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Positive Solutions for a Singular Elliptic Equation Arising in a Theory of Thermal Explosion

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Abstract: In this paper, the thermal explosion model described by a nonlinear boundary value problem is studied. Firstly, we prove the comparison principle under nonlinear boundary conditions. Secondly, using the sub-super solution theorem, we prove the existence of a positive solution for the case $K(x) > 0$, as well as the monotonicity of the maximal solution on parameter λ. Thirdly, the uniqueness of the solution for $K(x) < 0$ is proved, as well as the monotonicity of the solutions on parameter λ. Finally, we obtain some new results for the existence of solutions, and the dependence on the λ for the case $K(x)$ is sign-changing.

Keywords: model of thermal explosion; uniqueness; subsolution and supersolution; the comparison principle

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1. Introduction

In this paper, we study the following problem

$$\begin{cases} -\triangle u + K(x)u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the constant $\alpha \in (0,1)$, $p \in (0,1)$, $g : [0,\infty) \to (0,\infty)$ is a nondecreasing C^1 function, Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and $\lambda > 0$ is a parameter.

The problem (1) is related to the stationary analogue of the equation

$$\begin{cases} u_t - \triangle u = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0, & x \in \Omega, \end{cases} \tag{2}$$

which is a classical problem of combustion theory, see [1–4]. Here, u is the appropriately scaled temperature in a bounded smooth domain Ω in $\mathbb{R}^2$, and $f(t,x)$ is the normalized reaction rate.

In [2], Gordon, Ko and Shivaji considered the following problem

$$\begin{cases} -\triangle u = \lambda f(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

using the method of sub-supersolutions by which they showed that the solution of this problem is unique for large and small values of parameter λ, whereas for intermediate values of λ, solutions are multiple, provided that the nonlinear term f satisfies

Hypothesis 1 (H1). *$f : [0,+\infty) \to (0,+\infty)$ is a C^1 nondecreasing function, and*

Hypothesis 2 (H2). $\lim_{s\to+\infty} \frac{f(s)}{s} = 0.$

Using variational methods (see [5]), Ko and Prashanth considered the following problem

$$\begin{cases} -\triangle u = \lambda e^{u^\alpha}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & \text{on } \partial\Omega, \end{cases} \quad (4)$$

and showed that there exists $0 < \Lambda < +\infty$, and that the problem (5) has at least two positive solutions if $0 < \lambda < \Lambda$, no solution if $\lambda > \Lambda$, and at least one positive solution when $\lambda = \Lambda$. In [6], Rasouli considered the following system

$$\begin{cases} -\triangle u = \lambda f(v), & \text{in } \Omega, \\ -\triangle v = \lambda g(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + a(u)u = 0, & \text{on } \partial\Omega, \\ \boldsymbol{n} \cdot \nabla v + b(v)v = 0, & \text{on } \partial\Omega, \end{cases} \quad (5)$$

and established some existence and multiplicity results via the method of sub-supersolutions if nonlinearity f and g satisfies

Hypothesis 3 (H3). *$f, g : [0, +\infty) \to (0, +\infty)$ are nondecreasing function, and*

Hypothesis 4 (H4). $\lim_{s\to+\infty} \frac{f(Ag(s))}{s} = 0$ *for all $A > 0$.*

Another interesting work comes from [7], in which Shi and Yao considered the following problem

$$\begin{cases} -\triangle u + K(x)u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (6)$$

and obtained the existence, uniqueness, regularity and the dependency on parameters of the positive solutions under various assumptions for $K(x)$.

Notice that the condition that nonlinearity f (or f and g) is nondecreasing is very important in [2,5,6]. However, in [7], $f(x, u) = -K(x)u^{-\alpha} + \lambda u^p$ is nonlinear and lacks monotonicity as regards u. One natural question is whether problem (1) (note $f(x, u) = -K(x)u^{-\alpha} + \lambda u^p$) has the existence, as well as the uniqueness and the dependency, on parameters of the positive solutions under various assumptions for $K(x)$. This paper is devoted to answering the above question.

Throughout this paper, we always assume that $K \in C^{2,\beta}(\overline{\Omega})$ and the heat-loss parameter $g(u)$ satisfies the following hypothesis:

Hypothesis 5 (H5). *$g : [0, \infty) \to (0, \infty)$ is a $C^{1,\beta}$ nondecreasing bounded function with $\beta \in (0, 1)$ and satisfies*

$$\inf_{u\in[0,+\infty)} g(u) = g(0) = g_0 > 0.$$

Here, $C^{1,\beta}$ means that, for any x, y satisfies $|g'(x) - g'(y)| \le C|x - y|^\beta$.

The rest of the paper is organized as follows. In Section 2, we cite some lemmas. Due to the change of boundary conditions in the process of the citation, some theorems cannot be directly cited, so we provide the proof of this part of lemma. In Section 3, we record the conclusions of the paper and provide the relevant proofs.

2. Preliminaries

In this section, we will list and prove some lemmas. In order to obtain our results, we consider the following problems

$$\begin{cases} -\triangle u = f(x,u), & \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla u + g(u)u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$, g is satisfying (H5).

The definitions of the sub-supersolutions of problem (7) are listed as follows.

Definition 1. *$\overline{\Omega}$ is the closure of the set Ω. A function $\overline{u} : \overline{\Omega} \to \mathbb{R}$ is called a supersolution of problem (7) if $\overline{u} \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ and*

$$\begin{cases} -\triangle u \geq f(x,u), & \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla u + g(u)u \geq 0, & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Definition 2. *A function $\underline{u} : \overline{\Omega} \to \mathbb{R}$ is called a subsolution of (7) if $\underline{u} \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ and*

$$\begin{cases} -\triangle u \leq f(x,u), & \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla u + g(u)u \leq 0, & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Now, we point out the following lemma.

Lemma 1 ([8])**.** *Assume f is continuous on $\overline{\Omega} \times \mathbb{R}$, $\frac{\partial f}{\partial u}$ is continous and g satisfies condition $(H5)$. If $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution of problem (7) with $\underline{u} \leq \overline{u}$, then the problem (7) has at least one solution u in the order interval*

$$\underline{u} \leq u \leq \overline{u}, \quad on\ \overline{\Omega}.$$

Moreover, problem (7) has a minimal solution $u_{\min}$ and a maximal $u_{\max}$ in $[\underline{u}, \overline{u}]$.

In order to compare the supersolution and subsolution more conveniently, we list the following lemmas.

Lemma 2. *Let $w_1, w_2 \in C^{2,\beta}(\Omega) \bigcap C^{1,\beta}(\overline{\Omega})$ satisfy $-\triangle w_1 \leq -\triangle w_2$, in Ω, $\boldsymbol{n}\cdot\nabla w_1 + g(w_1)w_1 \leq \boldsymbol{n}\cdot\nabla w_2 + g(w_2)w_2$, on $\partial\Omega$. Then, $w_1 \leq w_2$ in $\overline{\Omega}$.*

Lemma 3. *Let $f : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ be a continuous function such that $\dfrac{f(x,s)}{s}$ is strictly decreasing for $s \in (0,\infty)$ at each $x \in \Omega$. Let $w, v \in \overline{u} \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$ satisfy:*

(a) *$\triangle w + f(x,w) \leq 0 \leq \triangle v + f(x,v)$ in Ω;*

(b) *$w, v > 0$ in Ω and $\boldsymbol{n}\cdot\nabla v + g(v)v \leq c \leq \boldsymbol{n}\cdot\nabla w + g(w)w$, on $\partial\Omega$ where c is a nonnegative constant;*

(c) *$\triangle v \in L^1(\Omega)$.*

Then, $v(x) \leq w(x)$ in Ω. If one "$\leq$" in the above condition (a) *is replaced by "$<$", we have $v(x) < w(x)$ for all $x \in \Omega$.*

Proof of Lemma 3. We prove this lemma by contradiction. If $v \leq w$ is not true, then there exist $\varepsilon_0, \delta_0 > 0$ and a ball $B \subset\subset \Omega$ such that

$$v(x) - w(x) > \varepsilon_0,\ x \in B$$

and

$$\int_B vw\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)dx > \delta_0. \tag{10}$$

Here, $\subset\subset$ means strict inclusion. We know that $\frac{f(x,w)}{w} - \frac{f(x,v)}{v}$ because $\frac{f(x,s)}{s}$ is strictly decreasing. Due to $w, v > 0$ in Ω, it is true to obtain (10).

