - \overline{g}(\overline{\nabla}_X NZ, JY) + \overline{g}(\overline{\nabla}_X NZ, T_1Y)$$
$$+\overline{g}(T_1Y, \overline{\nabla}_X Z) + \overline{g}(A_{NY}X, Z).$$

By using $\overline{g}(T_1Y + Y, Z) = 0$, we obtain

$$\sin^2\theta_1 \overline{g}(\nabla_X Y, Z) = \cos^2\theta_1 \overline{g}(\overline{\nabla}_X T_1 Y, Z) - \overline{g}(A_{NT_1Y}Z + A_{NY}T_2Z, X)$$
$$+\overline{g}(JNZ, \overline{\nabla}_X Y) - \overline{g}(A_{NZ}X, T_1Y) + \overline{g}(T_1Y, \overline{\nabla}_X Z) + \overline{g}(A_{NY}Z, X).$$

By using (8) and (23), we find

$$\overline{g}(JNZ, \overline{\nabla}_X Y) = \overline{g}(tNZ + nNZ, \overline{\nabla}_X Y)$$
$$= \sin^2\theta_2 \overline{g}(\nabla_X Y, Z + T_2Z) + \overline{g}(NZ - NT_2Z, \overline{\nabla}_X Y)$$
$$= \sin^2\theta_2 \overline{g}(\nabla_X Y, Z) + \sin^2\theta_2 \overline{g}(\nabla_X Y, T_2Z)$$
$$-\overline{g}(\overline{\nabla}_X NZ, Y) + \overline{g}(\overline{\nabla}_X NT_2Z, Y)$$

and from

$$\overline{g}(\overline{\nabla}_X T_1 Y, Z) = -\overline{g}(JY - NY, \overline{\nabla}_X Z) = \overline{g}(\overline{\nabla}_X Y, JZ) - \overline{g}(\overline{\nabla}_X NY, Z)$$
$$= \overline{g}(\overline{\nabla}_X Y, T_2Z) - \overline{g}(Y, \overline{\nabla}_X NZ) - \overline{g}(\overline{\nabla}_X NY, Z)$$
$$= \overline{g}(\overline{\nabla}_X Y, T_2Z) + \overline{g}(Y, A_{NZ}X) + \overline{g}(A_{NY}X, Z)$$

we have

$$(\sin^2\theta_1 - \sin^2\theta_2)\overline{g}(\nabla_X Y, Z) = (1 - \sin^2\theta_1)\overline{g}(\nabla_X Y, T_2Z)$$
$$+ \cos^2\theta_1 \overline{g}(A_{NZ}Y + A_{NY}Z, X) + \sin^2\theta_2 \overline{g}(\nabla_X Y, T_2Z)$$
$$-\overline{g}(A_{NT_1Y}Z + A_{NY}T_2Z + A_{NZ}T_1Y + A_{NT_2Z}Y, X)$$
$$-\overline{g}(Y, \overline{\nabla}_X NZ) + \overline{g}(T_1Y, \nabla_X Z) + \overline{g}(A_{NY}Z, X)$$

and from here, we obtain (26).

In the same manner, we find (27). □

Lemma 2. *Let M be a pointwise semi-slant submanifold in a locally Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$, with pointwise slant distributions D_1 and D_2 having slant functions θ_1 and θ_2.*

(i) If the slant functions are $\theta_1 = 0$ and $\theta_2 = \theta$, we obtain

$$\sin^2\theta \overline{g}(\nabla_X Y, T_2Z + Z) = -\overline{g}(\nabla_X Y, T_2Z) - \overline{g}(\nabla_X Z, T_1Y) \quad (30)$$
$$-2\overline{g}(A_{NZ}Y, X) + \overline{g}(A_{NZ}T_1Y + A_{NT_2Z}Y, X),$$

for any $X, Y \in D^T$ and $Z \in D^\theta$, and

$$\sin^2\theta \overline{g}(\nabla_Z W, T_1X + X) = \overline{g}(\nabla_Z W, T_1X) + \overline{g}(\nabla_Z X, T_2W) \quad (31)$$
$$+(\cos^2\theta + 1)\overline{g}(A_{NW}X, Z) - \overline{g}(A_{NT_2W}X + A_{NW}T_1X, Z),$$

for any $X \in D^T$ *and* $Z, W \in D^\theta$.