Let

$$M = \max\left\{1,\ \|\triangle v\|_{L^1(\Omega)}\right\}$$

and

$$\varepsilon = \min\left\{1,\ \varepsilon_0,\ \frac{\delta_0}{4M}\right\}.$$

Let ψ be a smooth function on $\mathbb{R}$ such that $\psi(t) = 0$ if $t \leq \frac{1}{2}$, $\psi(t) = 1$ if $t \geq 1$, $\psi(t) \in (0,1)$ if $t \in \left(\frac{1}{2}, 1\right)$ and $\psi'(t) \geq 0$ for $t \in \mathbb{R}$. For $\varepsilon > 0$, we define the function $\psi_\varepsilon(t)$ by

$$\psi_\varepsilon(t) = \psi\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

It then follows from (a) and the fact that $\psi_\varepsilon(t) \geq 0$ for $t \in \mathbb{R}$ that

$$(w\triangle v - v\triangle w)\psi_\varepsilon(v-w) \geq vw\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)\psi_\varepsilon(v-w), \quad \text{in } \Omega.$$

By the continuity of w, v and ψ_ε, set up a subdomain Ω^*

$$\Omega^* = \{x | v(x) > w(x), x \in \Omega\},$$

with a smooth boundary. It is easy to see that $vw\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right) > 0$, in Ω^*/B.

Therefore,

$$\begin{aligned}
&\int_{\Omega^*} (w\triangle v - v\triangle w)\psi_\varepsilon(v-w)dx \\
&\geq \int_{\Omega^*} \big(w(-f(x,v)) + vf(x,w)\big)\psi_\varepsilon(v-w)dx \\
&= \int_{\Omega^*} wv\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)\psi_\varepsilon(v-w)dx \\
&= \int_{\Omega^*/B} wv\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)\psi_\varepsilon(v-w)dx \\
&+ \int_B wv\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)\psi_\varepsilon(v-w)dx \\
&\geq \int_B wv\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)dx > \delta_0.
\end{aligned} \tag{11}$$

Define

$$\Psi_\varepsilon(t) = \int_0^t s\psi_\varepsilon'(s)ds, \quad t \in \mathbb{R},$$

then it is easy to verify that

$$0 \leq \Psi_\varepsilon(t) \leq 2\varepsilon,\ t \in \mathbb{R} \quad \text{and} \quad \Psi_\varepsilon(t) = 0,\ if\ t < \frac{\varepsilon}{2}. \tag{12}$$

On one hand, if $B \subset \Omega^* \subset\subset \Omega$, we have $v(x) = w(x)$ for all $x \in \partial\Omega^*$. Hence, the following result can be obtained through the divergence theorem

$$\begin{aligned}
&\int_{\Omega^*}(w\triangle v - v\triangle w)\psi_\varepsilon(v-w)dx \\
&= \int_{\partial\Omega^*} w\psi_\varepsilon(v-w)\cdot \boldsymbol{n}\cdot\nabla v ds - \int_{\Omega^*}\nabla v\nabla w\psi_\varepsilon(v-w)dx \\
&- \int_{\Omega^*} w\psi_\varepsilon'(v-w)\nabla v(\nabla v-\nabla w)dx - \int_{\partial\Omega^*} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds \\
&+ \int_{\Omega^*}\nabla v\nabla w\psi_\varepsilon(v-w)dx + \int_{\Omega^*} v\psi_\varepsilon'(v-w)\nabla w(\nabla v-\nabla w)dx \\
&= \int_{\Omega^*} v\psi_\varepsilon'(v-w)(\nabla v-\nabla w)(\nabla w-\nabla v)dx \\
&+ \int_{\Omega^*}(v-w)\psi_\varepsilon'(v-w)(\nabla v-\nabla w)\nabla v dx \leq \int_{\Omega^*}\nabla v\nabla(\Psi_\varepsilon(v-w))dx \\
&= \int_{\partial\Omega^*}\Psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds - \int_{\Omega^*}\Psi_\varepsilon(v-w)\triangle v dx < \frac{\delta_0}{2}.
\end{aligned}$$

This is a contradiction to (11). Thus, $v \leq w$ is true.

On the other hand, if $B \subset \Omega^*$ and $\partial\Omega^* = S_1 \cup S_2$ where $S_1 = \partial\Omega^* \cap \partial\Omega$ and $S_2 = \partial\Omega^* \subset\subset \Omega$, then $v(x) = w(x)$ holds in S_2 and

$$\begin{aligned}
&\int_{\partial\Omega^*} w\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds \\
&= \int_{S_1} w\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds + \int_{S_2} w\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds \\
&= \int_{S_1} w\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\int_{\partial\Omega^*} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds \\
&= \int_{S_1} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds + \int_{S_2} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds \\
&= \int_{S_1} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds.
\end{aligned} \tag{14}$$

From (13) and (14), we have

$$\begin{aligned}
&\int_{\Omega^*}(w\triangle v - v\triangle w)\psi_\varepsilon(v-w)dx \\
&= \int_{S_1} w\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds - \int_{\Omega^*}\nabla v\nabla w\psi_\varepsilon(v-w)dx \\
&- \int_{\Omega^*} w\psi_\varepsilon'(v-w)\nabla v(\nabla v-\nabla w)dx - \int_{S_1} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds \\
&+ \int_{\Omega^*}\nabla v\nabla w\psi_\varepsilon(v-w)dx + \int_{\Omega^*} v\psi_\varepsilon'(v-w)\nabla w(\nabla v-\nabla w)dx \\
&= \int_{\Omega^*} v\psi_\varepsilon'(v-w)(\nabla v-\nabla w)(\nabla w-\nabla v)dx \\
&+ \int_{\Omega^*}(v-w)\psi_\varepsilon'(\nabla v-\nabla w)(v-w)\nabla v dx \\
&+ \int_{S_1} w\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds - \int_{S_1} v\psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla w ds \\
&\leq \int_{\Omega^*}\nabla v\nabla(\Psi_\varepsilon(v-w))dx \\
&+ \int_{S_1}\psi_\varepsilon(v-w)(w\cdot\boldsymbol{n}\cdot\nabla v - v\cdot\boldsymbol{n}\cdot\nabla w)dx \\
&= \int_{\partial\Omega^*}\Psi_\varepsilon(v-w)\cdot\boldsymbol{n}\cdot\nabla v ds - \int_{\Omega^*}\Psi_\varepsilon(v-w)\triangle v dx \\
&+ \int_{S_1}\psi_\varepsilon(v-w)(w\cdot\boldsymbol{n}\cdot\nabla v - v\cdot\boldsymbol{n}\cdot\nabla w)dx \\
&< \frac{\delta_0}{2} + \int_{S_1}\psi_\varepsilon(v-w)(w\cdot\boldsymbol{n}\cdot\nabla v - v\cdot\boldsymbol{n}\cdot\nabla w)dx.
\end{aligned} \tag{15}$$