(ii) If the slant functions are $\theta_1 = \theta$ *and* $\theta_2 = 0$, *we obtain*

$$\sin^2\theta \overline{g}(\nabla_X Y, T_2 Z + Z) = \overline{g}(\nabla_X Y, T_2 Z) - \overline{g}(\nabla_X T_1 Y, Z) \tag{32}$$

$$+(\cos^2\theta + 1)\overline{g}(A_{NY}X, Z) - \overline{g}(A_{NT_1Y}Z + A_{NT_2Z}Y, X) - \overline{g}(A_{NY}T_2 Z, X),$$

for any $X, Y \in D^\theta$ *and* $Z \in D^T$, *and*

$$\sin^2\theta \overline{g}(\nabla_Z W, T_1 X + X) = -\overline{g}(\nabla_Z W, T_1 X) + \overline{g}(\nabla_Z T_2 W, X) \tag{33}$$

$$-2\overline{g}(A_{NX}Z, W) + \overline{g}(A_{NT_2W}Z, X) + \overline{g}(A_{NT_1X}Z, W) + \overline{g}(A_{NX}T_2 W, Z),$$

for any $X \in D^\theta$ *and* $Z, W \in D^T$.

Lemma 3. *Let* M *be a pointwise hemi-slant submanifold in a locally Golden Riemannian manifold* $(\overline{M}, \overline{g}, J)$, *with pointwise slant distributions* D_1 *and* D_2 *having slant functions* θ_1 *and* θ_2.

(i) If the slant functions are $\theta_1 = \frac{\pi}{2}$ *and* $\theta_2 = \theta$, *we obtain*

$$\cos^2\theta \overline{g}(\nabla_X Y, T_2 Z + Z) = \overline{g}(\nabla_X Y, T_2 Z) \tag{34}$$

$$+\overline{g}(A_{NZ}Y + A_{NY}Z, X) - \overline{g}(A_{NT_2Z}Y + A_{NY}T_2 Z, X),$$

for any $X, Y \in D^\perp$ *and* $Z \in D^\theta$, *and*

$$\cos^2\theta \overline{g}(\nabla_Z W, X) = -\overline{g}(\nabla_Z X, T_2 W) \tag{35}$$

$$-(\cos^2\theta + 1)\overline{g}(A_{NX}W + A_{NW}X, Z) + \overline{g}(A_{NT_2W}X + A_{NX}T_2 W, Z),$$

for any $X \in D^\perp$ *and* $Z, W \in D^\theta$.

(ii) If the slant functions are $\theta_1 = \theta$ *and* $\theta_2 = \frac{\pi}{2}$, *we obtain*

$$\cos^2\theta \overline{g}(\nabla_X Y, Z) = -\overline{g}(\nabla_X Z, T_1 Y) \tag{36}$$

$$-(\cos^2\theta + 1)\overline{g}(A_{NZ}Y + A_{NY}Z, X) + \overline{g}(A_{NT_1Y}Z + A_{NZ}T_1 Y, X),$$

for any $X, Y \in D^\theta$ *and* $Z \in D^\perp$, *and*

$$\cos^2\theta \overline{g}(\nabla_Z W, T_1 X + X) = \overline{g}(\nabla_Z W, T_1 X) \tag{37}$$

$$+\overline{g}(A_{NX}W + A_{NW}X, Z) - \overline{g}(A_{NT_1X}W + A_{NW}T_1 X, Z),$$

for any $X \in D^\theta$ *and* $Z, W \in D^\perp$.

5. Warped Product Pointwise Bi-Slant Submanifolds in Golden Riemannian Manifolds

In [13], the authors of this paper introduced the Golden warped product Riemannian manifold and provided a necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden. Moreover, the subject was continued in the papers [14,25], where the authors characterized the metallic structure on the product of two metallic manifolds in terms of metallic maps and provided a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds (of dimensions $n_1 > 0$ and $n_2 > 0$, respectively) and let π_1, π_2 be the projection maps from the product manifold $M_1 \times M_2$ onto M_1 and M_2, respectively. We denote by $\widetilde{\varphi} := \varphi \circ \pi_1$ the lift to $M_1 \times M_2$ of a smooth function φ on M_1. Then, M_1 is called *the base*, and M_2 is called *the fiber* of $M_1 \times M_2$. The unique element $\widetilde{X}$ of $\Gamma(T(M_1 \times M_2))$ that is π_1-related to $X \in \Gamma(TM_1)$ and to the zero vector field on M_2 will be called the *horizontal lift of* X, and the unique element $\widetilde{V}$ of $\Gamma(T(M_1 \times M_2))$ that is π_2-related to $V \in \Gamma(TM_2)$ and to the zero vector field on M_1 will be called the *vertical lift of* V. We denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

For $f : M_1 \to (0, \infty)$, a smooth function on M_1, we consider the Riemannian metric g on $M := M_1 \times M_2$:

$$g := \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2. \tag{38}$$

Definition 3. *The product manifold M of M_1 and M_2 together with the Riemannian metric g is called the warped product of M_1 and M_2 by the warping function f [26].*

A warped product manifold $M := M_1 \times_f M_2$ is called trivial if the warping function f is constant. In this case, M is the Riemannian product $M_1 \times M_2$, where the manifold M_2 is equipped with the metric $f^2 g_2$ (which is homothetic to g_2) [4].