From condition (b) and the property of S_1, we have $\boldsymbol{n} \cdot \nabla v + g(v)v \leq c \leq \boldsymbol{n} \cdot \nabla w + g(w)w$ and $v(x) \geq w(x)$ on $\partial\Omega$ for $S_1 \subset \partial\Omega$, which together with $(H5)$ implies that

$$\big(\boldsymbol{n} \cdot \nabla v + g(v)v\big) \cdot w \leq cw, \ \big(\boldsymbol{n} \cdot \nabla w + g(w)w\big) \cdot v \geq cv \geq cw, \ \ x \in S_1$$

and

$$g(v(x)) \geq g(w(x)), \ \ x \in S_1.$$

Hence,

$$\begin{aligned}
&\int_{S_1} \psi_\varepsilon(v-w)\big(w \cdot \boldsymbol{n} \cdot \nabla v - v \cdot \boldsymbol{n} \cdot \nabla w\big)dx \\
&\leq \int_{S_1} \psi_\varepsilon(v-w)\big(cw - g(v)vw + g(w)wv - cw\big)dx \\
&= \int_{S_1} \psi_\varepsilon(v-w)\big(-g(v)vw + g(w)wv\big)dx \\
&= \int_{S_1} \psi_\varepsilon(v-w)vw\big(g(w) - g(v)\big)dx \\
&\leq \int_{S_1} vw\big(g(w) - g(v)\big)dx \leq 0.
\end{aligned} \tag{16}$$

From (15) and (16), we have

$$\begin{aligned}
&\int_{\Omega^*} (w\triangle v - v\triangle w)\psi_\varepsilon(v-w)dx \\
&< \frac{\delta_0}{2} + \int_{S_1} \psi_\varepsilon(v-w)(w \cdot \boldsymbol{n} \cdot \nabla v - v \cdot \boldsymbol{n} \cdot \nabla w)dx \\
&\leq \frac{\delta_0}{2} + 0 = \frac{\delta_0}{2}.
\end{aligned}$$

This is a contradiction with (11).

Consequently, $v \leq w$ in Ω.

If one "$\leq$" in condition (a) is replaced by "$<$", we show that $v(x) < w(x)$ for all $x \in \Omega$.

In fact, suppose that there exists a $x_0 \in \Omega$ with $v(x_0) = w(x_0)$. Choose $r_0 > 0$ small enough $B(x_0, r_0) \subset \Omega$. From $v(x) \leq w(x)$ in $B(x_0, r_0)$, one has $w(x_0) - v(x_0) = \min_{x \in B(x_0, r_0)}(w(x) - v(x))$, which implies $\triangle(w(x_0) - v(x_0)) \geq 0$. On the other hand, from condition (a) and $f(x_0, w(x_0)) = f(x_0, v(x_0))$, we have $\triangle(w(x_0) - v(x_0)) < 0$. This is a contradiction.

The proof is completed. □

Now, we list some well-known results.

The following problem (see [9])

$$\begin{cases} -\triangle u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{17}$$

has the smallest eigenvalue λ_1 with the corresponding eigenfunction $\varphi_1 \in C^{2+\beta}(\bar{\Omega})$.

In the following, we present the existence and uniqueness of the positive solutions of the following problem

$$\begin{cases} -\triangle u = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = b \geq 0, & \text{on } \partial\Omega, p \in (0,1). \end{cases} \tag{18}$$

Lemma 4. *Let $p \in (0,1)$, $\lambda > 0$. Then, the boundary value problem (18) has a unique positive solution $u^* \in C^{2,\beta}(\Omega) \cap C^1(\overline{\Omega})$.*

Proof. We start with a proof of existence of a positive solution of (18).

First, let e be the unique solution of

$$\begin{cases} -\triangle u = 1, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g_0 u = b & \text{on } \partial\Omega, \end{cases} \tag{19}$$

where $g(0) = g_0 > 0$. Define $\overline{u} = Me$, where $M \geq \max\{1, (\lambda e_0^p)^{\frac{1}{1-p}}\}$ is a positive constant. Here, $e_0 = \max_{x \in \overline{\Omega}} e(x)$. It is easy to see that

$$\overline{u} > 0, \quad \text{in} \quad \Omega$$

and

$$\begin{aligned} -\triangle \overline{u} &= M(-\triangle e) = M \geq \lambda \overline{u}^p, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla \overline{u} + g(\overline{u})\overline{u} &\geq \boldsymbol{n} \cdot \nabla Me + g_0(Me) = M\big(\boldsymbol{n} \cdot \nabla e + g_0 e\big) \geq b, & \text{on } \partial\Omega, \end{aligned}$$

that is, $\overline{u} = Me$ satisfies

$$\begin{cases} -\triangle \overline{u} \geq \lambda \overline{u}^p, & \text{in } \Omega, \\ \overline{u} > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla \overline{u} + g(\overline{u})\overline{u} \geq b, & \text{on } \partial\Omega, \end{cases}$$

which guarantees that $\overline{u} = Me$ is the supersolution of (18).

Second, let us now show that there is a positive subsolution $\underline{u}$ for problem (18). We now choose $\underline{u} = \varepsilon\varphi_1$, where φ_1 is given by the previous (17) and ε is small enough such that

$$\underline{u}(x) < \overline{u}(x), \text{ and } \lambda_1(\varepsilon\varphi_1)^{1-p} < \lambda, \text{ in } \Omega.$$

Thus,

$$-\triangle \underline{u} = -\triangle(\varepsilon\varphi_1) = \varepsilon(-\triangle\varphi_1) = \lambda_1 \varepsilon\varphi_1 = \lambda_1(\varepsilon\varphi_1)^{1-p}(\varepsilon\varphi_1)^p < \lambda \underline{u}^p, \quad \text{in } \Omega.$$

Moreover, by Hopf's maximum principle, $\boldsymbol{n} \cdot \nabla\varphi_1 < 0$, we have

$$\boldsymbol{n} \cdot \nabla \underline{u} + g(\underline{u})\underline{u} = \varepsilon \boldsymbol{n} \cdot \nabla\varphi_1 + 0 < 0, \quad \text{on } \partial\Omega.$$

Hence, $\underline{u} = \varepsilon\varphi_1$ satisfies

$$\begin{cases} -\triangle \underline{u} \leq \lambda \underline{u}^p, & \text{in } \Omega, \\ \underline{u} > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla \underline{u} + g(\underline{u})\underline{u} \leq b, & \text{on } \partial\Omega, \end{cases}$$

which implies that $\underline{u}$ is indeed a positive subsolution of (18).

Therefore, by Lemma 1, we know that (18) has a positive solution.

Next, let us show that the positive solution of problem (18) is unique.

Let $u_1, u_2 > 0$ be solutions of (18), satisfying $u_1 \neq u_2$. We have

$$\triangle u_1 + \lambda u_1^p = 0 = \triangle u_2 + \lambda u_2^p, \quad \text{in } \Omega,$$

and

$$\boldsymbol{n} \cdot \nabla u_1 + g(u_1)u_1 = b = \boldsymbol{n} \cdot \nabla u_2 + g(u_2)u_2, \quad \text{on } \partial\Omega$$

where $\dfrac{\lambda s^p}{s}$ is strictly decreasing. By Lemma 3, we obtain $u_1 \leq u_2$ in Ω, and $u_2 \leq u_1$ in Ω. Now, we obtain $u_1 = u_2$, in Ω, the uniqueness is proved.

The proof is completed. □

If $g(u) \equiv g_0$ for all $u \in [0, +\infty)$, we have a direct conclusion as follows.

Lemma 5. *Let $p \in (0,1)$, $\lambda > 0$. Then, the following problem*

$$\begin{cases} -\triangle u = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla u + g_0 u = b \geq 0, & \text{on } \partial\Omega \end{cases} \tag{20}$$

has a unique positive solution $u^{**} \in C^{2,\beta}(\Omega) \cap C(\overline{\Omega})$, *where* $g_0 = g(0)$.

Lemma 6. *There exists a* $\delta_\lambda > 0$ *such that*

$$u^*(x) \geq \delta_\lambda, u^{**}(x) \geq \delta_\lambda, \ \ \forall x \in \overline{\Omega}, \tag{21}$$

where u^* *and* u^{**} *are the solutions of problem* (18) *and problem* (20), *respectively.*

Proof. Let $\delta_{1,\lambda} := \min_{x \in \overline{\Omega}} u^*(x)$. If $\delta_{1,\lambda} = 0$, there exists a $x_0 \in \partial\Omega$ such that $u^*(x_0) = 0$. By the strong maximum theorem, one has $\boldsymbol{n} \cdot \nabla u^*(x_0) < 0$, which implies that

$$\boldsymbol{n} \cdot \nabla u^*(x_0) + g(u^*(x_0))u^*(x_0) < 0, \ \ x \in \partial\Omega. \tag{22}$$

This is a contradiction. Thus, $\delta_{1,\lambda} > 0$.