In the next considerations, we shall denote by $(f \circ \pi_1)^2 =: f^2$, $\pi_1^* g_1 =: g_1$ and $\pi_2^* g_2 =: g_2$, respectively.

Lemma 4 ([4])**.** *For $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, we have on $M := M_1 \times_f M_2$ that*

$$\nabla_X Z = \nabla_Z X = X(\ln f) Z, \tag{39}$$

where ∇ denotes the Levi-Civita connection on M.

The warped product $M_1 \times_f M_2$ of two pointwise slant submanifolds M_1 and M_2 in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is called a *warped product pointwise bi-slant submanifold*. Moreover, the pointwise bi-slant submanifold $M_1 \times_f M_2$ is called *proper* if both of the submanifolds M_1 and M_2 are proper pointwise slant in $(\overline{M}, \overline{g}, J)$.

Now, we provide an example of a warped product pointwise bi-slant submanifold in a Golden Riemannian manifold.

Example 4. *Let $\mathbb{R}^6$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$. Let $i : M \to \mathbb{R}^6$ be the immersion given by*

$$i(f, u) := (f \sin u, f \cos u, f, f \cos u, f \sin u, u),$$

where $M := \{(f, u) \mid f > 0, u \in (0, \frac{\pi}{2})\}$.

We can find a local orthogonal frame on TM given by

$$Z_1 = \sin u \frac{\partial}{\partial x_1} + \cos u \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \cos u \frac{\partial}{\partial x_4} + \sin u \frac{\partial}{\partial x_5}$$

$$Z_2 = f \cos u \frac{\partial}{\partial x_1} - f \sin u \frac{\partial}{\partial x_2} - f \sin u \frac{\partial}{\partial x_4} + f \cos u \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}.$$

We define the Golden structure $J : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$J(X_1, X_2, X_3, X_4, X_5, X_6) := (\phi X_1, \phi X_2, \phi X_3, \overline{\phi} X_4, \overline{\phi} X_5, \overline{\phi} X_6),$$

where ϕ is the Golden number and $\overline{\phi} = 1 - \phi$.

We remark that J verifies $J^2 = J + I$ and $\langle JX, Y \rangle = \langle X, JY \rangle$, for any $X, Y \in \mathbb{R}^6$. Additionally, we have

$$JZ_1 = \phi \sin u \frac{\partial}{\partial x_1} + \phi \cos u \frac{\partial}{\partial x_2} + \phi \frac{\partial}{\partial x_3} + \overline{\phi} \cos u \frac{\partial}{\partial x_4} + \overline{\phi} \sin u \frac{\partial}{\partial x_5}$$

$$JZ_2 = \phi f \cos u \frac{\partial}{\partial x_1} - \phi f \sin u \frac{\partial}{\partial x_2} - \overline{\phi} f \sin u \frac{\partial}{\partial x_4} + \overline{\phi} f \cos u \frac{\partial}{\partial x_5} + \overline{\phi} \frac{\partial}{\partial x_6}.$$

We remark that $\langle JZ_1, Z_2 \rangle = \langle JZ_2, Z_1 \rangle = 0$, $\langle JZ_1, Z_1 \rangle = 2\phi + \overline{\phi} = \phi + 1$ and $\langle JZ_2, Z_2 \rangle = f^2(\phi + \overline{\phi}) + \overline{\phi} = f^2 + 1 - \phi$.

On the other hand, we obtain

$$\|Z_1\|^2 = 3, \ \ \|Z_2\|^2 = 2f^2 + 1,$$

$$\|JZ_1\|^2 = 2\phi^2 + \overline{\phi}^2 = \phi + 4, \ \ \|JZ_2\|^2 = f^2(\phi^2 + \overline{\phi}^2) + \overline{\phi}^2 = 3f^2 + 2 - \phi.$$

We denote by $D_1 := span\{Z_1\}$ *the slant distribution with the slant angle* θ_1 *and* $D_2 := span\{Z_2\}$ *the pointwise slant distribution with the slant angle* θ_2*, where*

$$\cos\theta_1 = \frac{\phi + 1}{\sqrt{3(\phi + 4)}}$$

and

$$\cos\theta_2 = \frac{f^2 + 1 - \phi}{\sqrt{(2f^2 + 1)(3f^2 + 2 - \phi)}}.$$

The distributions D_1 *and* D_2 *are integrable and, if* M_1 *and* M_2 *are the integral manifolds of the distributions* D_1 *and* D_2*, respectively, then* $M := M_1 \times_{\sqrt{2f^2+1}} M_2$ *with the Riemannian metric tensor*

$$g := 3df^2 + (2f^2 + 1)du^2$$

satisfy the conditions of the warped product of M_1 *and* M_2 *by the warping function* $\sqrt{2f^2 + 1}$*. Thus, we obtain a warped product pointwise bi-slant submanifold in the Golden Riemannian manifold* $(\mathbb{R}^6, \langle\cdot,\cdot\rangle, J)$*.*

In a similar manner as in [25], we obtain

Proposition 4. *Let* $M := M_1 \times_f M_2$ *be a warped product submanifold in a locally Golden Riemannian manifold* $(\overline{M}, \overline{g}, J)$ *with warping function* f*. Then, for any* $X, Y \in \Gamma(TM_1)$ *and* $Z, W \in \Gamma(TM_2)$*, we have*

$$\overline{g}(h(X,Y), NZ) = -\overline{g}(h(X,Z), NY), \tag{40}$$

$$\overline{g}(h(X,Z), NW) = 0, \tag{41}$$

and

$$\overline{g}(h(Z,W), NX) = T_1X(\ln f)\overline{g}(Z,W) - X(\ln f)\overline{g}(Z, T_2W). \tag{42}$$

Proof. For any $X, Y \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, by using (3), (5), (10), (39) and $\overline{\nabla}J = 0$, we obtain

$$\begin{aligned}
\overline{g}(h(X,Y), NZ) &= \overline{g}(\overline{\nabla}_X Y, JZ) - \overline{g}(\overline{\nabla}_X Y, T_2Z) \\
&= \overline{g}(\overline{\nabla}_X T_1Y, Z) + \overline{g}(\overline{\nabla}_X NY, Z) + \overline{g}(\overline{\nabla}_X T_2Z, Y) \\
&= -\overline{g}(\nabla_X Z, T_1Y) - \overline{g}(A_{NY}X, Z) + \overline{g}(Y, \nabla_X T_2Z) \\
&= -X(\ln f)\overline{g}(T_1Y, Z) - \overline{g}(h(X,Z), NY) + X(\ln f)\overline{g}(Y, T_2Z).
\end{aligned}$$

On the other hand, $\overline{g}(T_1Y, Z) = \overline{g}(JY, Z) = \overline{g}(Y, JZ) = \overline{g}(Y, T_2Z)$. Thus, we obtain (40).

For any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, by using (3), (5), (10), (39) and $\overline{\nabla}J = 0$, we obtain

$$\begin{aligned}
\overline{g}(h(X,Z), NW) &= \overline{g}(\overline{\nabla}_X Z, JW) - \overline{g}(\overline{\nabla}_X Z, T_2W) \\
&= \overline{g}(\nabla_X T_2Z, W) - \overline{g}(A_{NZ}X, W) - \overline{g}(\nabla_X Z, T_2W) \\
&= X(\ln f)[\overline{g}(T_2Z, W) - \overline{g}(Z, T_2W)] - \overline{g}(h(X,W), NZ)
\end{aligned}$$

and using

$$\overline{g}(T_2Z, W) - \overline{g}(Z, T_2W) = \overline{g}(JZ, W) - \overline{g}(Z, JW) = 0,$$

we obtain

$$\overline{g}(h(X,Z), NW) = -\overline{g}(h(X,W), NZ). \tag{43}$$

On the other hand, after interchanging Z by X, we have

$$\overline{g}(h(Z,X),NW) = \overline{g}(\nabla_Z T_1 X, W) - \overline{g}(A_{NX}Z, W) - \overline{g}(\nabla_Z X, T_2 W)$$

$$= T_1X(\ln f)\overline{g}(Z,W) - X(\ln f)\overline{g}(Z,T_2W) - \overline{g}(h(Z,W),NX) = \overline{g}(h(X,W),NZ)$$

and using (43), we obtain (41).

For any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, by using (3), (5), (10), (39) and $\overline{\nabla}J = 0$, we obtain

$$\begin{aligned}\overline{g}(h(Z,W),NX) &= \overline{g}(\overline{\nabla}_Z W, JX) - \overline{g}(\overline{\nabla}_Z W, T_1 X)\\ &= \overline{g}(\nabla_Z T_2 W, X) + \overline{g}(\overline{\nabla}_Z NW, X) - \overline{g}(\nabla_Z W, T_1 X)\\ &= -\overline{g}(T_2 W, \nabla_Z X) - \overline{g}(A_{NW}Z, X) + \overline{g}(W, \nabla_Z T_1 X)\\ &= -X(\ln f)\overline{g}(Z, T_2 W) + T_1 X(\ln f)\overline{g}(Z,W)\end{aligned}$$

and we obtain (42). □

6. Warped Product Pointwise Semi-Slant or Hemi-Slant Submanifolds in Golden Riemannian Manifolds

In this section, we obtain some properties of the distributions in the case of pointwise semi-slant and pointwise hemi-slant submanifolds in locally Golden Riemannian manifolds.