Let $\delta_{2,\lambda} := \min_{x \in \overline{\Omega}} u^{**}(x)$. The same argument shows that $\delta_{2,\lambda} > 0$. Set $\delta_\lambda = \min\{\delta_{1,\lambda}, \delta_{2,\lambda}\}$.

The proof is completed. □

3. Main Theorems

Now, we list our main results in the following.

Theorem 1. *Assume that* $K(x) > 0$ *for all* $x \in \overline{\Omega}$, $0 < \alpha < 1$ *and* $0 < p < 1$. *Then, there exists* $\bar{\lambda} \in (0, +\infty)$ *such that*

(i) (1) *has at least one positive solution* $u_\lambda \in C(\bar{\Omega}) \cap C^{2+\beta}(\Omega)$ *for all* $\lambda > \bar{\lambda}$;

(ii) (1) *has no positive solution in* $C(\bar{\Omega}) \cap C^2(\Omega)$ *for all* $\lambda < \bar{\lambda}$;

(iii) For $\lambda > \bar{\lambda}$, (1) *has a maximal solution* $\bar{u}_\lambda$, *and* $\bar{u}_\lambda$ *is increasing with respect to* λ.

Theorem 2. *Assume that* $K(x) < 0$ *for all* $x \in \overline{\Omega}$, $0 < \alpha < 1$ *and* $0 < p < 1$. *Then,*

(i) (1) *has a unique solution in* $C(\bar{\Omega}) \cap C^2(\Omega)$ *for any* $\lambda > 0$;

(ii) u_λ *is increasing with respect to* λ.

Theorem 3. *Assume that* $K(x)$ *is a sign changing function and* $0 < \alpha < 1, 0 < p < 1$. *Then, there exists a* $\lambda' > 0$ *such that*

(i) (1) *has at least one positive solution in* $C(\bar{\Omega}) \cap C^2(\Omega)$ *for any* $\lambda > \lambda'$;

(ii) For $\lambda_2 > \lambda_1 > \lambda'$, *there exists solutions* u_{λ_1} *and* u_{λ_2} *satisfying problem* (1) *for* $\lambda = \lambda_1$ *and* $\lambda = \lambda_2$, *respectively, such that* $u_{\lambda_1} \leq u_{\lambda_2}$.

In our proof, the following result is needed in [7].

Lemma 7. *There exists a* $\lambda_* > 0$ *such that problem* (6) *has at least one positive solution* $u_\lambda \in C^{1,\gamma}(\overline{\Omega})$, *where* $\gamma = 1 - \alpha$ *when* $K(x) > 0$ *and* $\lambda > \lambda_*$.

We now present the proofs of our main theorems.

The Proof of Theorem 1. (i) We show that there exists a $\bar{\lambda} > 0$ such that problem (1) has at least one positive solution $u_\lambda \in C(\bar{\Omega}) \cap C^{2+\beta}(\Omega)$ for all $\lambda > \bar{\lambda}$ and has no positive solution for $\lambda < \bar{\lambda}$.

First, we consider the following problem

$$\begin{cases} -\triangle u + K(x)u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = \frac{1}{k}, & \text{on } \partial\Omega \end{cases} \tag{23}$$

and claim that problem (23) has at least one positive solution for each $k \geq 1$ and $\lambda > \lambda_*$ where λ_* is defined in Lemma 7.

$$\begin{cases} -\triangle u + K(x)(\max\{\frac{1}{k}, u\})^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla u + g(u)u = \frac{1}{k}, & \text{on } \partial\Omega. \end{cases} \tag{24}$$

For $\lambda > \lambda_*$ and $k = 1$, let $\overline{u}_1 = u_1^* \in C^{2,\beta}(\overline{\Omega})$ be the positive solution of the problem (18) for $b = 1$. Then, we have

$$-\triangle\overline{u}_1 + K(x)(\overline{u}_1)^{-\alpha} = -\triangle u_1^* + K(x)(u_1^*)^{-\alpha} > \lambda(u_1^*)^p = \lambda(\overline{u}_1)^p, \quad \text{in } \Omega$$

and

$$\boldsymbol{n}\cdot\nabla\overline{u}_1 + g(\overline{u}_1)\overline{u}_1 = \boldsymbol{n}\cdot\nabla(u_1^*) + g(u_1^*)(u_1^*) = b = 1, \quad \text{on } \partial\Omega,$$

which guarantees $\overline{u}_1$ is a supersolution of problem (23) for $k = 1$.

Since $\lim_{s\to+0} g(s)s = 0 < \frac{1}{k}$ for each $k > 0$, choose a decreasing sequence $\{s_k > 0\}$ such that $g(s_k)s_k < \frac{1}{k}$, $k = 1, 2, \cdots$. Let $\underline{u}_1 = \underline{u} + s_1$ where $\underline{u}$ be the positive solution of problem (6) in Lemma 7 and $g(s_1)s_1 < 1$. Since $\underline{u} \in C^{1,\beta}(\overline{\Omega})$ with $\underline{u}_1 = \underline{u}(x) + s_1 > 0$ for all $x \in \Omega$, one has that $\boldsymbol{n}\cdot\nabla\underline{u}_1 = \boldsymbol{n}\cdot\nabla\underline{u} = \boldsymbol{n}\cdot\nabla\underline{u} \le 0$ for all $x \in \partial\Omega$, which, together with $\underline{u}(x) = 0$ for all $x \in \partial\Omega$, implies that

$$\begin{aligned} \boldsymbol{n}\cdot\nabla\underline{u}_1 + g(\underline{u}_1)\underline{u}_1 &= \boldsymbol{n}\cdot\nabla(\underline{u} + s_1) + g(\underline{u} + s_1)(\underline{u} + s_1) \\ &= \boldsymbol{n}\cdot\nabla\underline{u} + g(s_1)s_1 \\ &\le 1, \end{aligned}$$

for all $x \in \partial\Omega$. Moreover, we have

$$\begin{aligned} -\triangle\underline{u}_1 + K(x)\underline{u}_1^{-\alpha} &= -\triangle(\underline{u} + s_1) + K(x)(\underline{u} + s_1)^{-\alpha} \\ &\le -\triangle\underline{u} + K(x)\underline{u}^{-\alpha} \\ &= \lambda\underline{u}^p, \quad \text{in } \Omega. \end{aligned} \tag{25}$$

Thus, $\underline{u}_1$ is a subsolution of problem (23) for $k = 1$.

The above proof shows that

$$\begin{aligned} &\triangle\underline{u}_1 + \lambda\underline{u}_1^p \ge 0 = \triangle\overline{u}_1 + \lambda(\overline{u}_1)^p, \text{ in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\underline{u}_1 + g(\underline{u}_1)\underline{u}_1 \le 1 = \boldsymbol{n}\cdot\nabla\overline{u}_1 + g(\overline{u}_1)\overline{u}_1, \text{ on } \partial\Omega. \end{aligned}$$

Since $\dfrac{\lambda s^p}{s} = \lambda s^{p-1}$ is strictly decreasing for $s \in (0, \infty)$, it follows from Lemma 3 that

$$0 < \underline{u}_1(x) \le \overline{u}_1(x), \text{ for } x \in \overline{\Omega}.$$

Hence, Lemma 1 guarantees that problem (23) has at least one solution u_1 satisfying

$$\underline{u}_1 \le u_1 \le \overline{u}_1, \quad \text{on } \overline{\Omega}, \tag{26}$$

for all $\lambda > \lambda_*$.