Definition 4. *Let $M := M_1 \times_f M_2$ be a warped product bi-slant submanifold in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ such that one of the components M_i ($i \in \{1,2\}$) is an invariant submanifold (respectively, anti-invariant submanifold) in $\overline{M}$ and the other one is a pointwise slant submanifold in $\overline{M}$, with the Wirtinger angle $\theta_x \in [0, \frac{\pi}{2}]$. Then, we call the submanifold M warped product pointwise semi-slant submanifold (respectively, warped product pointwise hemi-slant submanifold) in the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$.*

Now, we provide an example of a warped product semi-slant submanifold in a Golden Riemannian manifold.

Example 5. *Let $\mathbb{R}^7$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$ and consider the immersion $i : M \to \mathbb{R}^7$, given by*

$$i(f,u,v) := (f\cos u, f\sin u, f\cos v, f\sin v, f, u, v),$$

where $M := \{(f,u,v) \mid f > 0, u, v \in [0, \frac{\pi}{2}]\}$.

The local orthogonal frame on TM is given by

$$Z_1 = \cos u \frac{\partial}{\partial x_1} + \sin u \frac{\partial}{\partial x_2} + \cos v \frac{\partial}{\partial x_3} + \sin v \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_1},$$

$$Z_2 = -f\sin u \frac{\partial}{\partial x_1} + f\cos u \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2},$$

$$Z_3 = -f\sin v \frac{\partial}{\partial x_3} + f\cos v \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_3}.$$

We define the Golden structure $J : \mathbb{R}^7 \to \mathbb{R}^7$ by

$$J\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}\right) := \left(\phi\frac{\partial}{\partial x_1}, \phi\frac{\partial}{\partial x_2}, \overline{\phi}\frac{\partial}{\partial x_3}, \overline{\phi}\frac{\partial}{\partial x_4}, \overline{\phi}\frac{\partial}{\partial y_1}, \phi\frac{\partial}{\partial y_2}, \overline{\phi}\frac{\partial}{\partial y_3}\right),$$

for $k \in \{1,2,3,4\}$ and $l \in \{1,2,3\}$, where ϕ is the Golden number and $\overline{\phi} = 1 - \phi$. Since

$$JZ_1 = \phi \cos u \frac{\partial}{\partial x_1} + \phi \sin u \frac{\partial}{\partial x_2} + \overline{\phi} \cos v \frac{\partial}{\partial x_3} + \overline{\phi} \sin v \frac{\partial}{\partial x_4} + \overline{\phi} \frac{\partial}{\partial y_1},$$

$$JZ_2 = \phi Z_2, \quad JZ_3 = \overline{\phi} Z_3,$$

we remark that $\langle JZ_k, Z_l \rangle = 0$, for any $k \neq l$, where $k, l \in \{1,2,3\}$, and $\langle JZ_1, Z_1 \rangle = \phi + 2\overline{\phi} = 2 - \phi$.

We find that

$$\|Z_1\|^2 = 3, \quad \|Z_2\|^2 = \|Z_3\|^2 = f^2 + 1,$$

$$\|JZ_1\|^2 = \phi^2 + 2\overline{\phi}^2 = 5 - \phi, \ \|JZ_2\|^2 = \phi^2(f^2+1), \ \|JZ_3\|^2 = \overline{\phi}^2(f^2+1).$$

Denote by $D_1 := span\{Z_1\}$ the slant distribution with the slant angle θ, where $\cos\theta = \frac{2-\phi}{\sqrt{3(5-\phi)}}$ and by $D_2 := span\{Z_2, Z_3\}$ the invariant distribution (with respect to J).

If M_θ and M_T are the integral manifolds of the distributions D_1 and D_2, respectively, then $M := M_\theta \times_{\sqrt{f^2+1}} M_T$ with the metric

$$g := 3df^2 + (f^2+1)(du^2 + dv^2) = g_{M_\theta} + (f^2+1) g_{M_T}$$

is a warped product semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^7, \langle \cdot, \cdot \rangle, J)$.

Now, we provide an example of a warped product hemi-slant submanifold in a Golden Riemannian manifold.

Example 6. *Let $\mathbb{R}^5$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$ and consider the immersion $i : M \to \mathbb{R}^5$, given by*

$$i(f,u) := (f \sin u, f \cos u, \phi f \sin u, \phi f \cos u, -f),$$

where $M := \{(f,u) \mid f > 0, u \in (0, \frac{\pi}{2})\}$ and ϕ is the Golden number.

The local orthogonal frame on TM is given by

$$Z_1 = \sin u \frac{\partial}{\partial x_1} + \cos u \frac{\partial}{\partial x_2} + \phi \sin u \frac{\partial}{\partial x_3} + \phi \cos u \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5},$$

$$Z_2 = f \cos u \frac{\partial}{\partial x_1} - f \sin u \frac{\partial}{\partial x_2} + \phi f \cos u \frac{\partial}{\partial x_3} - \phi f \sin u \frac{\partial}{\partial x_4}.$$

We define the Golden structure $J : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$J(X_1, X_2, X_3, X_4, X_5) := (\phi X_1, \phi X_2, \overline{\phi} X_3, \overline{\phi} X_4, \phi X_5),$$

where $\overline{\phi} = 1 - \phi$. Since

$$JZ_1 = \phi\left(\sin u \frac{\partial}{\partial x_1} + \cos u \frac{\partial}{\partial x_2}\right) - \left(\sin u \frac{\partial}{\partial x_3} + \cos u \frac{\partial}{\partial x_4}\right) - \phi \frac{\partial}{\partial x_5},$$

$$JZ_2 = \phi f\left(\cos u \frac{\partial}{\partial x_1} - \sin u \frac{\partial}{\partial x_2}\right) - f\left(\cos u \frac{\partial}{\partial x_3} - \sin u \frac{\partial}{\partial x_4}\right),$$

we remark that $\langle JZ_2, Z_k \rangle = 0$, for any $k \in \{1,2\}$, and $\langle JZ_1, Z_1 \rangle = \phi$.

We find that

$$\|Z_1\|^2 = \phi^2 + 2 = \phi + 3, \quad \|Z_2\|^2 = f^2(\phi^2+1) = f^2(\phi+2),$$

$$\|JZ_1\|^2 = 2\phi^2 + 1 = 2\phi + 3, \quad \|JZ_2\|^2 = f^2(\phi^2+1) = f^2(\phi+2).$$

Denote by $D_1 := span\{Z_1\}$ *the slant distribution with the slant angle* θ*, where* $\cos\theta = \frac{\phi}{\sqrt{(2\phi+3)(\phi+3)}}$ *and by* $D_2 := span\{Z_2\}$ *the anti-invariant distribution (with respect to* J*).*

If M_θ *and* $M_\perp$ *are the integral manifolds of the distributions* D_1 *and* D_2*, respectively, then* $M := M_\theta \times_{f\sqrt{\phi+2}} M_\perp$ *with the metric*

$$g := (\phi+3)df^2 + f^2(\phi+2)du^2 = g_{M_\theta} + f^2(\phi+2)g_{M_\perp}$$

is a warped product hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^5, \langle\cdot,\cdot\rangle, J)$*.*

In a similar manner as in Theorem 2 from [25], we obtain

Theorem 1. *If* $M := M_T \times_f M_\theta$ *is a warped product pointwise semi-slant submanifold in a locally Golden Riemannian manifold* $(\overline{M}, \overline{g}, J)$ *with the pointwise slant angle* $\theta_x \in (0, \frac{\pi}{2})$*, for* $x \in M_\theta$*, then the warping function* f *is constant on the connected components of* M_T*.*

Proof. For any $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_\theta) \setminus \{0\}$, by using (10) in $\overline{\nabla}_Z JX = J\overline{\nabla}_Z X$ and (39), we obtain

$$TX(\ln f)Z + h(TX, Z) = T\nabla_Z X + N\nabla_Z X + th(X, Z) + nh(X, Z).$$

From the equality of the normal components of the last equation, it follows

$$h(TX, Z) = X(\ln f)NZ + nh(X, Z) \tag{44}$$

and replacing X with $TX = JX$ (for $X \in \Gamma(TM_T)$) in (44), we obtain

$$h(J^2X, Z) = TX(\ln f)NZ + nh(TX, Z).$$

Thus, we obtain

$$TX(\ln f)\overline{g}(NZ, NZ) = \overline{g}(h(J^2X, Z), NZ) - \overline{g}(nh(TX, Z), NZ)$$

$$= \overline{g}(h(TX, Z), NZ) + \overline{g}(h(X, Z), NZ) - \overline{g}(nh(TX, Z), NZ),$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$.