Let $\overline{u}_2 = u_1$ be defined in (26). Obviously,

$$-\triangle\overline{u}_2 + K(x)(\overline{u}_2)^{-\alpha} = -\triangle u_1 + K(x)u_1^{-\alpha} = \lambda u_1^p = \lambda\overline{u}_2^p, \quad \text{in } \Omega$$

and

$$\boldsymbol{n}\cdot\nabla\overline{u}_2 + g(\overline{u}_2)\overline{u}_2 = \boldsymbol{n}\cdot\nabla u_1 + g(u_1)u_1 = 1 \ge \frac{1}{2}, \quad \text{on } \partial\Omega,$$

which guarantees $\overline{u}_2$ is a supersolution of problem (23) for $k = 2$.

A similar argument in (25) shows that $\underline{u}_2 = \underline{u} + s_2$ is a subsolution of problem (23) for $k = 2$. Now, Lemma 1 guarantees that problem (23) has at least one solution u_2 satisfying

$$\underline{u}_2(x) \le u_2(x) \le \overline{u}_2(x) = u_1(x), \quad \text{on } \overline{\Omega},$$

for all $\lambda > \lambda_*$.

Since

$$\lim_{k\to+\infty} \underline{u}_k(x) = \underline{u}(x), x \in \overline{\Omega}.$$

For each $k \geq 1$, problem (23) has at least one positive solution u_k for all $\lambda > \lambda_*$ with

$$\underline{u}(x) \leq \cdots \leq u_k(x) \leq \cdots \leq u_1(x), \quad x \in \overline{\Omega}. \tag{27}$$

Second, we consider the property of $\{u_k\}$ defined in (27).

The monotonicity and boundedness of $\{u_k\}$ guarantee that there exists a positive solution u such that

$$\lim_{k\to+\infty} u_k(x) = u(x) \geq \underline{u}(x), x \in \overline{\Omega}.$$

A standard argument in [10] shows that for each bounded $\Omega_1 \subseteq \Omega$ with $C^{1,\beta}$-boundary,

$$\lim_{k\to+\infty} u_k(x) = u(x), \text{ uniformly for all } x \in \overline{\Omega}_1,$$

which implies that $u \in C^2(\Omega)$ satisfies that

$$-\triangle u + K(x)u^{-\alpha} = \lambda u^p, \quad x \in \Omega. \tag{28}$$

Moreover, from the boundedness of $\{u_k(x)\}$ on $\partial\Omega$ and $\boldsymbol{n} \cdot \nabla u_k + g(u_k)u_k = \dfrac{1}{k}$ on $\partial\Omega$, we know that $\{\boldsymbol{n} \cdot \nabla u_k\}$ are bounded on $\partial\Omega$, which, together with $\lim_{k\to+\infty} u_k(x) = u(x)$ on $\partial\Omega$, guarantees that

$$\lim_{k\to+\infty} \boldsymbol{n} \cdot \nabla u_k(x) = \boldsymbol{n} \cdot \nabla u(x) \text{ and } \lim_{k\to+\infty} \frac{1}{k} = 0, \text{ uniformly on } \partial\Omega.$$

Thus,

$$\boldsymbol{n} \cdot \nabla u(x) + g(u(x))u(x) = 0, \text{ on } \partial\Omega. \tag{29}$$

Combined (28) and (29), we know that u is a positive solution of problem (1) for $\lambda > \lambda_*$.

Finally, we present the existence of $\overline{\lambda}$.

Set

$$\vartheta := \{\lambda > 0 \mid \text{problem (1) has at least one positive solution } u_\lambda\}$$

and

$$\overline{\lambda} := inf \vartheta.$$

Now, we show that if $\lambda_0 > \overline{\lambda}$, then $[\lambda_0, +\infty) \subset \vartheta$. Since $\lambda_0 > \overline{\lambda}$, there exists a $\lambda' \in \vartheta$ with $\lambda' < \lambda_0$.

For $\lambda \geq \lambda_0$, we consider the existence of positive solutions of the following problem

$$\begin{cases} -\triangle u + K(x)(u)^{-\alpha} = \lambda u^p, & in\ \Omega, \\ u > 0, & in\ \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & on\ \partial\Omega. \end{cases} \tag{30}$$

Observe the following problem

$$\begin{cases} -\triangle u + K(x)u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = \dfrac{1}{k}, & \text{on } \partial\Omega, \end{cases} \tag{31}$$

It is easy to see that $u_{\lambda'}$ is a subsolution of problem (31) for each $k \geq 1$, where $u_{\lambda'}$ satisfies

$$\begin{cases} -\triangle u + K(x)(u)^{-\alpha} = \lambda' u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & \text{on } \partial\Omega. \end{cases}$$

By the same process as we constructed the supersolutions of problem (23), we can obtain the supersolutions of problem (31) for each $k \geq 1$.

A standard argument as we discuss problem (23) shows that problem (31) has at least one positive solution for each $k \geq 1$, and we can obtain that problem (30) has at least one positive solution for $\lambda \in [\lambda_0, +\infty)$, i.e., $[\lambda_0, +\infty) \subseteq \vartheta$.

Moreover, by the definition of $\overline{\lambda}$, problem (23) has no positive solution for all $\lambda < \overline{\lambda}$.

(ii) We show that problem (1) has a maximal solution $\overline{u}_\lambda$ for all $\lambda > \overline{\lambda}$ and $\overline{u}_{\lambda_1} < \overline{u}_{\lambda_2}$ for $\lambda_2 > \lambda_1 > \overline{\lambda}$.

First, we prove problem (1) has a maximal solution $\overline{u}_\lambda$ for all $\lambda > \overline{\lambda}$.

Let $\{w_i\}$ be a solution sequence of the following problem

$$\begin{cases} -\triangle v + K(x)w_{i-1}^{-\alpha} = \lambda w_{i-1}^p, & \text{in } \Omega, \\ \boldsymbol{n}\cdot\nabla v + g(v)v = \dfrac{1}{i}, & \text{on } \partial\Omega, \end{cases} \tag{32}$$

for $i = 1, 2, \cdots$, where w_0 is defined in (20) for above $\lambda > \overline{\lambda}$ and $b = 1$.

We claim that for any positive solution u_λ of (1), we have

$$u_\lambda \le \cdots\cdots \le w_i \le w_{i-1} \le \cdots\cdots \le w_0. \tag{33}$$

In fact, it is easy to see that

$$\triangle u_\lambda + \lambda u_\lambda^p = K(x)u_\lambda^{-\alpha} \ge 0 = \triangle w_0 + \lambda w_0^p, \quad \text{in } \Omega$$

and

$$\boldsymbol{n}\cdot\nabla u_\lambda + g(u_\lambda)u_\lambda = 0 \le 1 = \boldsymbol{n}\cdot\nabla w_0 + g_0 w_0 \le \boldsymbol{n}\cdot\nabla w_0 + g(w_0)w_0, \quad \text{on } \partial\Omega.$$

From Lemma 3, we have $u_\lambda < w_0$.

From (32) and (20), we have

$$\triangle w_1 + \lambda w_0^p = K(x)w_0^{-\alpha} > 0 = \triangle w_0 + \lambda w_0^p, \quad \text{in } \Omega$$

and

$$\boldsymbol{n}\cdot\nabla w_1 + g(w_1)w_1 = 1 = \boldsymbol{n}\cdot\nabla w_0 + g_0 w_0 \le \boldsymbol{n}\cdot\nabla w_0 + g(w_0)w_0, \quad \text{on } \partial\Omega,$$

which, together with Lemma 2, implies that $w_1 \le w_0$, in Ω.