From (41), we have $\overline{g}(h(TX, Z), NZ) = \overline{g}(h(X, Z), NZ) = 0$, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$ and by using (22), we obtain

$$TX(\ln f)\sin^2\theta[\overline{g}(TZ, Z) + \overline{g}(Z, Z)] = -\overline{g}(nh(TX, Z), NZ). \tag{45}$$

On the other hand, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, we have $TX \in \Gamma(TM_T)$ and $TZ \in \Gamma(TM_\theta)$, and from (41), we obtain

$$\overline{g}(h(TX, Z), NZ) = \overline{g}(h(TX, Z), NTZ) = 0.$$

Thus, by using (2) and (6), we have

$$\overline{g}(nh(TX, Z), NZ) = \overline{g}(h(TX, Z), nNZ) = \overline{g}(h(TX, Z), J^2Z - JTZ)$$

$$= \overline{g}(h(TX, Z), NZ) + \overline{g}(h(TX, Z), Z) - \overline{g}(h(TX, Z), NTZ) = 0$$

and using (45), we obtain

$$TX(\ln f)\tan^2\theta_x \overline{g}(TZ, TZ) = 0,$$

for any $Z \in \Gamma(TM_\theta)$ and $x \in M_\theta$.

Since $\theta_x \in (0, \frac{\pi}{2})$ and $TZ \neq 0$, we get $TX(\ln f) = 0$, for any $X \in \Gamma(TM_T)$, which implies that the warping function f is constant on the connected components of M_T. □

Theorem 2. *Let $M := M_\theta \times_f M_T$ be a warped product pointwise semi-slant submanifold in a locally Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the pointwise slant angle $\theta_x \in (0, \frac{\pi}{2})$, for $x \in M_\theta$. Then*

$$(A_{NT_1Y}X - A_{NT_1X}Y) \in \Gamma(TM_\theta),$$

for any $X, Y \in \Gamma(TM_\theta)$.

Proof. For any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_T) \setminus \{0\}$, from (32) and the symmetry of the shape operator, we have

$$\sin^2\theta \overline{g}([X,Y], T_2^2 Z) = \overline{g}([X,Y], T_2 Z) - \overline{g}(\nabla_X T_1 Y - \nabla_Y T_1 X, Z)$$
$$+(\cos^2\theta + 1)[\overline{g}(h(X,Z), NY) - \overline{g}(h(Y,Z), NX)] - \overline{g}(h(X,Z), NT_1 Y)$$
$$+\overline{g}(h(Y,Z), NT_1 X) + \overline{g}(h(X,Y), NT_2 Z) - \overline{g}(h(Y,X), NT_2 Z)$$
$$-\overline{g}(h(X, T_2 Z), NY) + \overline{g}(h(Y, T_2 Z), NX).$$

Using (3) and (40), we obtain

$$\overline{g}(\nabla_X T_1 Y - \nabla_Y T_1 X, Z) = \overline{g}(\nabla_X JY - \nabla_Y NX - \nabla_Y JX + \nabla_Y NX, Z)$$
$$= \overline{g}(\nabla_X Y, JZ) - \overline{g}(\nabla_Y X, JZ) + \overline{g}(A_{NY}X, Z) - \overline{g}(A_{NX}Y, Z)$$
$$= \overline{g}([X,Y], JZ) + \overline{g}(h(X,Z), NY) - \overline{g}(h(Z,Y), NX) = \overline{g}([X,Y], T_2 Z).$$

From (40), we obtain

$$\overline{g}(h(X,Z), NY) = \overline{g}(h(Y,Z), NX) = -\overline{g}(h(X,Y), NZ).$$

Thus, using the symmetry of the shape operator, we have

$$\overline{g}(h(X, T_2 Z), NY) - \overline{g}(h(Y, T_2 Z), NX) = -\overline{g}(h(X,Y) NT_2 Z) + \overline{g}(h(Y,X), NT_2 Z) = 0$$

and

$$\overline{g}(h(X,Z), NT_1 Y) - \overline{g}(h(Y,Z), NT_1 X) = \overline{g}(A_{NT_1Y}X - A_{NT_1X}Y, Z).$$

Thus, we obtain

$$0 = \sin^2\theta \overline{g}([X,Y], T_2^2 Z) = \overline{g}(A_{NT_1Y}X - A_{NT_1X}Y, Z).$$

□

A similar result valid for warped product hemi-slant submanifolds in a locally metallic Riemannian manifold [25], which can be proved following the same steps, holds in our setting, too.

Theorem 3. *If $M := M_\perp \times_f M_\theta$ (or $M := M_\theta \times_f M_\perp$) is a warped product pointwise hemi-slant submanifold in a locally Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the pointwise slant angle $\theta_x \in (0, \frac{\pi}{2})$, for $x \in M_\theta$, then the warping function f is constant on the connected components of $M_\perp$ if and only if*

$$A_{NZ}X = A_{NX}Z, \tag{46}$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$ (or $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, respectively).

Author Contributions: Both authors have contributed to the conceptualization and investigation. Both authors have read and agreed to the final version of the manuscript.