In addition, for any $s \ge 0$, since $\left(\lambda s^p - K(x)s^{-\alpha}\right)' = \lambda \cdot p \cdot s^{p-1} + \alpha K(x)s^{-\alpha-1} > 0$ and the function $\lambda s^p - K(x)s^{-\alpha}$ is an increasing function, we have

$$\begin{cases} \lambda w_1^p - K(x)w_1^{-\alpha} \le \lambda w_0^p - K(x)w_0^{-\alpha}, & \text{in } \Omega, \\ \lambda u_\lambda^p - K(x)u_\lambda^{-\alpha} \le \lambda w_0^p - K(x)w_0^{-\alpha}, & \text{in } \Omega, \end{cases}$$

which, together with

$$-\triangle w_2 = \lambda w_1^p - K(x)w_1^{-\alpha}, \lambda w_0^p - K(x)w_0^{-\alpha} = -\triangle w_1$$

and

$$-\triangle u_\lambda = \lambda u_\lambda^p - K(x)u_\lambda^{-\alpha}, \lambda w_0^p - K(x)w_0^{-\alpha} = -\triangle w_1,$$

implies that $-\triangle w_2 \le -\triangle w_1$ and $-\triangle u_\lambda \le -\triangle w_1$, in Ω. Moreover, for $x \in \partial\Omega$, we have

$$\boldsymbol{n}\cdot\nabla w_1 + g(w_1)w_1 = 1 \ge \frac{1}{2} \ge \boldsymbol{n}\cdot\nabla w_2 + g(w_2)w_2$$

and

$$\boldsymbol{n}\cdot\nabla w_1 + g(w_1)w_1 = 1 \ge 0 = \boldsymbol{n}\cdot\nabla u_\lambda + g(u_\lambda)u_\lambda.$$

Hence, Lemma 2 guarantees that $w_2 \le w_1$ and $u_\lambda \le w_1$ in $\overline{\Omega}$.

Repeating the previous steps, we know that (33) is true.

Using the same proof as (28) and (29), we obtain a $\overline{u}_\lambda$ defined by

$$\overline{u}_\lambda(x) = \lim_{i\to+\infty} w_i(x) \tag{34}$$

and $\overline{u}_\lambda$ is a solution of (1) and for any u_λ such that $\overline{u}_\lambda \ge u_\lambda$. Therefore, $\overline{u}_\lambda$ is a maximal solution of problem (1) for $\lambda > \overline{\lambda}$.

Second, we show that $\overline{u}_{\lambda_1} \le \overline{u}_{\lambda_2}$, where $\overline{u}_{\lambda_1}$ and $\overline{u}_{\lambda_2}$ are the corresponding maximal solution of problem (1) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

Let w_{λ_2} be the solution of the

$$\begin{cases} -\triangle u = \lambda_2 u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g_0 u = 1, & \text{on } \partial\Omega. \end{cases} \tag{35}$$

Let v_j be a sequence of solutions to the following problem

$$\begin{cases} -\triangle v + K(x) v_{j-1}^{-\alpha} = \lambda_2 v_{j-1}^p, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla v + g(v) v = \dfrac{1}{j}, & \text{on } \partial\Omega, \end{cases} \tag{36}$$

for $j = 1, 2, \cdots$, with $v_0 = w_{\lambda_2}$.

By the proof process of (34), we obtain

$$\begin{cases} \overline{u}_{\lambda_2} \le \cdots \le v_j \le v_{j-1} \le \cdots \le v_0, \\ \lim\limits_{j \to +\infty} v_j(x) = \overline{u}_{\lambda_2}. \end{cases} \tag{37}$$

Moreover, from (35) and (36), we have

$$\boldsymbol{n} \cdot \nabla \overline{u}_{\lambda_1} + g(\overline{u}_{\lambda_1}) \overline{u}_{\lambda_1} = 0 \le 1 = \boldsymbol{n} \cdot \nabla v_0 + g_0 v_0 \le \boldsymbol{n} \cdot \nabla v_0 + g(v_0) v_0, \quad \text{on } \partial\Omega$$

and from

$$\begin{aligned} \triangle \overline{u}_{\lambda_1} + \lambda_1 \overline{u}_{\lambda_1}^p &= K(x)(\overline{u}_{\lambda_1})^{-\alpha} \\ &> 0 \\ &= \triangle v_0 + \lambda_2 (v_0)^p \\ &\ge \triangle v_0 + \lambda_1 (v_0)^p, \quad \text{in } \Omega, \end{aligned}$$

we obtain $\triangle v_0 + \lambda_1 v_0^p \le 0 \le \triangle \overline{u}_{\lambda_1} + \lambda_1 \overline{u}_{\lambda_1}^p$, in Ω. Now, Lemma 3 implies that $\overline{u}_{\lambda_1} \le v_0$, in Ω. The monotonicity of $\lambda_1 s^p - K(x) s^{-\alpha}$ guarantees that $\lambda_1 \overline{u}_{\lambda_1}^p - K(x) \overline{u}_{\lambda_1}^{-\alpha} \le \lambda_1 v_0^p - K(x) v_0^{-\alpha}$. From

$$-\triangle \overline{u}_{\lambda_1} = \lambda_1 \overline{u}_{\lambda_1}^p - K(x)(\overline{u}_{\lambda_1})^{-\alpha} \le \lambda_1 v_0^p - K(x) v_0^{-\alpha} = -\triangle v_1, \quad \text{in } \Omega$$

and

$$\boldsymbol{n} \cdot \nabla \overline{u}_{\lambda_1} + g(\overline{u}_{\lambda_1}) \overline{u}_{\lambda_1} = 0 < 1 = \boldsymbol{n} \cdot \nabla v_1 + g(v_1) v_1, \quad \text{on } \partial\Omega,$$

we obtain $-\triangle \overline{u}_{\lambda_1} \le -\triangle v_1$, in Ω, which implies that $\overline{u}_{\lambda_1} \le v_1$ by Lemma 2.

Repeating the process, we have

$$\overline{u}_{\lambda_1} \le v_j, \ j = 0, 1, 2, \cdots,$$

which, together with (37), implies that $\overline{u}_{\lambda_1} \le \overline{u}_{\lambda_2}$. So $\overline{u}_\lambda$ is increasing with respect to λ.

The proof of Theorem 1 is now completed. □

The Proof of Theorem 2. (i) We prove the existence and uniqueness of positive solution problem (1).

First, we consider a generalized problem

$$\begin{cases} -\triangle u + K(x) u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u) u = b \ge 0, & \text{on } \partial\Omega, \end{cases} \tag{38}$$

where $b \ge 0$ and claim that problem (38) has a unique positive solution for each $b \ge 0$.

From Lemma 4, problem (18) has a unique positive solution u_λ, and Lemma 6 guarantees that there exists a $\delta_\lambda > 0$ such that $u_\lambda \ge \delta_\lambda$ for all $x \in \overline{\Omega}$.

In order to discuss problem (38), we consider a modified problem

$$\begin{cases} -\triangle u + K(x)(\max\{u, \delta_\lambda\})^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u) u = b, & \text{on } \partial\Omega, \end{cases} \tag{39}$$

where δ_λ is in Lemma 6 and show that the problem (39) has at least one positive solution.

Let $\overline{u}_1 = Cu_1 \in C^{2,\beta}(\overline{\Omega})$ where u_1 is the positive solution of problem (20) for $b \geq 0$, and $C > 1$ is big enough such that

$$C\lambda u_1^p + K(x)(\max\{Cu_1, \delta_\lambda\})^{-\alpha} \geq \lambda (Cu_1)^p.$$

Hence,

$$\begin{aligned} -\triangle \overline{u}_1 + K(x)(\max\{\overline{u}_1, \delta_\lambda\})^{-\alpha} &= C\lambda u_1^p + K(x)(\max\{Cu_1, \delta_\lambda\})^{\alpha} \\ &\geq \lambda (Cu_1)^p \\ &= \lambda \overline{u}^p, \quad \text{in } \Omega \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{n} \cdot \nabla \overline{u}_1 + g(\overline{u}_1)\overline{u}_1 &= \boldsymbol{n} \cdot \nabla Cu_1 + g(Cu_1) \cdot Cu_1 \\ &\geq C(\boldsymbol{n} \cdot \nabla u_1 + g_0 u_1) \\ &= C \cdot b \\ &\geq b, \quad \text{on } \partial\Omega, \end{aligned}$$

which implies that $\overline{u}_1$ is a supersolution of (39).

Set $\underline{u}(x) = u^*(x)$ where u^* is the unique positive solution of problem (18) for $b \geq 0$. We know that the problem

$$\begin{cases} -\triangle \underline{u} = \lambda \underline{u}^p, & \text{in } \Omega, \\ \underline{u} > 0, & \text{in } \Omega, \\ \mathbf{n}\nabla \underline{u} + g(\underline{u})\underline{u} = b, & \text{on } \partial\Omega, \end{cases}$$

which implies that $\underline{u}$ is a subsolution of problem (39). In fact,

$$-\triangle \underline{u} + K(x)(\max\{\underline{u}, \delta_\lambda\})^{-\alpha} \leq -\triangle \underline{u} + 0 \cdot (\max\{\underline{u}, \delta_\lambda\})^{-\alpha} = -\triangle \underline{u} = \lambda \underline{u}^p, \quad \text{in } \Omega$$

and

$$\boldsymbol{n} \cdot \nabla \underline{u} + g(\underline{u})\underline{u} = b, \quad \text{on } \partial\Omega.$$

Thus, $\underline{u}$ is a subsolution of problem (39).

Moreover, since

$$\begin{aligned} &\triangle \underline{u} + \lambda \underline{u}^p = 0 \geq K(x)(\max\{\overline{u}_1, \delta\})^{-\alpha} \geq \triangle \overline{u}_1 + \lambda \overline{u}_1^p && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla \underline{u} + g(\underline{u})\underline{u} = b \leq \boldsymbol{n} \cdot \nabla \overline{u}_1 + g(\overline{u}_1)\overline{u}_1 && \text{on } \partial\Omega \end{aligned}$$

and $\dfrac{\lambda s^p}{s} = \lambda s^{p-1}$ is strictly decreasing for $s \in (0, \infty)$, which implies that

$$\underline{u}(x) \leq \overline{u}_1(x) \ for \ x \in \overline{\Omega}$$

by Lemma 3.

Hence, problem (39) has at least one solution u satisfying

$$\delta_\lambda \leq \underline{u} \leq u \leq \overline{u}_1, \quad \text{on } \overline{\Omega}.$$

In addition, $\max\{u(x), \delta_\lambda\} = u(x)$ for all $x \in \overline{\Omega}$ guarantees that u is the solution of problem (38) as well.

Next, let us show that the solution of problem (38) is unique.

We assume that u_1 and u_2 are the solutions of problem (38). In other words, u_1, u_2 respectively satisfy

$$\begin{cases} -\triangle u_1 + K(x)u_1^{-\alpha} = \lambda u_1^p, & \text{in } \Omega, \\ u_1 > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u_1 + g(u_1)u_1 = b, & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\triangle u_2 + K(x)u_2^{-\alpha} = \lambda u_2^p, & \text{in } \Omega, \\ u_2 > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u_2 + g(u_2)u_2 = b, & \text{on } \partial\Omega. \end{cases}$$

Since

$$(\frac{f(x,s)}{s})' = \lambda(p-1)s^{p-2} + \frac{(\alpha+1)K(x)}{s^{\alpha+2}} < 0,$$

we know that $\frac{f(x,s)}{s}$ is decreasing. Therefore, Lemma 3 guarantees that $u_1 \geq u_2$ and $u_1 \leq u_2$. Thus, $u_1 = u_2$, which implies that the solution of problem (38) is unique when $K(x) < 0$.

Hence, problem (38) has a unique positive solution for each $\lambda > 0$ when $K(x) < 0$ and $b \geq 0$.

Let $b = 0$. We know that problem (1) has a unique positive solution when $K(x) < 0$ for $x \in \overline{\Omega}$.

(ii) We show that u_λ is increasing with respect to λ where u_λ is the solution of problem (1).

We assume that $\lambda_2 > \lambda_1 > 0$ and u_{λ_1}, u_{λ_2} are the corresponding unique solutions. Obviously,

$$\begin{aligned} \triangle u_{\lambda_2} - K(x)u_{\lambda_2}^{-\alpha} + \lambda_2 u_{\lambda_2}^p &= 0 \\ &= \triangle u_{\lambda_1} - K(x)u_{\lambda_1}^{-\alpha} + \lambda_1 u_{\lambda_1}^p \\ &< \triangle u_{\lambda_1} - K(x)u_{\lambda_1}^{-\alpha} + \lambda_2 u_{\lambda_1}^p, \end{aligned}$$

for $x \in \Omega$ and

$$\boldsymbol{n} \cdot \nabla u_{\lambda_2} + g(u_{\lambda_2})u_{\lambda_2} = 0 = \boldsymbol{n} \cdot \nabla u_{\lambda_1} + g(u_{\lambda_1})u_{\lambda_1}, \quad \text{on } \partial\Omega.$$

Lemma 3 implies that $u_{\lambda_1}(x) \leq u_{\lambda_2}(x)$ in $\bar{\Omega}$ since $\frac{-K(x)s^{-\alpha} + \lambda_2 s^p}{s}$ is strictly decreasing for $s > 0$ at each $x \in \Omega$. Moreover, by the extension of Lemma 3, we have $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, $x \in \Omega$. Thus, u_λ is increasing with respect to λ.

The proof of Theorem 2 is completed. □

The Proof of Theorem 3. (i) We show that there exists a $\overline{\lambda} > 0$ such that problem (1) has at least one positive solution for all $\lambda > \overline{\lambda}$ when $K(x)$ is sign-changing.

Let $\underline{K} = \min_{x\in\overline{\Omega}} K(x)$ and $\overline{K} = \max_{x\in\overline{\Omega}} K(x)$. Obviously, $\underline{K} \leq K(x) \leq \overline{K}$ and $\underline{K} < 0 < \overline{K}$. Now, Theorem 1 guarantees that there exist a $\overline{\lambda} > 0$ such that the following problem

$$\begin{cases} -\triangle u + \overline{K}u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 0, & \text{on } \partial\Omega \end{cases} \tag{40}$$

has a maximal solution u_λ for all $\lambda > \overline{\lambda}$.

Let $\underline{v} := u_\lambda$ for $\lambda > \overline{\lambda}$.

First, we consider an approximate problem $P_k(\lambda)$ of problem (1) as follows

$$\begin{cases} -\triangle u + K(x)(u)^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = \frac{1}{k}, & \text{on } \partial\Omega. \end{cases} \tag{41}$$

Obviously,

$$\begin{aligned} &-\triangle \underline{v} + K(x)\underline{v}^{-\alpha} = -\triangle u_\lambda + K(x)u_\lambda^{-\alpha} \leq -\triangle u_\lambda + \overline{K}u_\lambda^{-\alpha} = \lambda u_\lambda^p = \lambda \underline{v}^p, && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla \underline{v} + g(\underline{v})\underline{v} = \boldsymbol{n} \cdot \nabla \overline{u}_\lambda + g(\overline{u}_\lambda)\overline{u}_\lambda = 0 < \frac{1}{k}, && \text{on } \partial\Omega, \end{aligned}$$

which implies that $\underline{v}$ is a subsolution of (41) for each $k \geq 1$.

According to the previous proof Theorem 2,

$$\begin{cases} -\triangle u + \underline{K}u^{-\alpha} = \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u)u = 1, & \text{on } \partial\Omega \end{cases}$$

has a unique positive solution $\overline{w}_1$. Therefore, we obtain

$$\begin{aligned} &-\triangle \overline{w}_1 + K(x)(\overline{w}_1)^{-\alpha} \geq -\triangle \overline{w}_1 + \underline{K}(\overline{w}_1)^{-\alpha} = \lambda \overline{w}_1^p, && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla \overline{w}_1 + g(\overline{w}_1)\overline{w}_1 = 1, && \text{on } \partial\Omega, \end{aligned}$$

which implies that $\overline{w}_1$ is a supersolution of (41) for $k = 1$.