Funding: The APC was funded by Stefan cel Mare University of Suceava, Romania.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest. The funder had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Chen, B.-Y. Geometry of warped products as Riemannian submanifolds and related problems. *Soochow J. Math.* **2020**, *28*, 125–156.
2. Chen, B.-Y. Geometry of warped product CR-Submanifolds in a Kaehler manifold. *Monatsh. Math.* **2001**, *133*, 177–195. [CrossRef]
3. Chen, B.-Y. Geometry of warped product CR-Submanifolds in a Kaehler manifold II. *Monatsh. Math.* **2001**, *134*, 103–119. [CrossRef]
4. Chen, B.-Y. *Differential Geometry of Warped Product Manifolds and Submanifolds*; World Scientific: Singapore, 2017.
5. Hretcanu, C.E.; Blaga, A.M. Submanifolds in metallic Riemannian manifolds. *Differ. Geom. Dyn. Syst.* **2018**, *20*, 83–97.
6. Hretcanu, C.E.; Crasmareanu, M. Metallic structures on Riemannian manifolds. *Rev. Unión Mat. Argent.* **2013**, *54*, 15–27.
7. Crasmareanu, M.; Hretcanu, C.E. Golden differential geometry. *Chaos Solitons Fractals* **2008**, *38*, 1229–1238. [CrossRef]
8. Hretcanu, C.E.; Crasmareanu, M.C. Applications of the Golden ratio on Riemannian manifolds. *Turkish J. Math.* **2009**, *33*, 179–191.
9. Hretcanu, C.E.; Crasmareanu, M. On some invariant submanifolds in a Riemannian manifold with Golden structure. *An. Stiint. Univ. Cuza Iasi Mat. (N. S.)* **2007**, *53* (Suppl. 1), 199–211.
10. Blaga, A.M.; Hretcanu, C.E. Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. *Novi Sad J. Math.* **2018**, *48*, 55–80. [CrossRef]
11. Hretcanu, C.E.; Blaga, A.M. Slant and semi-slant submanifolds in metallic Riemannian manifolds. *J. Funct. Spaces* **2018**, *2018*, 2864263. [CrossRef]
12. Hretcanu, C.E.; Blaga, A.M. Hemi-slant submanifolds in metallic Riemannian manifolds. *Carpathian J. Math.* **2019**, *35*, 59–68. [CrossRef]
13. Blaga, A.M.; Hretcanu, C.E. Golden warped product Riemannian manifolds. *Lib. Math.* **2017**, *37*, 39–49.
14. Blaga, A.M.; Hretcanu, C.E. Remarks on metallic warped product manifolds. *Facta Univ.* **2018**, *33*, 269–277.
15. Erdogan, F.E. Transversal Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds. *Turkish J. Math.* **2018**, *42*, 3133–3148. [CrossRef]
16. Erdogan, F.E.; Perktaş, S.Y.; Acet, B.E.; Blaga, A.M. Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds. *J. Geom. Phys.* **2019**, *142*, 111–120. [CrossRef]
17. Perktaş, S.Y.; Erdogan, F.E.; Acet, B.E. Lightlike submanifolds of metallic semi-Riemannian manifolds. *Filomat* **2020**, *34*, 1781–1794.
18. De Spinadel, V.W. The metallic means family and forbidden symmetries. *Int. Math. J.* **2002**, *2*, 279–288.
19. Goldberg, S.I.; Petridis, N.C. Differentiable solutions of algebraic equations on manifolds. *Kodai Math. Sem. Rep.* **1973**, *25*, 111–128. [CrossRef]
20. Goldberg, S.I.; Yano, K. Polynomial structures on manifolds. *Kodai Math. Sem. Rep.* **1970**, *22*, 199–218. [CrossRef]
21. Blaga, A.M.; Hretcanu, C.E. Metallic conjugate connections. *Rev. Unión Mat. Argent.* **2018**, *59*, 179–192. [CrossRef]
22. Li, H.; Liu, X.; Semi-slant submanifolds of a locally product manifold. *Georgian Math. J.* **2005**, *12*, 273–282. [CrossRef]
23. Chen, B.-Y.; Garay, O. Pointwise slant submanifolds in almost Hermitian manifolds. *Turkish J. Math.* **2012**, *36*, 630–640.
24. Chen, B.-Y.; Uddin, S. Warped product pointwise bi-slant submanifolds of Kaehler manifolds. *Publ. Math. Debr.* **2018**, 1–17. In-print. [CrossRef]
25. Hretcanu, C.E.; Blaga, A.M. Warped product submanifolds in metallic Riemannian manifolds. *Tamkang J. Math.* **2020**, *51*, 161–186. [CrossRef]
26. Bishop, R.L.; O'Neill, B. Manifolds of negative curvature. *Trans. Am. Math. Soc.* **1969**, *145*, 1–49. [CrossRef]