Obviously,

$$\triangle \overline{w}_1 + \lambda \overline{w}_1^p = \underline{K} \cdot \overline{w}_1^{-\alpha} \leq 0 \leq \overline{K}\underline{v}^{-\alpha} = \triangle \underline{v} + \lambda \underline{v}^p, \quad \text{in } \Omega$$

and

$$\boldsymbol{n} \cdot \nabla \overline{w}_1 + g(\overline{w}_1)\overline{w}_1 = 1 \geq 0 = \boldsymbol{n} \cdot \nabla \underline{v} + g(\underline{v})\underline{v}, \quad \text{on } \partial\Omega,$$

Lemma 3 guarantees that $\underline{v} \leq \overline{w}_1$ in Ω because $\frac{\lambda s^p}{s}$ is a strictly decreasing function for $s \in (0, \infty)$. Therefore, by Lemma 1, there exists a minimal solution u_1 satisfying problem (41) for $k = 1$ with $\underline{v} \leq u_1 \leq \overline{w}_1$.

It is easy to see that u_1 satisfies

$$\begin{aligned} &-\triangle u_1 + K(x)(u_1)^{-\alpha} \geq -\triangle u_1 + \underline{K}(u_1)^{-\alpha} = \lambda u_1^p, && \text{in } \Omega, \\ &\boldsymbol{n} \cdot \nabla u_1 + g(u_1)u_1 = 1 > \frac{1}{2}, && \text{on } \partial\Omega, \end{aligned}$$

which implies that u_1 is a supersolution of (41) for $k = \frac{1}{2}$. For $k = \frac{1}{2}$, $\underline{v}$ is a subsolution of the problem (41). Lemma 1 guarantees the minimum solution u_2 of the existing problem (41), meeting $\underline{v} \leq u_2 \leq u_1$.

Repeating the process, we can obtain a solution sequence $\{u_k\}$ such that

$$\underline{v} \leq \cdots\cdots \leq u_k \leq u_{k-1} \leq \cdots\cdots \leq u_1 \leq w_1, \tag{42}$$

where u_k satisfies problem (41) for $k \geq 1$.

Second, we consider the property of $\{u_k\}$ defined in (42).

The monotonicity and boundedness of $\{u_k\}$ guarantee that there exists a positive solution u such that

$$\lim_{k \to +\infty} u_k(x) = u(x) \geq \underline{v}(x), x \in \overline{\Omega}.$$

A standard argument in [10] shows that for each bounded $\Omega_1 \subseteq \Omega$ with $C^{1,\beta}$-boundary,

$$\lim_{k \to +\infty} u_k(x) = u(x), \text{uniformly for all } x \in \overline{\Omega}_1,$$

which implies that $u \in C^2(\Omega)$ satisfies that

$$-\triangle u + K(x)u^{-\alpha} = \lambda u^p, \;\; x \in \Omega. \tag{43}$$

Moreover, from the boundedness of $\{u_k(x)\}$ on $\partial\Omega$ and $\boldsymbol{n} \cdot \nabla u_k + g(u_k)u_k = \frac{1}{k}$ on $\partial\Omega$, we know that $\{\boldsymbol{n} \cdot \nabla u_k\}$ are bounded on $\partial\Omega$, which, together with $\lim_{k\to+\infty} u_k(x) = u(x)$ on $\partial\Omega$, guarantees that

$$\lim_{k \to +\infty} \boldsymbol{n} \cdot \nabla u_k(x) = \boldsymbol{n} \cdot \nabla u(x) \text{ and } \lim_{k \to +\infty} \frac{1}{k} = 0, \text{uniformly on } \partial\Omega.$$

Thus,

$$\boldsymbol{n} \cdot \nabla u(x) + g(u(x))u(x) = 0, \text{ on } \partial\Omega. \tag{44}$$

Combined (43) and (44), we know that u is a positive solution of problem (1) for $\lambda > \lambda'$.

(ii) For above $\lambda' > 0$, let $\lambda' < \lambda_1 < \lambda_2$. Theorem 1 implies that, for any λ_i, the following problem

$$\begin{cases} -\triangle v + \overline{K} v^{-\alpha} = \lambda_i v^p, & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla v + g(v) v = 0, & \text{on } \partial\Omega \end{cases}$$

has a maximal solution v_{λ_i} with $v_{\lambda_1} \leq v_{\lambda_2}$ for $\lambda_1 < \lambda_2$. Similarly, Theorem 2 implies that for any λ_i, the following problem

$$\begin{cases} -\triangle v + \underline{K} v^{-\alpha} = \lambda_i v^p, & \text{in } \Omega, \\ v > 0, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla v + g(v) v = 1, & \text{on } \partial\Omega \end{cases}$$

has a unique positive solution w_{λ_i} with $w_{\lambda_1} \leq w_{\lambda_2}$. Obviously, v_{λ_i} and w_{λ_i} are subsolution and supersolution of $P_k(\lambda_i)$ as follows

$$\begin{cases} -\triangle u + K(x) u^{-\alpha} = \lambda_i u^p, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u + g(u) u = \dfrac{1}{k}, & \text{on } \partial\Omega \end{cases} \tag{45}$$

respectively. According to the proof of (i), for λ_1, there exists $\{u_{\lambda_1}^{(k)}\}$ such that

$$v_{\lambda_1} \leq \cdots\cdots \leq u_{\lambda_1}^{(k)} \leq u_{\lambda_1}^{(k-1)} \leq \cdots\cdots \leq u_{\lambda_1}^{(1)} \leq w_{\lambda_1}$$

and

$$u_{\lambda_1}(x) = \lim_{k\to+\infty} u_{\lambda_1}^{(k)}(x), x \in \overline{\Omega}$$

where $u_{\lambda_1}^{(k)}$ is a minimal solution of problem (45) for $k \geq 1$ in $[v_{\lambda_1}, w_{\lambda_2}]$. For λ_2, the same argument shows that there exists $\{u_{\lambda_2}^{(k)}\}$ such that

$$v_{\lambda_2} \leq \cdots\cdots \leq u_{\lambda_2}^{(k)} \leq u_{\lambda_2}^{(k-1)} \leq \cdots\cdots \leq u_{\lambda_2}^{(1)} \leq w_{\lambda_2}$$

and

$$u_{\lambda_2}(x) = \lim_{k\to+\infty} u_{\lambda_2}^{(k)}(x), x \in \overline{\Omega},$$

where $u_{\lambda_2}^{(k)}$ is a minimal solution of problem (45) for $k \geq 1$ in $[v_{\lambda_2}, w_{\lambda_2}]$.

Since $u_{\lambda_2}^{(k)}$ satisfies

$$\begin{cases} -\Delta u_{\lambda_2}^{(k)} + K(x)(u_{\lambda_2}^{(k)})^{-\alpha} = \lambda_2 (u_{\lambda_2}^{(k)})^p > \lambda_1 (u_{\lambda_2}^{(k)})^p, & \text{in } \Omega, \\ \boldsymbol{n} \cdot \nabla u_{\lambda_2}^{(k)} + g(u_{\lambda_2}^{(k)}) u_{\lambda_2}^{(k)} = \dfrac{1}{k}, & \text{on } \partial\Omega, \end{cases}$$

we know that $u_{\lambda_2}^{(k)}$ is a supersolution of problem $P_k(\lambda_1)$, which implies $u_{\lambda_1}^{(k)} \leq u_{\lambda_2}^{(k)}$(note that $u_{\lambda_1}^{(k)}$ is a minimal solution in $[v_{\lambda_1}, w_{\lambda_2}]$) for each $k \geq 1$. Hence,

$$u_{\lambda_1} = \lim_{k\to+\infty} u_{\lambda_1}^{(k)} \leq \lim_{k\to+\infty} u_{\lambda_2}^{(k)} = u_{\lambda_2}$$

for $\lambda_1 < \lambda_2$.

So u_λ is increasing with respect to λ, and the proof of Theorem 3 is now completed. □

4. Concluding Remarks

This article has a positive impact on the problem of thermal explosion in physics. For example, the left side of the first equation in problem (2) is the heat conduction equation, which is widely used in physics. Before our research, some articles discussed the existence of positive solutions for our problem when the nonlinear term is nonnegative and is continuous at zero. In this paper, a new comparison method for an elliptic problem with a nonlinear boundary condition is designed. Based on our comparison theorem and sub-super method, we obtained the existence of the solution and the dependence of the solutions on parameters when the nonlinear term is sign-changing and singular at zero.

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